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Small group effectiveness, per capita boundedness and nonemptiness of approximate cores

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This paper is dedicated to Roko Aliprantis who has contributed greatly to the advancement of economics through his own research, his influence on others, his intellectual generosity, and through the editorship of *Economic Theory*.

Abstract

Small groups of players of a cooperative game with side payments are “effective” if almost all gains to group formation can be realized by groups of players bounded in absolute size. Per capita payoffs are bounded if the average payoff to players has a uniform upper bound, independent of the size of the total player set. It is known that in the context of games with side payments derived from pregames (which induce a common underlying structure on the potential gains to groups of players from cooperation in any game) small group effectiveness implies nonemptiness of approximate cores and the approximation can be made arbitrarily close as the player set is increased in size. Moreover, per capita boundedness, along with thickness (implying that there are many substitutes for each player) yields the same result. In this paper, using extensions of the concepts of small group effectiveness and per capita boundedness to games without side payments (NTU games), we obtain results analogous to those for games with side payments. As the prior results, the results of the current paper can be applied to economies with non-convexities, non-monotonicities, production, indivisibilities, clubs, and local public goods.

JEL Classification: C71

Keywords: NTU core; Games with many players; Small group effectiveness; Approximate core; Clubs; Local public goods

1. Small group effectiveness in games and economies

It has been shown that cooperative side-payment games with many players have nonempty approximate cores and indeed, are approximated by market games—games derived from economies where all participants have continuous, concave utility functions (Wooders, 1994a).[1] The game theoretic structure used to obtain these results is that of a pregame, which specifies the total payoff achievable by a group of players as a function of the numbers of players in the group and their characteristics. The minimal conditions required to obtain this result are (1) small group

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[1] This definition of market games is due to Shapley and Shubik (1969), which demonstrates an equivalence of market games and “totally balanced” games with side payments.

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effectiveness—almost all gains to collective activities can be achieved by cooperation only within groups of players bounded in size and (2) essential superadditivity—any payoff vector achievable by a partition of a set of players into groups is feasible for the set of players.[2] Moreover, if there are many substitutes for each player, boundedness of average feasible payoff to players – per capita boundedness – suffices for nonemptiness of approximate cores.[3]

This paper turns to NTU games and initiates a study aimed at obtaining analogous results to those described above. The advantage of our framework is that the results can be applied to a diversity of economies, including: economies with private goods, indivisibilities and non-monotonicities; economies with public goods, both local and pure and; economies with coalition production. The recent research of Bonnisseau and Iehle (2007) provides new motivation. Bonnisseau and Iehle derive a number of new relationships between cores and other solution concepts of game and economic equilibrium. Thus, to the extent that these relationships continue to hold for approximate cores, nonemptiness of approximate cores gains new significance.[4]

In this paper we demonstrate that the condition of uniform upper-boundedness of the set of equal treatment payoffs, called per capita boundedness and introduced in Wooders (1983) for NTU games with a fixed distribution of player types, suffices for nonemptiness of approximate cores when an "exceptional set" of players is ignored.[5] We then introduce a concept of small group effectiveness (SGE) for NTU games and demonstrate nonemptiness of uniform approximate cores of games with sufficiently many players. The concept is an extension of a concept with the same name for games with side payments introduced in Wooders (1992, 1994a, b).[6] Our results are extensions of the results of Wooders (1983) and our proofs rely heavily on results from that paper.

To motivate our research, we first consider TU games and provide a number of motivating examples. We are grateful to Jean-Marc Bonnisseau for suggesting this addition to the paper.

Further discussion of the literature is contained throughout the paper, in its concluding section, and in the cited papers.

2. TU games, small group effectiveness and per capita boundedness

2.1. Transferable utility games: some standard definitions

Let (N, v) be a pair consisting of a finite set N, called the *player set*, and a function v, called the *worth function*, from subsets of N to the non-negative real numbers with $v(\phi) = 0$. The pair (N, v) is a *TU game* (also called a game with side payments). Nonempty subsets of N are called *groups*.[7]

A *payoff vector* for a game (N, v) is a vector $x \in \mathbb{R}^N$.[8] A payoff vector x is *feasible* if

$$x(N) \overset{\text{def}}{=} \sum_{i \in N} x^i \leq \sum v(S^k) \tag{1}$$

for some partition $\{S^1, \ldots, S^K\}$ of N.

[2] Or, in other words, an option open to a group of players is to cooperate only within elements of a partition of the group.

[3] This result, for games with side payments, first appeared in Wooders (1979b) and, for games without side payments, in Wooders (1983).

[4] Both Predtetchinski and Herings (2004) and Bonnisseau and Iehle (2007) demonstrate necessary and sufficient conditions for nonemptiness of cores of NTU games. Bonnisseau and Iehle relate their findings to a number of techniques and concepts for markets, including, for example, the social coalitional equilibrium, introduced in Ichiishi (1981), and the partnered core, introduced in Reny and Wooders (1996).

[5] For games with side payments, per capita boundedness is equivalent to finiteness of the supremum of the per capita (or average) payoff to players, introduced in Wooders (1979b) and used in a number of papers, including, for example, Shubik and Wooders (1982).

[6] Earlier results, dating back to Wooders (1977, 1979a), use stronger conditions.

[7] To state our assumptions on the model we use the term "groups" instead of "coalitions" as we interpret the model as pertaining to socio-economic structures rather than only to the cooperative behavior suggested by the word "coalition". When we wish, however, to suggest cooperation by the members of a group, we also use the term "coalition".

[8] We regard vectors in finite dimensional Euclidean space $\mathbb{R}^T$ as functions from T to $\mathbb{R}$, and write x_i for the ith component of x, etc. If $S \subset T$ and $x \in \mathbb{R}^T$, we shall write $x_S := (x_i : i \in S)$ for the restriction of x to S. We write 1_S for the element of $\mathbb{R}^S$ all of whose coordinates are 1, or simply 1 if no confusion can arise.

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Given $\varepsilon \geq 0$, a payoff vector $x \in \mathbb{R}^N$ is in the *weak ε-core* of the game (N, v) if it is feasible and if there is a group of players $N^0 \subset N$ such that

$$\frac{|N \setminus N^0|}{|N|} \leq \varepsilon \tag{2}$$

and, for all groups $S \subset N^0$,[9]

$$x(S) \geq v(S) - \varepsilon|S| \tag{3}$$

where $|S|$ is the cardinality of the set S. The payoff vector x is in the *uniform ε-core* (or simply in the *ε-core*) if if is feasible and if (3) holds for *all* groups $S \subset N$.

Let (N, v) be a game and let $i, j \in N$. Then i and j are *substitutes* if, for all subsets $S \subset N$ with $i, j \notin S$, it holds that

$$v(S \cup \{i\}) = v(S \cup \{j\}).$$

Let (N, v) be a game and let $x \in \mathbb{R}^N$ be a payoff vector for the game. If for all players i and j who are substitutes it holds that $x_i = x_j$ then x has the *equal treatment property*. Note that if there is a partition of N into T subsets, say $N_1, \ldots, N_T$, where all players in each subset N_t are substitutes for each other, then we can *represent* x by a vector $y \in \mathbb{R}^T$ where, for each t, it holds that $y_t = x_i$ for all $i \in N_t$.

2.2. Essential superadditivity

In this paper we wish to treat games where the worth of a group of players is independent of the total player set in which it is embedded and an option open to a group is to achieve the total worths realizable by a partition of the total player set into smaller groups; that is, we treat games that are essentially superadditive. This is built into our the definition of feasibility above, (1). An alternative approach would be to assume that v is the 'superadditive cover' of some other worth function v'. Given a not-necessarily-superadditive function v', for each group S define $v(S)$ by:

$$v(S) = \max \sum v'(S^k) \tag{4}$$

where the maximum is taken over all partitions $\{S^k\}$ of S; the function v is the *superadditive cover* of v'. Then the notion of feasibility for superadditive games, requiring that a payoff vector x is feasible only if

$$x(N) \leq v(N), \tag{5}$$

gives an equivalent set of feasible payoff vectors to those of the game (N, v') with the definition of feasibility given by (1).

The following Proposition may be well known and is easily proven.[10]

Proposition 1. *Given $\varepsilon \geq 0$, let (N, v') be a game. A payoff vector $x \in \mathbb{R}^N$ is in the weak, respectively uniform, ε-core of (N, v') if and only if it is in the weak, respectively uniform, ε-core of the superadditive cover game, say (N, v), where v is defined by* (4).

In view of the above Proposition, for ease in notation we shall simply assume that (N, v) is superadditive and typically use the definition of feasibility given by (5).[11] We stress, however, that for the results of this paper there is no gain or loss in invoking the assumption of superadditivity rather than essential superadditivity and in fact this holds for numerous papers dealing with coalition economies (for example, papers on coalition economies, clubs, or Tiebout

[9] It would be possible to use two different values for epsilon in expressions (2) and (3). For simplicity, we have chosen to take the same value for epsilon in both expressions.

[10] This result was already well understood in Gillies (1959) and applications have appeared in a number of papers in the theoretical literature of game theory; see, for example, Aumann and Dreze (1974); Kaneko and Wooders (1982).

[11] For some applications, such as those with clubs or local public goods, it is important to keep in mind the underlying groups supporting outcomes in the core—that is, groups S such that $x(S) \leq v'(S)$. But unless one has something to say about such groups, there is no gain in keeping track of groups supporting outcomes in the core or in ε-cores.

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economies in Demange and Wooders, 2005, such as Jaramillo et al., 2005, or Conley and Smith, 2005, or many others on the theory of Tiebout economies or economies with clubs).

2.3. TU pregames

First, let us provide a simple example of a pregame based on the well known Shapley–Shubik glove game. In the example, we introduce the reader to the general notation and concepts used to describe our model.

Example 1: A glove pregame. Suppose there are two types of players, players who each own a RH (right-hand) glove and players who each own a LH (left-hand) glove. A (RH, LH) pair of gloves is worth 1.00. Formally, in the notation to be used below, let $\Omega = \{\omega_1, \omega_2\}$ denote a set of attributes, where ω_1 denotes the attribute "is endowed with a RH glove" and ω_2 denotes the attribute "is endowed with a LH glove". A population (S, α) is a pair consisting of a finite set S of players and a function α, called an attribute function, from S to Ω. For this example, α simply tells us which players have RH gloves and which players have LH gloves.[12] Let $F(\Omega)$ denote the set of all populations, that is, the set of all pairs (S, α) consisting of a finite set S and an attribute function $\alpha :\rightarrow \Omega$. Given $(S, \alpha) \in F(\Omega)$, define[13]

$$\Psi(S, \alpha) = \min\{|\alpha^{-1}(\omega_1)|, |\alpha^{-1}(\omega_2)|\}.$$

The pair (Ω, Ψ) is an example of a pregame. Note that the pregame is not a game since we do not yet have a set of players. Even this simple pregame, however, can be used to induce a countable collection of games. To induce a game, let (N, α) denote a population. For each nonempty group $S \subset N$ define $v_\alpha(S) = \Psi(S, \alpha_S)$, where α_S is the restriction of α to S, and define $v_\alpha(\emptyset) = 0$. The pair (N, v_α) is then a game induced by the pregame and, as to be expected since it is a glove game, the worth function of the induced game is:

$$v_\alpha(S) = \min \left\{ \begin{array}{l} \#\text{ of players with RH gloves in } S, \\ \#\text{ of players with LH gloves in } S \end{array} \right\}. \quad \square$$

We now provide the formal model of a TU pregame. Let (Ω, d) be a compact metric space of player *attributes* (or types) equipped with a metric denoted by d. Let $F(\Omega)$ denote the set of all pairs (S, α) where S is a finite, nonempty set (of players) and $\alpha :\rightarrow \Omega$ is an *attribute function*. Then, for each player $i \in S$, $\alpha(i)$ provides a complete description of the relevant characteristics of player i. The pair (S, α) is called a *population* and may be thought of as a listing of players with their ascribed attributes. For ω in Ω, the set of players in S with attribute ω is $\alpha^{-1}(\omega)$ and $|\alpha^{-1}(\omega)|$ is the number of players in S with that attribute.

A *pregame with side payments* (or a TU pregame) is an ordered pair (Ω, Ψ) where Ω is a space of attributes, and Ψ is a function (the *worth function* of the pregame) which associates to each population (S, α) in $F(\Omega)$ a non-negative real number, called the *worth of the population* (S, α). In interpretation, $\Psi(S, \alpha)$ is the total worth (or value or payoff) of a group of players S, given that the attributes of the members of S are described by the attribute function α.

Let (Ω, Ψ) be a pregame, let N be a finite set, and let α be an attribute function mapping N into Ω. The *derived game* (N, v_α) is the game with the characteristic function defined by

$$v_\alpha(S) \stackrel{\text{def}}{=} \Psi(S, \alpha_S)$$

for every nonempty subset S of N, where α_S denotes the restriction of α to S. It follows from the definition of a pregame that, in any game (N, v_α) derived from a pregame (Ω, Ψ), any two players $i, j \in N$ with the same attributes $(\alpha(i) = \alpha(j))$ are substitutes, that is, players with the same attributes are substitutes. Thus, the worth of a group of players depends on the attributes and not on the names of its members.

The pregame framework can capture a variety of economic situations. We provide two more examples. The first example illustrates a situation with ever-increasing returns to population size. The second illustrates a situation with a compact metric space of attributes.

[12] Suppose, for example, that $S = \{1, 2, 3\}$ and $\alpha(1) = \omega_1$, $\alpha(2) = \omega_1$, and $\alpha(3) = \omega_2$; the population S then consists of two players, 1 and 2, who are each endowed with a RH glove and one player, 3, who is endowed with a LH glove.
[13] If, for example, $S = \{1, 2, 3\}$, $\alpha(1) = \omega_1$, $\alpha(2) = \omega_1$ and $\alpha(3) = \omega_2$, then $\Psi(S, \alpha) = 1$.

Example 2: A marriage pregame with a compact metric space of player attributes. Le $\Omega = \{(t, \omega) : t \in \{m, f\} \text{ and } \omega \in [0, 1]\}$. In interpretation, a pair $(t, \omega) \in \Omega$ describes a player by his gender t and his socio-economic status ω. We define a metric d on Ω so that two players with difference genders are "far apart" in attribute space and players who are of the same gender differ by the Euclidean distances between their socio-economic status's.[14] The worth of a population (S, α) will be what it can achieve by partitioning into male–female pairs. Specifically, suppose that for population (S, α):

(a) If $|S| = 1$ then $\Psi(S, \alpha) = 0$ (groups consisting of only a single player earn zero);
(b) If $S = \{i, j\}, \alpha(i) = (m, \omega_1)$ and $\alpha(j) = (f, \omega_2)$ for some $\omega_1, \omega_2 \in [0, 1]$ then $\Psi(S, \alpha) = h(\omega_1, \omega_2)$ where h is some non-negative real valued, continuous function. For example, we might take $h(\omega_1, \omega_2) = \omega_1 + \omega_2$ or $h(\omega_1, \omega_2) = \omega_1\omega_2$. (groups consisting of male–female pairs have positive payoffs, depending on their attributes);
(c) Otherwise, let $\Psi(S, \alpha)$ equal the maximum total sum achievable by partitioning S into male–female pair and singleton groups.

In this example small groups are strictly effective in the sense that partitions of the total player set into groups bounded in size (by two) can realize all gains to collective activities. This feature is not necessary for our results; the example, however, illustrates another sort of case that will be covered by our assumptions.

Note that players who are close in terms of the metric on attribute space are close substitutes, that is, they are nearly equally valuable as population members. If the Euclidean distance between (ω_1, ω_2) and (ω_1, ω_2) is small, then a male–female pair with socio-economic status's ω_1 and ω_2 has nearly the same worth as a pair with status's ω_1', ω_2' □.

While our results require only essential superadditivity, for ease in notation and exposition, and since it makes no difference to the results, we will assume superadditivity. First, given two populations (S, α) and (T, β), write $S \vee T$ for the disjoint union of S and T and $\alpha \vee \beta$ for the function from $S \vee T$ to Ω defined by:

$$\alpha \vee \beta(i) \stackrel{\text{def}}{=} \begin{cases} \alpha(i) \text{ if } i \in S \\ \beta(i) \text{ if } i \in T \end{cases}.$$

Superadditivity: A pregame (Ω, Ψ) is superadditive if for all (S, α), (T, β) in $F(\Omega)$ we have

$$\Psi(S, \alpha) + \Psi(T, \beta) \le \Psi(S \vee T, \alpha \vee \beta).$$

Superadditivity is a natural assumption when one of the options open to a group of players is to partition into disjoint subgroups and the worth of each subgroup is independent of the worths to the remaining subgroups.

We require that players who have similar attributes are approximately substitutes, a continuity assumption. We call this 'per capita continuity' since the distance between the worths of two populations (which differ only in the attributes of the players) is divided by the number of players in the population.

Per capita continuity: Let (Ω, Ψ) be a pregame. We assume that:

Given $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that for any two populations (S, α) and (S, β) (with the same set of players) if $d(\alpha(i), \beta(i)) \le \delta(\varepsilon)$ for all $i \in S$, then

$$\frac{\Psi(S, \alpha)}{|S|} - \frac{\Psi(S, \beta)}{|S|} \le \varepsilon. \tag{6}$$

Throughout the following, we shall assume that pregames satisfy superadditivity and per capita continuity. Note that from the definition of a pregame it follows that in any induced game players with the same attributes are substitutes for each other.

[14] For example, we could let d' denote the usual Euclidean distance on [0,1] and let $d((t, \omega), (t', \omega')) = \delta_{t,t'} + d'(\omega, \omega')$ where $\delta_{t,t'} = 2$ if $t \ne t'$ and $\delta_{t,t'} = 0$ otherwise.

2.4. Small group effectiveness and per capita boundedness

To motivate the following concepts, we provide another example. This example satisfies both the conditions, to be introduced below, of per capita boundedness and small group effectiveness.

Example 3: An increasing returns to group size pregame. Let $\Omega = \{\omega\}$; there is only one attribute. Since there is only one attribute, an attribute function must assign all players the same attribute ω and the worth of a population (S, α) will depend only on $|S|$, the number of players in the population. Let

$$\Psi(S, \alpha) = |S| - \frac{1}{|S|}.$$

This pregame exhibits ever-increasing returns to group size. But per capita payoffs are bounded, that is, there is a constant C, in this case 1, such that

$$\frac{\Psi(S, \alpha)}{|S|} \leq C$$

for all populations (S, α). Also, given $\varepsilon > 0$, there is a population (S^0, α) so that for all populations (S, α) with $|S| > |S^0|$, it holds that

$$0 \leq \frac{\Psi(S, \alpha)}{|S|} - \frac{\Psi(S^0, \alpha)}{|S^0|} \leq \varepsilon;$$

thus, almost all (within ε per capita) gains to collective activities can be realized by groups bounded in size by $|S^0|$, that is, small groups are effective. □

A pregame (Ω, Ψ) satisfies *per capita boundedness*, PCB, if there is a constant C such that for all populations (S, α) it holds that

$$\frac{\Psi(S, \alpha)}{|S|} \leq C, \tag{7}$$

that is, per capita payoffs are bounded over all populations (S, α) or, in other words, the supremum of average worths is finite.

If a pregame (Ω, Ψ) satisfies per capita boundedness then, as Theorem 1 in the next section will demonstrate, for both TU and NTU games, given $\varepsilon > 0$ all derived games with sufficiently many players have nonempty weak ε-cores. As the following example illustrates, however, when there is more than one type of player this result may not hold for uniform ε-cores.

Example 4. Consider a pregame (Ω, Ψ) where $\Omega = \{\omega_1, \omega_2\}$ and Ψ is the superadditive cover of the function Ψ' defined by:

$$\Psi'(S, \alpha) = \begin{cases} |S| & \text{if } |\alpha^{-1}(\omega_1)| = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if a group of players S contains two players with attribute ω_1 then the worth of the group is equal to the number of players. Otherwise, the worth of S is zero.

Now consider a sequence of games (S^ν, v_{α^ν}) where $|\alpha^{\nu^{-1}}(\omega_1)| = 3$ and $|\alpha^{\nu^{-1}}(\omega_2)| = \nu$ for all ν. Note that if the uniform ε_0-core were nonempty, it would have to contain an equal-treatment payoff vector.[15] For the purpose of

[15] It is well known and easily demonstrated that the uniform ε-core of a TU game is nonempty if and only if it contains an equal treatment payoff vector. This follows from the fact that the uniform ε-core is a convex set.

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demonstrating a contradiction, suppose that $x^\nu = (x_1^\nu, x_2^\nu)$ represents an equal treatment payoff vector in the uniform ε-core of (S^ν, v_{α^ν}). The following inequalities must hold:

$$3x_1^\nu + vx_2^\nu \le \nu + 3$$
$$2x_1^\nu + vx_2^\nu \ge \nu + 3, \quad \text{and}$$
$$x_1^\nu \ge \frac{3}{4},$$

which is impossible. A payoff vector which assigns each player zero is, however, in the weak ε-core for $\varepsilon > (3/\nu + 3)$. It is not very appealing, however, in situations such as this to ignore a relatively small group of players who can have a large effect on per capita payoffs. This leads us to the next concept. □

In general, given a population (S, α), a partition of the population is a collection of (sub) populations $\{(S^k, \alpha^k)\}$ where $\{S^k\}$ is a partition of S and where, for each k, α^k is the restriction of α to S^k.

A pregame (Ω, Ψ) satisfies *small group effectiveness*, SGE, if it is superadditive[16] and if, given any real number $\varepsilon > 0$, there is an integer $\eta_0(\varepsilon)$ such that for each population (S, α), for some partition $\{(S^k, \alpha^k)\}$ of (S, α) into subpopulations with $|S^k| \le \eta_0(\varepsilon)$ for each subpopulation (S^k, α^k) in the partition it holds that

$$\Psi(S, \alpha) - \sum_k \Psi(S^k, \alpha^k) \le \varepsilon|S|.$$

Thus, for every population (S, α), almost all (within ε per capita) gains to collective activities can be realized by aggregating collective activities within groups of players bounded in absolute size.

Small group effectiveness is a natural relaxation of the condition that *all* gains to collective activities can be realized by groups of players uniformly bounded in size, now commonly called *strict small group effectiveness.* [17] This condition is satisfied by Example 3 above but not by Example 4. The beauty of small group effectiveness is that, in addition to being as nonrestrictive, when there are many players of each type, as per capita boundedness, it allows us to approximate games with many players and potentially large effective groups by games with groups bounded in size (as will be demonstrated in the proofs).

If there are sufficiently many players of each of a finite number of player types (that is, the players attributes are from some finite set) then per capita boundedness is equivalent to small group effectiveness (Wooders, 1994a, Theorem 4) – besides bounding per capita payoffs, the function of SGE is to ensure that players with few substitutes cannot have significant impacts on average payoffs (as players with attribute ω_1 can have in Example 4). The following result is stated and proven for the case where Ω is a finite set and demonstrates more generally that when there are many substitutes for each player, then SGE is equivalent to PCB.

2.4.1. Theorem: (Wooders, 1994a) With "thickness", per capita boundedness ≈ small group effectiveness
Let (Ω, Ψ) be a pregame where Ω is a finite set.

1. *Suppose that (Ω, Ψ) satisfies PCB. Then for each pair of real numbers $\rho > 0$ and $\varepsilon > 0$ there is an integer $\eta(\rho, \varepsilon)$ such that, for every population (S, α) with $(|\alpha^{-1}(\omega)|/|S|) > \rho$ or $|\alpha^{-1}(\omega)| = 0$ for each $\omega \in \Omega$, for some partition $\{(S^k, \alpha^k)\}$ of (S, α) with $|S^k| \le \eta(\rho, \varepsilon)$ for each k, it holds that*

$$\Psi(S, \alpha) - \sum_k \Psi(S^k, \alpha^k) \le \varepsilon|S|;$$

that is, if the domain of the pregame is restricted so that the percentage of players of each type that appears in the population is bounded away from zero, then the pregame satisfies small group effectiveness on this domain.
2. *Suppose that (Ω, Ψ) satisfies SGE. Then (Ω, Ψ) satisfies PCB.*

[16] We stress that essential superadditivity would suffice.
[17] A condition of strict small group effectiveness was introduced in Wooders (1977). The condition there dictated that, for some bound $\mathbb{B}$, any payoff that could be improved upon could be improved upon by a group of players containing no more than $\mathbb{B}$ members of any type. See Winter and Wooders (1990) and Kovalenkov and Wooders (2003, 2005) for some implications of strict small group effectiveness..

Note that for Example 3, PCB and SGE are equivalent; since there is only one type of player, any game with many players must have many players of each type.

Our Theorems in the next section will demonstrate that, for all sufficiently large games derived from pregames, (1) per capita boundedness implies nonemptiness of weak ε-cores and, with an additional condition, (2) small group effectiveness implies nonemptiness of uniform ε-cores.

3. NTU game and pregames

3.1. NTU games

The primary difference between a TU and a NTU game is that, for the TU case, the payoff possibilities for a group of players are determined by a real number, the worth of the group. For a NTU game the payoff possibilities are described by a set, whose elements are feasible payoff vectors.

A *NTU game* (in coalitional function form) is a pair (N, V) where N is a finite set (the set of *players*) and V is a set-valued function that assigns to each nonempty subset S of N (a *group* or *coalition*) a nonempty subset $V(S)$ of $\mathbb{R}^S$, called a *payoff possibilities set* or simply a *payoff set*, with the following properties:

$V(S)$ is a closed subset of $\mathbb{R}^S$, comprehensively generated[18] by $V(S) \cap \mathbb{R}^S_+$;
$0 \in V(S)$;
$V(S) \cap \mathbb{R}^S_+$ is bounded.

A *payoff vector for a game* (N, V) is a vector x in $\mathbb{R}^N$. A payoff vector x is *feasible* for N if there exists a partition $\{S^k\}$ of N with the property that $x_{S^k} \in V(S^k)$ for each k. With this definition of feasibility, we say that the game (N, V) is *essentially superadditive*.

Given $\varepsilon \geq 0$, a payoff vector x is in the *weak ε-core* of (N, V) if it is feasible and if there is a subset $N^0 \subset N$ such that $(|N^0|/|N|) \leq \varepsilon$ and, for every subset S of $N \setminus N^0$, $x_S + \varepsilon 1_S \notin \text{ int } V(S)$. A payoff vector x *uniform ε-core* of a game (N, V) if it is feasible for N and if, for every subset S of N, $x_S + \varepsilon 1_S \notin \text{ int } V(S)$.[19]

Informally, a feasible payoff vector x is in the weak ε-core if no set of players can improve upon x by more than ε for each player in the set, provided that we ignore an exceptional set of players, which we shall call “leftovers”, that constitutes, at most, a small fraction of the entire player set. A feasible payoff vector x is in the uniform ε-core if no set of players can improve upon x by more than ε for each player in the set.

Our notion of the weak ε-core is analogous to notions of approximate cores and approximate equilibria that are used in the literature of private goods exchange economies (see, for example, Hildenbrand, 1974, p. 202). The notion of the uniform ε-core also appears in the literature, typically in the presence of assumptions on the convexity of $V(N)$ or on some degree of transferability of utility (or, in other words, “nonlevelness” of payoff possibilities sets). Payoffs in either the weak ε-core or the uniform ε-core may be interpreted as stable if players are satisficing or approximately optimizing, or if there are costs to coalition formation.

As noted and illustrated in Wooders (1983), to go from weak ε-cores to uniform ε-cores can be done in some circumstances with convexity assumptions on payoff sets or with “nonleveledness” of payoff sets,[20] insuring a degree of “side-paymentness”. See Kaneko and Wooders (1996) for game theoretic applications and Wooders (1988) or Allouch and Wooders (2007) for examples of application to economic models of an economy with local public goods/clubs.

Our results will also show that there are payoff vectors in approximate cores that treat substitute players equally.[21] For NTU games our definition of substitutes requires that if i and j are substitutes then they make the same contribution to any group they might join and, if they both belong to one group and a payoff vector x is feasible for the group, then

[18] That is, $x \in V(S)$ if and only if there is some $y \in V(S) \cap \mathbb{R}^S_+$ such that $y \geq x$.

[19] It would be possible to include the requirement that x is Pareto-optimal in the sense that there does not exist another feasible payoff y for N with $y \geq x$, $y \neq x$. We do not do so, however, since it does not seem consistent with the notion of an approximate core.

[20] Called the QTU property in Wooders (1983).

[21] In fact, we demonstrate that, for the weak ε-core, there are payoff vectors that treat most similar players similarly and, for the uniform ε-core, there are payoff vectors that treat all similar players nearly equally, where “similar players” have “similar” attributes. These results are obtained by approximating the compact set of attributes by a finite set of types and then using continuity.

x' is also feasible for the group, where x' is derived from x by interchanging the payoffs of i and j. More formally, consider a NTU game (N, V). Two players $i, j \in N$ are *substitutes* if

1. For any $S \subset N$ such that $i, j \notin S$ if $x \in V(S \cup \{i\})$ then $x' \in V(S \cup \{j\})$ where x' is defined by $x'_j = x_i$ and $x'_\ell = x_\ell$ for all $\ell \in S$.
2. For any $S \subset N$ such that $i, j \in S$ if $x \in V(S)$ then $x' \in V(S)$ where x' is defined by $x'_j = x_i$, $x'_i = x_j$ and $x'_\ell = x_\ell$ for all $\ell \in S$, $\ell \neq i, j$.

Let (N, V) be a game and let $x \in V(N)$. Then x has the *equal treatment property* if, and only if, for all players i and j who are substitutes, it holds that $x_i = x_j$.

The same sort of remarks concerning superadditivity as those made in Section 2.2 continue to hold for NTU games. For details, if desired, we refer the reader to the Vanderbilt Working Paper version of this paper.

3.2. NTU pregames

To define a NTU pregame, we repeat some definitions from TU pregames:

1. Ω denotes a compact metric space (the space of attributes) with the distance function d.
2. $F(\Omega)$ denotes the set of all pairs (S, α), called populations, where S is a finite non-empty set (of players) and $\alpha : S \to \Omega$ is a function (an attribute function).
3. For ω in Ω, the set of players in S with attribute ω is $\alpha^{-1}(\omega)$ and $|\alpha^{-1}(\omega)|$ is the number of players in S with that attribute.

The difference between TU and NTU games arises in the definition of the worth or payoff possibilities set of a group of players.

A (*NTU*) *pregame* is an ordered pair (Ω, ϕ) where Ω is a space of attributes and ϕ is a function (the worth function or payoff possibilities function) that associates to each population (S, α) in $F(\Omega)$ a subset $\phi(S, \alpha)$ of $\mathbb{R}^S$, called a payoff possibilities set or simply a *payoff set*, that is closed, comprehensively generated by $\phi(S, \alpha) \cap \mathbb{R}^S_+$, contains the origin, and has bounded intersection with the positive orthant $\mathbb{R}^S_+$. In interpretation, $\phi(S, \alpha)$ represents the set of payoff vectors corresponding to all possible payoff vectors that the group of players S can achieve for its members, given that the attributes of the members of S are as described by the attribute function α.

Example 5. To illustrate a NTU pregame, we will convert the glove game of the preceding section into an NTU pregame. Let $\Omega = \{\omega_1, \omega_2\}$ denote a set of attributes, where ω_1 denotes the attribute "is endowed with a RH glove" and ω_2 denotes the attribute "is endowed with a LH glove". As previously, given a population (S, α), the attribute function α assigns one glove to each player. For a population (S, α) consisting of only one player we define

$$\phi(S, \alpha) = \{x \in \mathbb{R} : x \leq 0\}.$$

For a population (S, α) consisting of a pair of players with attributes ω_1, ω_2 we define

$$\phi(S, \alpha) = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq \frac{1}{2}, x_2 \leq \frac{1}{2}\right\}.$$

Using this data, we can construct the payoff possibilities for any population (S, α) by taking $\phi(S, \alpha)$ as the set of payoff possibilities achievable by the population (S, α) when it is partitioned into groups consisting of pairs of players where one player is endowed with a RH glove and the other player is endowed with a LH glove and groups consisting of only one player.

We could also modify the example to assume that payoff sets for the pregame are given by the convex hulls of the payoff sets $\phi(S, \alpha)$, In this case, for example, if there were three players in a derived game, player 1 endowed with a RH glove and players 2 and 3 endowed with LH gloves, then, since the payoff vectors $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, \frac{1}{2})$ are feasible, the payoff vector $(\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ would be feasible. Alternatively, we could allow some transfers to be made between players, but not necessarily at a one-to-one rate. □

We will require that a pregame satisfies superadditivity. Superadditivity for the NTU case is in interpretation the same as for the TU case and the discussion of essential superadditivity of TU games applies also to NTU games.

The function ϕ is *superadditive* if for all (S, α), (T, β) in $F(\Omega)$ we have

$$\phi(S, \alpha) \times \phi(T, \beta) \subset \phi(S \vee T, \alpha \vee \beta).$$

This is superadditivity in the usual sense: the union of two disjoint groups can always obtain for itself anything that the groups could obtain separately.

Let (Ω, ϕ) be a pregame, let N be a finite set, and let α be an attribute function mapping N into Ω. The derived game (N, V_α) is the game with V_α defined by

$$V_\alpha(S) \stackrel{\text{def}}{=} \phi(S, \alpha_S)$$

for every nonempty subset S of N, where α_S denotes the restriction of α to S. From the definition of a pregame it follows that a pregame (Ω, ϕ) satisfies *substitution*, the property that, for all populations (S, α), whenever $\alpha(i) = \alpha(j)$ for two players i, j in S then i and j are substitutes.

Let (N, V_α) be a game derived from a pregame (Ω, ϕ). A payoff vector $x \in V_\alpha(N)$ has the *equal-treatment property* if

$$x_i = x_j \text{ whenever } \alpha(i) = \alpha(j).$$

3.3. Nonemptiness of approximate cores and equal treatment

In this section we introduce two concepts of small group effectiveness and show that they both imply nonemptiness of approximate cores.

Weak approximate core property: A NTU pregame (Ω, ϕ) has the *weak approximate core property* if: Given any real number $\varepsilon > 0$ there is an integer $n_1(\varepsilon)$ such that for all populations (S, α) with $|S| \geq n_1(\varepsilon)$, the weak ε-core of (S, V_α) is nonempty.

We actually will demonstrate that there exists equal-treatment payoff vectors in weak ε-cores of derived games with sufficiently many players.[22]

Weak equal treatment approximate core property: A NTU pregame (Ω, ϕ) has the *weak equal-treatment approximate core property* if: given any $\varepsilon > 0$ there is an integer $\eta_2(\varepsilon)$ such that for all populations (S, α) with $|S| \geq \eta_2(\varepsilon)$, there is a payoff vector x in the weak ε-core of (S, V_α) with the properties that for some $S' \subset S$ with $(|S'|/|S|) > 1 - \varepsilon$,

1. $x_{S'} \in V_\alpha(S')$,
2. $x_W + \varepsilon 1_W \notin \operatorname{int} V_\alpha(W)$ for any group W of S', and
3. $x_{S'}$ has the equal-treatment property.

Our first Theorem will use the following continuity condition:

Per capita continuity with respect to attributes: A NTU pregame (Ω, ϕ) satisfies *per capita continuity with respect to attributes* if: For every $\varepsilon > 0$ there is a $\delta > 0$ such that for all populations $(S, \alpha), (S, \beta) \in F(\Omega)$ with $\mathrm{d}(\alpha(i), \beta(i)) < \delta$ for all i, it holds that

$$H_1(\phi(S, \alpha), \phi(S, \beta)) < \varepsilon|S|$$

where H_1 is the Hausdorff distance[23] relative to the metric

$$||x - y||_1 = \sum_i |x(i) - y(i)|.$$

Note that the two populations (S, α) and (S, β) have the same set of players but the attributes of the players may have changed. Per capita continuity with respect to attributes dictates that players whose attributes are close in attribute space are approximate substitutes in induced games.

[22] It is also interesting to ask when all payoffs in approximate cores treat similar players approximately equally. This is addressed in Wooders (1983, Theorem 3), Wooders (1994b) and Kovalenkov and Wooders (2003) and other works.

[23] We refer the reader to Hildenbrand (1974, p. 16) for a definition of the Hausdorff distance.

Per capita boundedness: A pregame (Ω, ϕ) satisfies *per capita boundedness* if: There is a constant K such that for any population (S, α) and its derived game (S, V_α), if $x \in V_\alpha(S)$ and x has the equal treatment property, then $x_i \leq K$ for all $i \in S$.

The reader can observe that the above example and those introduced in the preceding section all satisfy per capita boundedness.

Theorem 1. *Let (Ω, ϕ) be a NTU pregame satisfying per capita boundedness and per capita continuity with respect to attributes. Then (Ω, ϕ) has the weak equal-treatment approximate core property.*

Theorem 2, demonstrating nonemptiness of uniform equal treatment ε-cores, requires small group effectiveness.

Small group effectiveness: A pregame (Ω, ϕ) satisfies *small group effectiveness* if: For every $\varepsilon > 0$ there is an integer $\eta_3(\varepsilon)$ such that for every pair (S, α) in $F(\Omega)$,

$$H_1(\phi^{etp}(S, \alpha_S), \cup\Pi\phi^{etp}(S^k, \alpha_{S^k})) \leq \varepsilon|S|,$$

where the union is taken over all partitions $\{S^k\}$ of S with $|S^k| \leq \eta_3(\varepsilon)$ for each member S^k of the partition and where $\phi^{etp}(S, \alpha_S)$ denotes the set of payoff vectors in $\phi(S, \alpha_S)$ with the equal treatment property (and similarly for $\phi^{etp}(S^k, \alpha_{S^k})$).

We also use the term "strict small group effectiveness" in our proofs. The pregame (Ω, ϕ) satisfies this condition if, for some sufficiently large (but finite) value of $\eta_3(\varepsilon)$, in the above definition ε can be set equal to zero.

It is easy to prove that small group effectiveness implies per capita boundedness. Thus we have the following Corollary to Theorem 1.

Corollary 1. *Let (Ω, ϕ) be an NTU pregame satisfying small group effectiveness and per capita continuity with respect to attributes. Then (Ω, ϕ) has the weak equal-treatment approximate core property.*

Small group effectiveness, by itself, is not enough to ensure nonemptiness of uniform approximate cores for derived games with many players. It is also necessary that leftover players can be compensated. For this purpose, we require the following assumption.

Compensation: A pregame (Ω, ϕ) satisfies *compensation* if there is is a positive real number $0 < c \leq 1$, such that, for any population (S, α) and derived game (S, V_α), if x is individually rational (that is, $x_i \geq \max\{y \in \mathbb{R} : y \in V_\alpha(\{i\})\}$ for each $i \in S$) then $y \in V_\alpha(S)$, where, for some subset $S' \subset S$ and each $i \in S'$, $y_i = x_i + (c'/|S'|)|S \setminus S'|$ and, for all $i' \in S \setminus S'$, $y_{i'} = x_{i'} - c'$ where c' is any positive real number less than c (that is, $0 \leq c' \leq c$).

Compensation ensures that, given a payoff vector x that is individually rational for the players in a derived game (S, V_α), it is possible to construct another feasible payoff vector y by taking away a small positive amount c' from each of the players in a subset $S \setminus S'$ of S and increasing the payoff to players in S' by $(c'/|S'|)|S \setminus S'|$ for each player. Note that compensation, with $c = 1$, is satisfied in any TU game.

Example 6. Consider a Shubik "bridge pregame". There is only one attribute so any attribute function α assigns all players the same attribute. Any group of four players can realize a payoff of \$1.00 each. Thus, for a population, say $S = \{1, 2, 3, 4\}$, consisting of four players, we have

$$\phi(S, \alpha) = \{(x_1, x_2, x_3, x_4) : (x_1, x_2, x_3, x_4) \leq (1, 1, 1, 1)\}$$

If the number of players is not a multiple of four, however, not all players can play bridge—after as many bridge tables as possible are formed, there may be leftover players. Suppose a group consisting of fewer than four players can realize only zero for each of its members. Then, if a group S does not have a multiple of four members, it holds that any equal treatment payoff vector x assigns each player no more than zero. Note, however, that there will be at most three players leftover, that is, for any group S, for some non-negative integer r, it holds that $|S| = 4r + L$ where $0 \leq L \leq 3$. This implies that $4r$ players can play bridge but L players will be leftover. Compensation and small group effectiveness ensure that it is possible to 'tax' those players accommodated at bridge tables and make transfers to the leftovers until the leftover players are as well treated as those who get to play bridge.[24] □

[24] In general, comprehensiveness is also required.

Concepts of uniform approximate core properties are defined analogously to the week approximate core properties.

Theorem 2. *Let* (Ω, ϕ) *be an NTU pregame satisfying small group effectiveness and per capita continuity. If, in addition,* (Ω, ϕ) *satisfies compensation then* (Ω, ϕ) *has the uniform equal-treatment approximate core property.*

A result such as the above could also be obtained using assumptions of convexity of payoff sets, as in Wooders (1983). In some interesting situations to which the results of this paper can be applied – economies with clubs, coalitions, and/or local public goods, for example – convexity of payoff sets is restrictive while the ability to "transfer" some payoff by transferring private goods from members of one club to another club seems relatively non-restrictive. (See, for example, Allouch and Wooders, 2007).

4. Relationships to the literature and concluding remarks

Shapley and Shubik (1966) showed that private goods exchange economies with many players, all with quasi-linear preferences, have nonempty approximate cores. The framework used in this paper is an outgrowth of that introduced in Wooders (1977, 1983) for TU and NTU games (respectively) with finite number of player types. In Wooders (1983) mild conditions are determined under which large NTU games with a fixed distribution of a finite number of player types have nonempty uniform ε-cores. Effectively, in that paper the Lemmas show that large replica games (games with the same percentage of players of each of a finite number of types) have nonempty equal-treatment (weak) approximate cores. This fact, plus convexity, is then used to show nonemptiness of uniform approximate cores of all sufficiently large replica games. Shubik and Wooders (1983) continue this research by defining the weak ε-core and, using the lemmas of the earlier work, show that large replica games have nonempty weak ε-cores.

In the current paper we first state a Lemma showing that, with a finite number of player types, any large player set is approximately a large replica of a fixed player set. To show that small group effectiveness implies nonemptiness of approximate cores, in the finite-types case we use the Lemma to approximate a large player set by a replica of a relatively small player set, which enables us to use the results of the prior papers, especially Wooders (1983). Since, for the general case, the space of types is assumed to be a compact metric space, we can use continuity and compactness to approximate large games by ones with a finite number of player types.

To obtain nonemptiness of uniform ε-cores, we introduce the condition of small group effectiveness for NTU games. This condition is an extension of the condition of Wooders (1992, 1994a, b) to NTU games and, along with the compensation property, implies that small groups of players can be compensated for not being in preferred groups. In the TU case, the compensation property is built into the framework; the standard definition of games with side payments allows compensation of leftover players. For the NTU case, we need to make the additional assumption of compensation. We stress that compensation applies to games derived from club economies or economies with local public goods, where there is some infinitely divisible good(s) that can be transferred between clubs, that everyone wants, and that everyone has, while convexity of payoff possibilities sets may well not hold; see, for example, Allouch and Wooders (2007).

A number of extensions and variations of the main results of Wooders (1983) have been obtained. The restriction to a finite set of types was first relaxed in working paper versions of Kaneko and Wooders (1996). Wooders and Zame (1987),[25] as a consequence of the result that NTU Shapley values are in approximate cores of games with many players, demonstrate nonemptiness of approximate cores (both weak and uniform) under a condition of boundedness of individual marginal contributions to coalitions.[26] This condition, however, also used in Wooders and Zame (1984)

[25] And in an unpublished typescript, with working title "Approximate cores of games with many players". In that paper, Wooders and Zame (1989) demonstrate nonemptiness of approximate cores of NTU games with many players, without the replication restriction of Wooders (1983). As in their 1984 paper, they rely upon a condition of boundedness of marginal contributions. Their condition, however, is more difficult to apply and also is more restrictive than both per capita boundedness and small group effectiveness.

[26] An NTU pregame (Ω, ϕ) satisfies *boundedness of marginal contributions* if that there is a constant M for the pregame such that:
if $(S, \alpha) \in F(\Omega)$ and $(\{i\}, \beta) \in F(\Omega)$, where $\{i\}$ is a singleton, and if $x \in \phi(S \vee \{i\}, \alpha \vee \beta)$
but $x_S \notin \operatorname{int} \phi(S, \alpha)$, then

$$\sum_{j \in S \cup \{i\}} x_j - \sum_{j \in S} x_j \leq M.$$

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for TU games, is stronger than required. Roughly, the Wooders-Zame condition bounds marginal contributions while small group effectiveness, introduced in Wooders (1992), bounds average contributions; an example making this point for TU games appears in Wooders (1994b).

Most recently, Kovalenkov and Wooders, in a series of papers, derive conditions under which games in parameterized collections have nonempty approximate cores; see Kovalenkov and Wooders (2001, 2003, 2005) and references therein. In the Kovalenkov–Wooders papers, a collection of games is parameterized by (a) the number of approximate types of players and the goodness of the approximation and (b) the size of nearly effective groups of players and their distance from exact effectiveness. All games described by the same parameters are members of the same collection. The conditions required on a parameterized collection of games to ensure nonemptiness of approximate cores are merely that most players have many close substitutes and all or almost all gains to collective activities can be realized by groups of players bounded in size (small group effectiveness). The Kovalenkov–Wooders approach has the advantage that the results apply to given games. The framework of parameterized collections of games, however, is not readily adaptable to limiting results, such as those of Wooders (1994a) for games with side payments or for continuum limit results for economies. This motivates the continued study of games derived from NTU pregames.

We emphasize that the results of this paper can be applied to a variety of economic situations, including economies with only private goods with indivisibilities and nonconvexities, with clubs or local public goods as in Conley and Wooders (2001), with coalition production, and so on. In a number of papers on clubs and/or local public good economies, including Conley and Wooders, approximate cores have been decentralized by price-taking economic equilibrium. Allouch and Wooders (2007) takes full advantage of nonemptiness of approximate cores under per capita boundedness in application to club economies with possibly ever-increasing returns to club size. Whether the results can be applied or extended to apply to the broad classes of economies such as those considered in Aliprantis and Burkinshaw (1991) is an open question.

In conclusion, we remark that under stronger conditions of small group effectiveness (in particular, when within ε of all gains to coalitions can be realized by groups of players bounded in size, rather than ε per capita), then stronger forms of the results could be obtained, such as nonemptiness of strong ε-cores.[27] In the TU case, if strict small group effectiveness is satisfied, then all sufficiently large games derived from a (TU) pregame have nonempty strong ε-cores. Fewer economic models would satisfy the required conditions, however.

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Appendix A

We proceed as follows: First, we provide some definitions and notation and state a handy approximation Lemma for the case of pregames with a finite set of attributes. (These concepts could also be stated for the case of a compact metric space of attributes, but this is not necessary.) Before proving Theorem 1, however, we provide a discussion of some prior results for games with a fixed distribution of player types. We then prove Theorem 1 by appealing to results on nonemptiness of approximate cores of replica games – games with a fixed finite set of player types – from Wooders (1983); Shubik and Wooders (1983). Then, using per capita continuity to approximate games by games satisfying the conditions of the prior papers, we conclude the proof. We then turn to the proof of Theorem 2. Again, we use the prior results, discussed in more detail the following section.

Let us first recall the definition of balancedness, a concept which plays a significant role in our proofs. Given a game (N, V), we first modify the payoff possibilities sets for groups $S \subset N$ to be subsets of $\mathbb{R}^N$ with coordinates associated with nonmembers of S unconstrained. Formallly, for group $S \subset N$ define $V'(S)$ by

$$V'(S) = \{x \in \mathbb{R}^N : x_S \in V(S)\}.$$

Now define payoff sets for the *balanced cover game*, say $(N, \tilde{V})$, by:

$$\tilde{V}(S) = V'(S) \text{ for all } S \subset N, S \neq \emptyset, S \neq N$$

[27] The strong ε-core requires feasibility and that no coalition can improve by more than ε per capita.

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and

$$\tilde{V}(N) = \cup\cap_{S\in B} V(S)$$

where the union is taken over all balanced collections of subsets of N. A collection B of subsets of N is *balanced* if there exists non-negative weights w_S for $S \in B$, called balancing weights, such that $\sum_{\substack{S\in B\\ i\in S}} w_s = 1$ for each $i \in N$. A balanced cover game is a balanced game, as defined in Scarf (1967). Since $\{N\}$ is a partition of N the balanced cover game generated by a balanced game is the balanced game itself.

A.1. Pregames with a finite set of attributes

Let (Ω, ϕ) be a pregame where Ω is a finite set, say $\Omega = \{\omega_1, \ldots, \omega_T\}$. Let (S, α) be a population. For each $t = 1, \ldots, T$, let $s_t = |\alpha^{-1}(\omega_t)|$ and let $s = (s_1, \ldots, s_T)$. The vector s is called the *profile of* (S, α). Observe that each population (S, α) generates a *profile* $s \in \mathbb{Z}_+^T$. Also, a profile determines a population (unique up to a re-naming players of the same types). Observe also that with any partition of the set S, say $\{S^1, \ldots, S^K\}$ we can associate a collection of profiles, say $\{s^1, \ldots, s^{K'}\}$, where s^k is in the collection if and only if it is the profile of some member of $\{S^1, \ldots, S^K\}$.

Kovalenkov and Wooders (2003, Lemma 2) provide a stronger version of the following Lemma.

Lemma 1. *Let (Ω, ϕ) be a pregame where $\Omega = \{\omega_1, \ldots, \omega_T\}$. Let $\{s^\nu\}$ be a sequence of profiles such that $||s^\nu|| \to \infty$ as $\nu \to \infty$ and $(1/||s^\nu||)s^\nu \to s$ for some $s \in \mathbb{R}^T$. Then given any $\varepsilon > 0$ there is a profile h and an integer $\nu(\varepsilon)$ such that for each $\nu \geq \nu(\varepsilon)$, for some integer r_ν and some profile ℓ^ν we have $r_\nu h + \ell^\nu = s^\nu$ and $||\ell^\nu||/||s^\nu|| < \varepsilon$. Moreover, when s is rational-valued, we can take $h = ms$ for some integer m such that ms is integer-valued.*

For the proof of Theorem 2, it will be useful to describe a balanced family of subsets of a population by a condition on profiles of members of the family. Roughly, balanced collections can be described by profiles and weights (possibly greater than one) and, conversely, profiles and non-negative weights can generate balanced collections of subsets of populations. Details of the following arguments appear in prior papers in the literature (cf. Wooders, 1983; Kovalenkov and Wooders, 2003).

Let (S, α) be a population where α maps S into a finite set $\Omega = \{\omega_1, \ldots, \omega_T\}$. Let $\{B^\ell\}$ denote a balanced family of subsets of S and let $\{w_\ell\}$ denote a set of balancing weights for the collection. Let b^ℓ denote the profile of B^ℓ. It follows that $\sum w_\ell b^\ell = s$ where s denotes the profile of (S, α). Moreover, there may exist $B^{\ell'}$ and $B^{\ell''}$, with $\ell' \neq \ell''$, such that $b^{\ell'} = b^{\ell''}$. Obviously, it holds that $\sum_{\ell\neq\ell',\ell''} w_\ell b^\ell + (w_{\ell'} + w_{\ell''})b^{l'} = s$. Thus, we can describe a balanced collection of subsets by a collection of (distinct) profiles $\bar{b}^\ell$ with weights $\bar{w}^\ell \in \mathbb{R}_+$ satisfying. Also, it holds that if a set of nonnegative real numbers $\bar{w}_\ell$ and a collection of profiles $\bar{b}^\ell$ satisfy the condition that $\sum \bar{w}_\ell \bar{b}^\ell = s$ then one can generate (nonuniquely except in special cases) a balanced collection of subsets of S where the profile of each member of the subset is that of some $\bar{b}^\ell$ with a positive weight.

A.1.1. Replica games

Let (Ω, ϕ) be a pregame where, again, Ω is a finite set, say $\Omega = \{\omega_1, \ldots, \omega_T\}$. Let $\{(N_r, \alpha_r)\}_{r=1}^\infty$ be a sequence of populations, where, for each $t = 1, \ldots, T$, it holds that

$$|\alpha_r^{-1}(\omega_t)| = r|\alpha^{-1}(\omega_t)|;$$

that is, the r th population in the sequence contains r times as many players with the same attribute (players of the same 'type') as the first game in the sequence and is called a *replica game*. Let (N_r, V_r) denote the rth replica game.

Let $E(r) \subset \mathbb{R}^T$ represent the set of equal treatment payoff vectors for the game (N_r, V_r) and let $\tilde{E}(r)$ represent the set of equal treatment payoff vectors for the balanced cover game derived from the game (N_r, V_r). From Scarf (1967) it follows that there is an equal treatment payoff vector in the core of the balanced cover of the game (N_r, V_r); this payoff vector can be represented by an element of $\tilde{E}(r)$. (Of course this payoff vector may not be feasible for the original game, that is, it is not necessarily contained in $E(r)$.)

Per capita boundedness implies that, for all replication numbers r, $E(r) \cap \mathbb{R}_+^T$ is contained in a compact set. It follows that, for all r, $\tilde{E}(r) \cap \mathbb{R}_+^T$ is contained in the same compact set (from Wooders, 1983, Lemma 5).

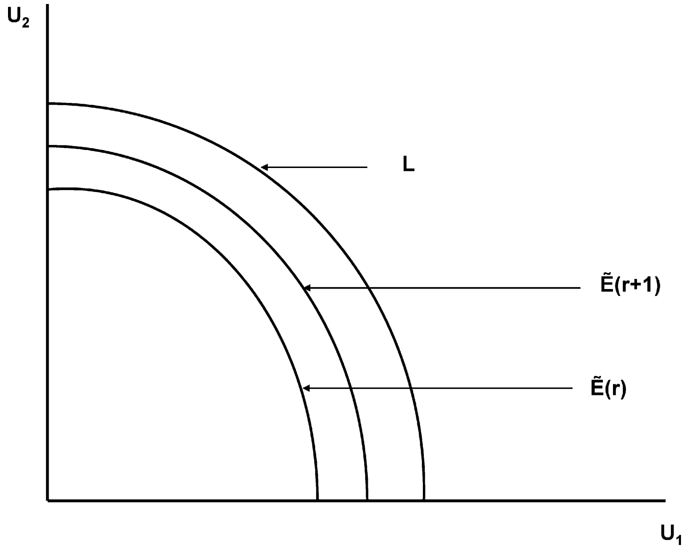

Fig. A.1. Increasing equal-treatment balanced cover payoff possibilities.

From the properties of balanced cover games it holds that $\tilde{E}(r) \subset \tilde{E}(r+1)$ for all r (Wooders, 1983, Lemma 7). These two facts imply that the closed limit with respect to Hausdorff distance of the sequence of sets $\{\tilde{E}(r)\}$ exists; let L denote this limit.[28]

These relationships are depicted in Fig. A.1.

A critical part of arguments underlying nonemptiness of approximate cores of games with many players connects equal treatment payoffs of balanced cover games to equal treatment payoffs for larger replications of the game itself (Wooders, 1983, Lemma 5). Specifically, given the replication number $\hat{r}$ there is an integer $m(\hat{r})$ such that

$$\tilde{E}(\hat{r}) \subset E(m(\hat{r})\hat{r}),$$

a consequence of the fact that 'minimal balanced collections' have rational weights. (Minimal balanced collections were introduced in Shapley, 1967; their definition is also stated in Wooders, 1983, p. 290, and used in the proof of Lemma 5 of that paper).

From superadditivity, for all positive integers ℓ we have $E(r) \subset E(\ell r)$ (Wooders, 1983, Lemma 3 applied to the sets of equal treatment payoffs). Thus, one obtains, for all positive integers ℓ,

$$\tilde{E}(\hat{r}) \subset E(\ell m(\hat{r})\hat{r}),$$

as depicted in Fig. A.2. (Again, only essential superadditivity is required.)

It now follows that given $\varepsilon > 0$ there is a replication number $\hat{r}^*$ with the property that the Hausdorff distance between L and $\tilde{E}(\hat{r}^*)$ is less than ε for all $r \geq \hat{r}^*$. Moreover, for the sequence $\{(N_{\ell m(\hat{r}^*)\hat{r}^*}, V_{\ell m(\hat{r}^*)\hat{r}^*})\}_{\ell=1}^{\infty}$ it holds that if $x \in \mathbb{R}^T$ represents an equal treatment payoff in the core of $(N_{\hat{r}^*}, \tilde{V}_{\hat{r}^*})$ then x represents a feasible payoff vector for the game $(N_{\ell m(\hat{r}^*)\hat{r}^*}, V_{\ell m(\hat{r}^*)\hat{r}^*})$ for each $\ell = 1, 2, 3 \ldots$. It is then easy to show that x represents a equal treatment payoff in the uniform ε-core of $(N_{\ell m(\hat{r}^*)\hat{r}^*}, V_{\ell m(\hat{r}^*)\hat{r}^*})$ for each $\ell = 1, 2, 3 \ldots$.

We now turn to the question of how to handle the 'leftovers'. That is, given any player set N_r for $r \geq \hat{r}^*$, we can select a subset of players, say S, containing the same number of players of each type as $N_{\ell m(\hat{r}^*)\hat{r}^*}$ for ℓ chosen to be as large as possible. This leaves the set of leftovers $N_r \setminus S = L$. Observe that $|L| \leq (m(\hat{r}^*) - 1)|N_{m(\hat{r}^*)\hat{r}^*}|$ and this bound is independent of the replication number r.

There are a number of ways to proceed. One is to assume, as in Wooders (1983), that the payoff possibilities sets are convex.[29] Given x as above, simply consider payoff vectors y that assign a group of players with the same profile

[28] See Hildenbrand, 1974, p 16, for example, for a definition of the Hausdorff limit.

[29] It is only required, in fact, that one can convexify payoffs between players of the same type.

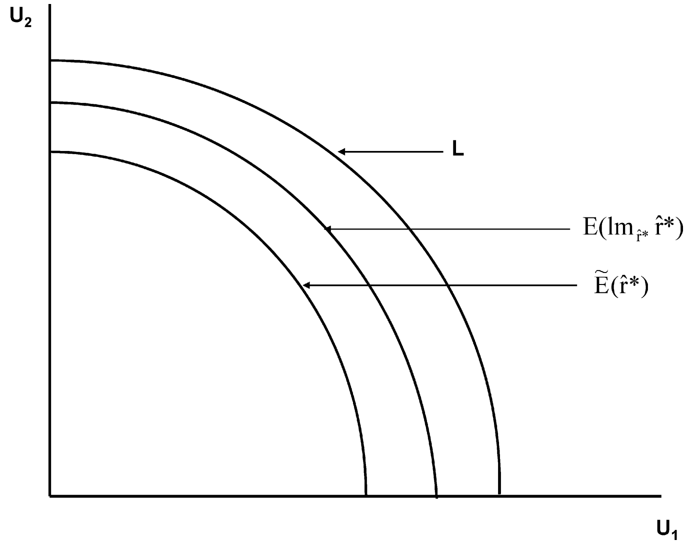

Fig. A.2. Equal treatment payoff possibilities for the balanced cover games and for a subsequence of the games.

as $N_{\ell m(\hat{r}^*)\hat{r}^*}$ the payoffs given by x and assign all the leftovers zero. Consider the payoff vector, say y^*, that is the average of all the vectors y. For all sufficiently large r, y^* will be in the 2ε-core of the game (N_r, V_r). Another way is to define the weak ε-core and take r sufficiently large so that $(|L|/|N_r|) \leq \varepsilon$, as is done in the proof of Theorem 1 of this paper. A third way, taken to prove Theorem 2 of this paper is to assume the compensation property and then 'tax' the non-leftovers to subsidize the leftovers; this has the same implication as assuming convexity—it permits payoffs of similar players to be equalized.

A.2. Proof of Theorem 1

Proof of Theorem 1. Suppose the conclusion of the Theorem is false. Suppose first, for the purposes of obtaining a contradiction, that there is an $\varepsilon_0 > 0$ and a sequence of populations $\{(S^\nu, \alpha^\nu)\}$ such that, for each positive integer ν, $|S^\nu| \geq \nu$ and the ε_0-core of (S^ν, V_{α^ν}) is empty.

We consider first the case where Ω is finite, say $\Omega = \{\omega_1, \ldots, \omega_T\}$. Let s^ν denote the profile of (S^ν, α^ν). By passing to a subsequence if necessary we may assume that $(1/||s^\nu||)s^\nu$ converges, say to $s \in \mathbb{R}^T_+$.

We now consider the case where, additionally, s is rational-valued. Let m_1 be an integer such that $m_1 s$ is integer-valued. Define $g = m_1 s$. Consider a sequence of populations $(\bar{S}^\ell, \beta^\ell)$ where $|\bar{S}^\ell| = ||\ell g||$ and β^ℓ satisfies the property that the profile of $(\bar{S}^\ell, \beta^\ell)$ is ℓg, i.e.

$$|\{\beta^\ell(i) = \omega_t : i \in \bar{S}^\ell\}| = \ell g(\omega_t) \text{ for each } t = 1, \ldots, T.$$

Taking the definition of a payoff set for each subset of $\bar{S}^\ell$ as given by ϕ, the pair $(\bar{S}^\ell, \beta^\ell)$ describes a NTU game in coalitional form, say $(\bar{S}^\ell, W^\ell)$. We can now apply results from Wooders (1983) and Shubik and Wooders (1983) to the sequence $\{(\bar{S}^\ell, W^\ell)\}$ and conclude that for some positive integer m_0, for all positive integers r the (equal treatment) uniform $(\varepsilon_0/2)$-core of $(\bar{S}^{rm_0}, W^{rm_0})$ is nonempty.

From Lemma A.1, there is an integer $\eta((\varepsilon_0/2), g)$ such that for all $\nu \geq \eta((\varepsilon_0/2), g)$, for some integer r_ν we have

$$r_\nu(m_0 g) + \ell^\nu = s^\nu \text{ and } \frac{||\ell^\nu||}{||s^\nu||} < \frac{\varepsilon_0}{2}.$$

Let $x^\nu \in V^\nu(S^{\alpha^\nu})$ have the property that, for some subset $\bar{S}^\nu \subset S^\nu$ with profile $r_\nu(m_0 g)$, it holds that $x^\nu_{\bar{S}^\nu}$ is in the uniform $(\varepsilon_0/2)$-core of a subgame with player set $\bar{S}^\nu$ and where $\bar{S}^\nu$ has profile $r_\nu(m_0 g)$. Then x^ν is in the weak ε-core of the game induced by the population (S^ν, α^ν), which yields a contradiction for the case of s having rational components.

We next relax the assumption that s has rational components. Let h be a profile satisfying the conditions of Lemma A.1; for each ν sufficiently large, for some integer r_ν and profile ℓ^ν we have $r_\nu h + \ell^\nu = s^\nu$ and $(||\ell^\nu||/||s^\nu||) < (\varepsilon_0/2)$.

From the case considered above, for all ν sufficiently large, a game with the profile of the total player set equal to $r_\nu h$ has a nonempty uniform $(\varepsilon_0/2)$-core. This, and the fact that $(||\ell^\nu||/||s^\nu||) < (\varepsilon_0/2)$, yields a contradiction.

Now we turn to the general case where Ω is a compact metric space. We approximate Ω by a finite set of types, and apply our result above to obtain a contradiction. Let $\delta(\varepsilon_0/2)$ be a positive real number with the property that if $(S,\alpha),(S,\beta) \in F(\Omega)$ with $\mathrm{d}(\alpha(i),\beta(i)) \le \delta(\varepsilon_0/2)$ for each $i \in S$, then

$$H_1(\phi(S,\alpha),\phi(S,\beta)) < \frac{\varepsilon_0}{2}|S|.$$

From per capita continuity there exists such a $\delta(\varepsilon_0/2)$. Let $\Omega_1,\ldots,\Omega_T$ be a partition of Ω such that if $\omega,\omega' \in \Omega_t$ for any t, then $\mathrm{d}(\omega,\omega') < \delta(\varepsilon_0/2)$. For each $t=1,\ldots,T$ arbitrarily select $\omega_t \in \Omega_t$. For each (S^ν,α^ν) define another population (S^ν,β^ν) where $\beta^\nu(i)=\omega_t$ for all $i \in S^\nu$ with $\alpha^\nu(i) \in \Omega_t$. From our result for the finite-type case, for all sufficiently large ν, say $\nu \ge \nu_0$, each derived game (S^ν,V_{β^ν}) has a nonempty weak $(\varepsilon_0/2)$-core. For $\nu \ge \nu_0$, let x^ν be in the weak $(\varepsilon_0/2)$-core of (S^ν,V_{β^ν}). From continuity and comprehensiveness we have $y^\nu = x^\nu - (\varepsilon_0/2)1_{S^\nu} \in \phi(S^\nu,\alpha^\nu)$. It follows that y^ν is in the weak ε_0-core of a game (S^ν,V_{α^ν}), a contradiction. This completes the proof of Theorem 1. □

A.3. Proof of Theorem 2

Proof of Theorem 2. Suppose that the Theorem is false. Then there is a positive real number $\varepsilon_0 > 0$ and a sequence of populations $\{(S^\nu,\alpha^\nu)\}$ and derived games (S^ν,V_{α^ν}) with the properties that $|S^\nu| \to \infty$ as $\nu \to \infty$ and the uniform ε_0-core of each derived game is empty for each game in the sequence. Without any loss of generality we may assume that $\varepsilon_0 < c$, given in the definition of compensation.

Case A: *Ω is finite and strict small group effectiveness is satisfied*. We first consider the case where Ω is a finite set and, in addition, all gains to collective activities can be realized by groups bounded in size. Let $\Omega=\{\omega_1,\ldots,\omega_T\}$. Suppose that there is a bound $\mathbb{B}$ such that for all populations (S^ν,α^ν) it holds that:

$$H_1(\phi^{etp}(S^\nu,\alpha_S),\cup\Pi\phi^{etp}(S^{k\nu},\alpha_{S^{k}\nu}))=0,$$

where the union is taken over all partitions $\{S^{k\nu}\}$ of S^ν with $|S^{k\nu}| \le \mathbb{B}$ for each element $S^{k\nu}$ in the partition.

We will use the following notation and observations. Let $(p^1,p^2,\ldots,p^L)$ be the set of all vectors $p^\ell \in \mathbb{Z}_+^T$ with the properties that $\sum_{t=1}^T p_t^\ell \le \mathbb{B}$. Notice that each p^ℓ can be interpreted as the profile of some population whose members have attributes in $\{\omega_1,\ldots,\omega_T\}$ and which contains no more than $\mathbb{B}$ players. Also observe that given any population (S,α) and any partition $\{S^k\}$ as above, we can describe the partition by a vector $(m_1,\ldots,m_L)$ where m_ℓ is the number of elements S^k in the partition with $|\alpha^{-1}(\omega_t)\cap S^k| = p_t^\ell$ for each ω_t; that is, m_ℓ is the number of subsets in the partition with profile p^ℓ. Since $\{S^k\}$ is a partition of S, it will hold that $\sum_{\ell=1}^L m_\ell p_t^\ell = |\alpha^{-1}(\omega_t)\cap S|$. Moreover, any balanced collection $\{B^\nu\}$ of subsets of S, each satisfying $|B^\nu| \le \mathbb{B}$, can be described by a vector $(\gamma_1,\ldots,\gamma_L) \in \mathbb{R}_+^T$ where $\gamma_\ell \in \mathbb{R}_+$ is the (total) balancing weight assigned to subsets B^ν with profile p^ℓ (and $\sum_{\ell=1}^L \gamma_\ell p_t^\ell = |\alpha^{-1}(\omega_t \cap S)|$ for each ω_t, $t=1,\ldots,T$).[30]

We now consider a sequence of balanced cover games $\{(S^\nu,\tilde{V}_{\alpha^\nu})\}$ derived from the games (S^ν,V_{α^ν}). For each ν, let $x^\nu \in \mathbb{R}^{S^\nu}$ be an equal treatment payoff vector in the core of the game $(S^\nu,\tilde{V}_{\alpha^\nu})$. Since the game is balanced there exists such a payoff vector.

From the definition of $(S^\nu,\tilde{V}_{\alpha^\nu})$ there is a balanced collection of subsets $\{B^{\nu q}\}$ of S^ν such that $x^\nu \in V_{\alpha^\nu}(B^{\nu q})$ for each subset in the collection. As discussed above, each subset $B^{\nu q}$ can be represented by a vector $p^\ell \in \mathbb{Z}_+^T$, where p_t^ℓ is the the number of players with attribute ω_t in $B^{\nu q}$. Since the collection is balanced, there are balancing weights, $\gamma^{\nu\ell} \in \mathbb{R}_+$ for $\ell=1,\ldots,L$, where $\gamma^{\nu\ell}$ is the total weight assigned to subsets $B^{\nu q}$ with profile p^ℓ and where $\gamma^{\nu\ell}=0$ if there are no subsets in the collection $\{B^{\nu q}\}$ with profile p^ℓ. For each ν let $r^{\nu\ell}$ denote the largest integer less than or equal to $\gamma_{\nu\ell}$.

[30] Describing subsets of players by their profiles, and partitions of a group of players by profiles of elements of the partition, are now common techniques in the theory of cooperative games with many players, in papers dating from Wooders (1977, 1983) to Kovalenkov and Wooders (2003) and beyond. The same holds holds for balanced collections of subsets of a set.

Let $\hat{S}^\nu \subset S^\nu$ be a group with the property that $\hat{S}^\nu$ is the disjoint union of $r^{\nu\ell}$ groups with profile p^ℓ, $\ell = 1, \ldots, L$. That is, there is a partition of $\hat{S}^\nu$ into $\sum_\ell r^{\nu\ell}$ groups where, for each ℓ, there are $r^{\nu\ell}$ subsets in the partition with profile p^ℓ. Denote these subsets by $\{B^{\nu qj}\}$ Observe that $x^\nu_{B^{\nu qj}} \in V_{\alpha^\nu}(B^{\nu qj})$. Since the game is superadditive, $x^\nu_{\hat{S}^\nu} \in V_{\alpha^\nu}(\hat{S}^\nu)$. Since x^ν is in the core of $(S^\nu, \tilde{V}_{\alpha^\nu})$, no subset of $\hat{S}^\nu$ can be improve upon $x^\nu_{\hat{S}^\nu}$.

Now observe that, since $r^{\nu\ell}$ is the largest integer less than or equal to $\gamma^{\nu\ell}$, it holds that $\sum_{\ell=1}^{L}(\gamma^{\nu\ell} - r^{\nu\ell}) \leq L$ and thus, $|S^\nu \setminus \hat{S}^\nu| \leq L\mathbb{B}$. that is, the number of players who cannot be accommodated in groups $B^{\nu qj}$ is uniformly bounded. From compensation, we can construct another payoff vector, $x^\nu - \varepsilon_0 1_{S^\nu}$ that is in the uniform ε_0-core of the game, which is a contradiction.

Case B: *Ω is finite and small group effectiveness (not necessarily strict) is satisfied*. Let $\Omega = \{\omega_1, \ldots, \omega_T\}$. From small group effectiveness (for NTU games), there is an integer $\eta_3(\varepsilon_0/8)$ such that for every population (S, α) it holds that

$$H_1(\phi^{etp}(S, \alpha), \cup\Pi\phi^{etp}(S^k, \alpha_{S^k})) < \frac{\varepsilon_0}{8}|S|$$

where the union is taken over all partitions $\{S^k\}$ of S with $|S^k| \leq \eta_3(\varepsilon_0/8)$ for each member S^k of the partition. Let $\mathbb{B} = \eta_3(\varepsilon_0/8)$.

We define another pregame (Ω, ϕ^B) satisfying strict small group effectiveness. For any population (S, α) with $|S| \leq \mathbb{B}$ define $\phi^B(S, \alpha) \stackrel{\text{def}}{=} \phi(S, \alpha^\nu)$ and for any population (S, α) with $|S| \leq \mathbb{B}$ define $\phi^B(S, \alpha) \stackrel{\text{def}}{=} \cup\Pi\phi(S^k, \alpha_{S^k})$ where the union is taken over all partitions $\{S^k\}$ of S with $|S^k| \leq \mathbb{B}$ for each member S^k of the partition.

Now consider the original sequence of populations (S^ν, α^ν) but with derived games (S^ν, W_{α^ν}) where $W_{\alpha^\nu}(S) = \phi^B(S^\nu, \alpha^\nu)$. From Case A, there is an integer ν_0 such that, for all $\nu \geq \nu_0$, there is a payoff vector y^ν in the uniform $(\varepsilon_0/2)$-core of the game (S^ν, W_{α^ν}). It is routine to verify that this vector is in the uniform ε_0-core of the game (S^ν, V_{α^ν}). Thus the conclusion of Theorem 2 holds for the class of pregames treated in this case.

Case C: *Ω is an arbitrary compact metric space and small group effectiveness (not necessarily strict) is satisfied*. Let $\delta(\varepsilon_0/2)$ be a positive real number with the property that for all populations $(S, \alpha), (S, \beta) \in F(\Omega)$ with $\mathrm{d}(\alpha(i), \beta(i)) < \delta$ for each i, it holds that

$$H_1(\phi(S, \alpha), \phi(S, \beta)) < \frac{\varepsilon_0}{2}|S|.$$

From per capita continuity there exists such a $\delta(\varepsilon_0/2)$. Let $\Omega_1, \ldots, \Omega_T$ be a partition of Ω such that if $\omega, \omega' \in \Omega_t$ for any t, then $\mathrm{d}(\omega, \omega') < \delta(\varepsilon_0/2)$. For each $t = 1, \ldots, T$ arbitrarily select $\omega_t \in \Omega_t$. Now from the sequence of games (S^ν, α^ν) construct another sequence of games (S^ν, β^ν) where, for each $t = 1, \ldots, T$, $\alpha^\nu(i) \in \Omega_t$ if and only if $\beta^\nu(i) = \omega_t$. The conclusion of Case B applies to the sequence of derived games (S^ν, V_{β^ν}). Thus, there is an integer ν_1 such that for all $\nu \geq \nu_1$ there is an equal treatment payoff vector z^ν in the $(\varepsilon_0/2)$-core of the game (S^ν, V_{β^ν}).

We claim that, for each $\nu \geq \nu_1$, the payoff vector $z^\nu - \varepsilon_0 1_{S^\nu}$ is in the equal treatment uniform ε_0-core of the game (S^ν, V_{α^ν}). To obtain a contradiction, suppose that for some (S^ν, V_{α^ν}) there is a coalition $C \subset S^\nu$ for which $z^\nu_C - \varepsilon_0 1_C \in int V_{\alpha^\nu}(C)$. Now let (C, β) be a population where $\beta(i) = \omega_t$ if and only if $\alpha^\nu(i) \in \Omega_t$, $t = 1, \ldots, T$. Then, from per capita continuity, it holds that $z^\nu_C - (\varepsilon_0/2)1_C \in int V_{\beta^\nu}(C')$, which is a contradiction to the conclusion of Case B. From per capita continuity it also follows that $z^\nu - \varepsilon 1_{S^\nu} \in V_{\alpha^n}(S^\nu)$. This is a contradiction for the general case and concludes the proof of the Theorem. □

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