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An extension of the KKMS Theorem

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Abstract

We extend the KKMS Theorem to show that the intersecting collection of sets of the theorem can be chosen to be both balanced and partnered. © 1998 Elsevier Science S.A.

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1. Introduction

The Knaster–Kuratowski–Mazurkiewicz Theorem, and its generalization, the KKMS Theorem, due to Shapley (1972), are important tools in mathematics and also in the general equilibrium theory of economic analysis. See, for example, discussions in Kannai (1992) or Ichiishi (1983). Recall that the KKMS Theorem states that, under certain conditions on a family of subsets of the simplex, the intersection of a 'balanced' family of subsets in the collection is nonempty. In this paper we show that the balanced family can be chosen to be 'partnered'. We also show that if the intersection of each balanced and partnered collection contains at most countably many points, then at least one of these balanced collections is 'minimally' partnered.

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The notion of partnership arises in cooperative game theory. Let N be a finite set (of players). A collection of subsets of N (coalitions) is partnered if each player i in N is in some coalition in the collection and whenever i is in all the coalitions containing j then j is in all the coalitions containing i. In models of economies and games, the partnership property of a solution concept ensures that there are no asymmetric dependencies between players. A solution payoff is partnered if whenever a player i needs another player j to realize his payoff, then j has a symmetric need for i. Consider, for example, the two-person divide the dollar game. Any division giving the entire dollar to one player displays an asymmetric dependency, since the player receiving the dollar needs the cooperation of the player getting nothing, but the player getting nothing can achieve this on his own. If a solution payoff is not partnered, there is an opportunity for one player to demand a larger share of the surplus from another player. Thus, a payoff that is not partnered exhibits a potential for instability.

A payoff is minimally partnered if no player needs any other player in particular. For example, consider a one-buyer, two-seller game where the sellers each have an object that the buyer is willing to buy for a dollar and the sellers each have a reservation value of zero for the object. The payoff that gives the entire surplus to the buyer is minimally partnered; the buyer needs neither seller, since from the buyer's perspective each seller is a perfect substitute for the other.

The economic motivation for our extension of the KKMS Theorem comes from a number of papers in the literature that have made use of the concept of partnership. This concept was first introduced to study solution concepts for games with transferable utility (TU) in Maschler and Peleg (1967) and Maschler et al. (1971). Further studies of solution concepts related to partnership in TU games appear in Albers (1979), Bennett (1983) and Reny et al. (1993). The first authors to investigate partnership in non-transferable utility (NTU) games were Bennett and Zame (1988). They show that a large class of NTU games possess undominated payoffs whose collection of supporting coalitions is partnered.

Partnership may significantly refine both the core and the set of competitive payoffs. For instance, Reny and Wooders (1996a) provide an example in which the core is a convex set containing a continuum of points, while the partnered core consists of but one point on the core's boundary. Reny and Wooders (1996a) refine the result of Bennett and Zame (1988) as well as the core nonemptiness theorem of Scarf (1967) by showing that balanced NTU games possess core payoffs having a partnered collection of supporting coalitions. [1]

[1] To obtain the Bennett and Zame (1988) result from that of Reny and Wooders (1996a,b) simply apply the latter result to the (not necessarily balanced) NTU game's balanced cover. Note also that when the given game is balanced, the result of Reny and Wooders (1996a,b) strictly strengthens that of Bennett and Zame (1988) since it ensures the existence of a payoff that is not only undominated and partnered, but also feasible.

An especially important economic result is due to Bennett and Zame (1988) who show that when preferences are strictly convex competitive payoffs are partnered. In an Arrow–Debreu economy (with merely convex preferences), although competitive payoffs need not be partnered, the result of Reny and Wooders (1996a) nonetheless can be applied to show that the partnered core of the economy is nonempty. Page and Wooders (1996) refine this result by allowing consumption sets to be unbounded below so long as a no arbitrage condition is satisfied. Another direction of application of partnership is taken in Reny and Wooders (1996b), who relate partnerships to commonwealths – organizations of not-necessarily-self-sufficient groups – through a notion of credible threats of secession.

The literature noted above motivates our interest in the mathematics underlying the partnered core and thus motivates our extension of the KKMS Theorem. The motivation for our result on minimal partnership also stems from economics, since it is a natural outcome in certain economic environments. For example, from the Bennett and Zame (1988) result it follows that all competitive equilibria of replica exchange economies with strictly convex preferences are minimally partnered.

2. Preliminaries

Let $N = \{1,2,\dots,n\}$ and let $\mathscr{P}$ be a collection of subsets of N. For each i in N let

$$\mathscr{P}_i = \{S \in \mathscr{P} : i \in S\}.$$

We say that $\mathscr{P}$ is *partnered* if for each i in N the set $\mathscr{P}_i$ is nonempty and for every i and j in N the following requirement is satisfied: [2]

$$\text{if } \mathscr{P}_i \subset \mathscr{P}_j \text{ then } \mathscr{P}_j \subset \mathscr{P}_i;$$

i.e., if all subsets in $\mathscr{P}$ that contain i also contain j then all subsets containing j also contain i. Let $\mathscr{P}[i]$ denote the set of those $j \in N$ such that $\mathscr{P}_i = \mathscr{P}_j$. We say that $\mathscr{P}$ is *minimally partnered* if it is partnered and for each $i \in N$, $\mathscr{P}[i] = \{i\}$.

Let $\mathscr{N}$ denote the set of nonempty subsets of N. For any $S \in \mathscr{N}$ let e^S denote the vector in $\mathbf{R}^n$ whose ith coordinate is 1 if $i \in S$ and 0 otherwise. For ease in notation we denote $e^{\{i\}}$ by e^i

Let Δ denote the unit simplex in $\mathbf{R}^n$. For every $S \in \mathscr{N}$ define

$$\Delta^S = \mathrm{co}\{e^i : i \in S\},$$

[2] The concept of a partnered collection of sets was introduced in Maschler and Peleg (1966, 1967) and further studied in Maschler et al. (1971). They used the term 'separating collection' rather than 'partnered collection'. Our use of "partnered" stems from Bennet (1983).

and

$$m^S = \frac{e^S}{|S|}.$$

Let $\mathscr{B}$ be a collection of subsets of N. The collection is *balanced* if there exist nonnegative weights $\{\lambda^S\}_{S \in \mathscr{B}}$ such that

$$\sum_{S \in \mathscr{B}} \lambda^S e^S = e^N.$$

Observe that the collection $\mathscr{B}$ is balanced if and only if

$$m^N \in \operatorname{co}\{m^S : S \in \mathscr{B}\}.$$

Our objective is to prove the following two theorems, the first of which extends the KKMS theorem (see, for instance, Shapley and Vohra, 1991).

Theorem 2.1. *Let $\{C^S : S \in \mathscr{N}\}$ be a collection of closed subsets of Δ such that*

$$\bigcup_{S \subseteq T} C^S \supseteq \Delta^T, \quad \text{for all } T \in \mathscr{N}. \tag{2.1}$$

Then there exists $x^* \in \Delta$ such that $\mathscr{S}(x^*) \equiv \{S \in \mathscr{N} : x^* \in C^S\}$ is balanced and partnered.

Theorem 2.2. *Let $\{C^S : S \in \mathscr{N}\}$ be a collection of closed subsets of Δ satisfying (2.1). If the set of $x^* \in \Delta$ such that $\mathscr{S}(x^*)$ is balanced and partnered is at most countable, then at least one $x^* \in \Delta$ renders $\mathscr{S}(x^*)$ balanced and* minimally *partnered.*

Remark 0. The countability requirement cannot merely be dispensed with. On the other hand, stimulated by the present result, Kannai and Wooders (1996) have shown that countability can be replaced by the weaker condition that the set in question be zero dimensional. The arguments of Kannai and Wooders (1996) require more sophisticated tools as they depend on degree theory. Whether it is possible that under our assumptions the set of points in the partnered core is either countable or zero-dimensional but not finite is an open question.

3. Proofs

3.1. Proof of Theorem 2.1

It should be noted that our method of proof makes substantial use of the elegant and powerful techniques developed in Shapley and Vohra (1991), and also relies on an ingenious construction found in Bennett and Zame (1988). A related argument can be found in Reny and Wooders (1996a).

The following lemma plays a central role in the proof of Theorems 2.1 and 2.2.

Lemma 3.1. *Let* $\{C^S : S \in \mathscr{N}\}$ *be a collection of closed subsets of* Δ *such that*

$$\bigcup_{S \subseteq T} C^S \supseteq \Delta^T, \quad \text{for all } T \in \mathscr{N}. \tag{3.1}$$

Suppose that for every i *and* j *in* N *there is a continuous function* $c_{ij} : \Delta \to \mathbf{R}_+$ *such that for all* $S \in \mathscr{N}$,

$$c_{ij} \text{ is identically zero on } C^S, \quad \text{whenever } i \notin S \text{ and } j \in S. \tag{3.2}$$

Then there exists $x^* \in \Delta$ *such that* $(x^*) \equiv \{S \in \mathscr{N} : x^* \in C^S\}$ *is balanced and such that for all* $i \in N$,

$$\sum_{j \in N} \left(c_{ij}(x^*) - c_{ji}(x^*) \right) = 0.$$

Remark 1. Putting c_{ij} identically equal to zero yields the KKMS Theorem.

Remark 2. The following functions, introduced by Bennett and Zame (1988), satisfy (3.2) and are particularly useful for the proofs of our theorems. For each pair of distinct i,j in N, define the function $c_{ij} : \Delta \to \mathbf{R}_+$ by

$$c_{ij}(x) = \min_{\substack{S: \\ i \notin S \\ j \in S}} \operatorname{dist}(x, C^S),$$

where dist is Euclidean distance. Note that since the distance from a closed set to a point depends continuously on the point, c_{ij} is a continuous function. For convenience set $c_{ii}(x) = 0$ for each i and all x.

Remark 3. In the TU game case, where $v(S)$ denotes the value of the coalition S, the lemma can be interpreted as yielding an existence result for a kernel-like solution. Indeed, for each x define $c_{ij}(x)$ as follows:

$$c_{ij}(x) = \min_{\substack{S: \\ i \notin S \\ j \in S}} \left(\sum_{k \in S} x_k - v(S) \right)^+,$$

where $\alpha^+ = \max(0, \alpha)$ for all $\alpha \in \mathbf{R}$. While the lemma does not guarantee an x such that for all i and j, $c_{ij}(x) = c_{ji}(x)$ as is (essentially) required for x to be in the kernel, it does guarantee that this condition hold for each player 'on average across all other players'. Consequently, Lemma 3.1 may help in establishing the existence of an NTU kernel-type solution.

Proof of Lemma 3.1. We break the proof into three steps.
Step 1. [Find x^*, a candidate for satisfying the conclusion of Lemma 3.1.]

For any $S \in \mathcal{N}$ and $x \in \Delta$, let

$$\eta_i^S(x) = \begin{cases} \sum_{j \in S} \left[c_{ij}(x) - c_{ji}(x) \right], & \text{if} \quad i \in S \\ 0, & \text{if} \quad i \notin S \end{cases}$$

and let $\eta^S(x)$ denote the vector in $\mathbf{R}^n$ whose ith coordinate is $\eta_i^S(x)$.

Let

$$\bar{\eta} = \max_{\substack{S \in \mathcal{N} \\ x \in \Delta \\ i \in N}} |\eta_i^S(x)| \tag{3.3}$$

and define

$$\Omega = \left\{ \omega \in \mathbf{R}^n : \sum_{i=1}^{n} \omega_i = 0 \text{ and } |\omega_i| \leq \bar{\eta}|, \forall i \right\}, \tag{3.4}$$

$$X = \left\{ x \in \mathbf{R}^n : \sum_{i=1}^{n} x_i = 1 \text{ and } x_i \geq -1, \forall i \right\}. \tag{3.5}$$

Observe that Ω and X are compact and convex. Let $h: X \to \Delta$ be defined by $h_i(x) \equiv (\max[x_i,0])/(\Sigma_{i=1}^n \max[x_i,0])$ for each $i \in N$.

Define the continuous function $f: X \times \Delta \times \Omega \to X$ by

$$f_i(x,p,\omega) \equiv h_i(x) + \frac{(1/n) - p_i - \omega_i}{1 + \bar{\eta}} \tag{3.6}$$

for each $i \in N$.

Define the correspondence, $F: X \to \Delta \times \Omega$ by

$$F(x) = \left\{ (m^S, \eta^S(h(x)) : h(x) \in C^S \text{ and } x_i \geq 0, \forall i \in S \right\}. \tag{3.7}$$

Observe that for each $x \in \Delta$, $F(x)$ is nonempty by virtue of (3.1). In addition, F is upper-hemicontinuous since $h(x)$ and $\eta^S(h(x))$ are continuous functions.

Consider the correspondence $f \times \mathrm{co}F : X \times \Delta \times \Omega \to X \times \Delta \times \Omega$. Since $f \times \mathrm{co}F$ is nonempty-valued, convex-valued and upper-hemicontinuous, by Kakutani's theorem it admits a fixed point, (x^*, p^*, ω^*). Consequently,

$$(p^*, \omega^*) \in \mathrm{co}\left\{ (m^S, \eta^S(h(x^*)) : h(x^*) \in C^S, \text{ and } x_i^* \geq 0, \forall i \in S \right\}, \tag{3.8}$$

and

$$x_i^* = h_i(x^*) + \frac{(1/n) - p_i^* - \omega_i^*}{1 + \bar{\eta}} \tag{3.9}$$

for all $i \in N$.

By (3.8) there exist nonnegative real numbers $((\alpha_S)_{S \in \mathcal{N}}$ satisfying

$$\sum_{S \in \mathcal{N}} \alpha_S = 1,$$

$$\omega^* = \sum_{S \in \mathcal{N}} \alpha_S \eta^S(h(x^*)), \tag{3.10}$$

$$p^* = \sum_{S \in \mathcal{N}} \alpha_S m^S, \tag{3.11}$$

and

$$\alpha_S > 0 \Rightarrow h(x^*) \in C^S, \text{ and } x_i^* \geq 0, \forall i \in S. \tag{3.12}$$

Let $x^* \in X$ be the desired candidate. (We will later show that x^* is in fact an element of Δ.)

Step 2. [Show that $\omega_i^* \leq 0$ for some i minimizing p_i^*.]

Let $y^* \equiv h(x^*) \in \Delta$ and let $M = \{m : p_m^* = \min_j\{p_j^*\}\}$. Now if $p_j^* > p_m^*$, then by Eq. (3.11) there exists $\alpha_S > 0$ such that $j \in S$ and $m \notin S$. But (3.12) then implies that $y^* \in C^S$. Hence, $c_{mj}(y^*) = 0$. Therefore for each $m \in M$, for all $S \in \mathcal{N}$,

$$\begin{aligned}
\eta_m^S(y^*) &= \sum_{j \in S} \left[c_{mj}(y^*) - c_{jm}(y^*)\right] \\
&= \sum_{\substack{j \in S \\ p_j^* = p_m^*}} \left[c_{mj}(y^*) - c_{jm}(y^*)\right] + \sum_{\substack{j \in S \\ p_j^* > p_m^*}} \left[c_{mj}(y^*) - c_{jm}(y^*)\right] \\
&= \sum_{\substack{j \in S \\ p_j^* = p_m^*}} \left[c_{mj}(y^*) - c_{jm}(y^*)\right] - \sum_{\substack{j \in S \\ p_j^* > p_m^*}} c_{jm}(y^*),
\end{aligned}$$

so that

$$\begin{aligned}
\sum_{m \in M} \eta_m^S(y^*) = \sum_{m \in M \cap S} \eta_m^S(y^*) &= \sum_{m \in M \cap S} \sum_{j \in M \cap S} \left[c_{mj}(y^*) - c_{jm}(y^*)\right] \\
&\quad - \sum_{m \in M \cap S} \sum_{j \in S \setminus M} c_{jm}(y^*) \\
&= - \sum_{m \in M \cap S} \sum_{j \in S \setminus M} c_{jm}(y^*) \\
&\leq 0,
\end{aligned}$$

where the first equality follows since $\eta_i^S(y^*) = 0$ whenever $i \notin S$, and the final inequality follows since each term in the sum is nonnegative.

Now from Eq. (3.10)

$$\omega^* = \sum_{S \in \mathcal{N}} \alpha_S \eta^S(y^*).$$

Hence,

$$\sum_{m \in M} \omega^*_m = \sum_{S \in \mathcal{N}} \alpha_S \sum_{m \in M} \eta^S_m(y^*) \le 0.$$

Consequently $\omega^*_{m'} \le 0$ for some $m' \in M$.

Step 3. [Show that x^* satisfies the conclusion of Lemma 3.1.]

We first show that $y^* = x^*$. To obtain this result, by the definition of h, it suffices to show that $x_i^* \ge 0$ for all i. So, suppose by way of contradiction that $x_i^* < 0$ for some i. Therefore $h_i(x^*) = 0$ and by (3.12) $\alpha_S = 0$ for all S containing i. Consequently, Eq. (3.10) implies that $\omega_i^* = 0$, and Eq. (3.11) implies that $p_i^* = 0$. But Eq. (3.9) then yields $x_i^* = (1/n)/(1+\bar{\eta}) > 0$, a contradiction. Hence, $x^* = h(x^*) = y^* \in \Delta$.

We may now conclude from Eq. (3.9) that

$$\omega_i^* = \frac{1}{n} - p_i^*, \quad \text{for all } i \in N.$$

However, by Step 2, there exists an $\bar{i} \in \mathcal{N}$ for which $\omega_{\bar{i}}^* \le 0$ and $p_{\bar{i}}^* \le p_i^*$ for all i. Thus we have

$$0 \ge \omega_{\bar{i}}^* = \frac{1}{n} - p_{\bar{i}}^* \ge \frac{1}{n} - p_i^*, \quad \text{for all } i \in N.$$

But since $p^* \in \Delta$ this implies that $p_i^* = 1/n$ for all $i \in N$ and therefore also that $\omega_i^* = 0$ for all $i \in N$.

Since $p^* = (1/n, 1/n, \dots, 1/n)$, (3.8) and $x^* = h(x^*) \in \Delta$ imply that $\{S \in \mathcal{N} : x^* \in C^S\}$ is balanced. It remains to show that $\eta_i^N(x^*) = 0$ for all $i \in N$. But this follows from the equalities below which hold for all $i \in N$.

$$\begin{aligned}
0 &= \omega_i^* \\
&= \sum_{S \in \mathcal{N}} \alpha_S \eta_i^S(x^*), \text{ by (3.10) and } y^* = x^* \\
&= \sum_{S \in \mathcal{N}} \alpha_S \sum_{j \in S} \left[c_{ij}(x^*) - c_{ji}(x^*) \right] \\
&= \sum_{j \in \mathcal{N}} \left(\sum_{\substack{S \in \mathcal{N} \\ j \in S}} \alpha_S \right) \left[c_{ij}(x^*) - c_{ji}(x^*) \right] \\
&= \frac{1}{n} \sum_{j \in N} \left[c_{ij}(x^*) - c_{ji}(x^*) \right], \text{ by (3.11) since } p^* = \left(\frac{1}{n}, \dots, \frac{1}{n} \right) \\
&= \frac{1}{n} \eta_i^N(x^*). \quad \square
\end{aligned}$$

Proof of Theorem 2.1. Define c_{ij} as in Remark 2. By Lemma 3.1 there is a payoff $x^* \in \Delta$ such that (x^*) is balanced and $\eta_i^N(x^*) = 0$ for all $i \in N$. But as shown in Bennett and Zame (1988) (see the proof of their lemma), for these choices of the c_{ij}, $\eta_i^N(x^*) = 0$ for all $i \in N$ implies that (x^*) is partnered. [3] □

Proof of Theorem 2.2. Again let the functions c_{ij} be as in Remark 2 of the preceding subsection. Let $x^1, x^2, \dots$ denote the (at most) countably many points such that for each k, (x^k) is balanced and partnered. By Theorem 2.1, there is at least one such point. It suffices to show that for some k

$$c_{ij}(x^k) = 0 \quad \text{for all } i,j \in N.$$

Let

$$A(x) = \left\{ \alpha \in \mathbf{R}_{++}^{n^2} : \text{for all } i \in N,\ \sum_{j \in N} \left(\alpha_{ij} c_{ij}(x) - \alpha_{ji} c_{ji}(x) \right) = 0 \right\}.$$

Since for every $\alpha \in \mathbf{R}_{++}^{n^2}$, $\alpha_{ij} c_{ij}(\cdot)$ satisfies (3.2), the lemma implies that for some x^* with $\mathscr{S}(x^*)$ balanced, α belongs to $A(x^*)$. But as in the proof of Theorem 2.1, this implies that $\mathscr{S}(x^*)$ is partnered. Hence, for every $\alpha \in \mathbf{R}_{++}^{n^2}$, there is a k such that

$$\alpha \in A(x^k).$$

Consequently,

$$\bigcup_{k=1}^{\infty} A(x^k) = \mathbf{R}_{++}^{n^2}.$$

The Baire Category Theorem (see, for instance, Friedman, 1982, p. 106, Theorem 3.4.2) then implies that there exists a k such that $A(x^k)$ is somewhere dense in $\mathbf{R}_{++}^{n^2}$. Consequently, the closure of $A(x^k)$ contains an open set A^0. For all $i \in N$ and all $\alpha \in A^0$, we have $\Sigma_{j \in N}(\alpha_{ij} c_{ij}(x^k) - \alpha_{ji} c_{ij}(x^k)) = 0$. But this implies that $c_{ij}(x^k) = 0$ for all pairs i and j. □

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[3] The proof of the lemma of Bennett and Zame (1988) relies only on the assumption that the sets C^S are closed for all subsets S of N, and this is used only to ensure that the minimum distance from a point to any of these sets is well-defined. Consequently, since we have assumed the sets C^S to be closed, we may apply their lemma here.

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