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An axiomatization of the core for finite and continuum games*

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Abstract. We provide a new axiomatization of the core of games in characteristic form. The games may have either finite sets of players or continuum sets of players and finite coalitions. Our research is based on Peleg's axiomatization for finite games and on the notions of measurement-consistent partitions and the f-core introduced by Kaneko and Wooders. Since coalitions are finite in both finite games and in continuum games, we can use the reduced game property and the converse reduced game property for our axiomatization. Both properties are particularly appealing in large economies.

1. Motivation for an axiom system for the core of finite and continuum games

Recently Peleg (1985, 1986) has provided an axiom system for the core of finite games. Peleg's axioms are based on the opportunities available to individual players and coalitions, with 2-person coalitions having a special role. Our purpose is to provide an axiomatization for finite and continuum games where the roles of individual players and coalitions are the same regardless of the type of player set. Our axiomatization applies to the class of games discussed by Peleg (1985, 1986) and to the class of continuum games introduced by Kaneko and Wooders (1986a).

The fact that both finite and continuum games with finite coalitions have cores described by one axiom system reflects a game-theoretic equivalence between the two sorts of games. Both sorts of games can be regarded as coalition structure games where coalitions are required to be finite.[1] Except for the size of the total

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[1] Cores of games with coalition structures are studied in Aumann and Dreze (1974) and Kaneko and Wooders (1982), for example, and, with a continuum of players and finite coalitions, in Kaneko and Wooders (1986a, b; 1990).

player set, finite games and continuum games with finite coalitions are the same. In particular, the concepts of the player and of a coalition are invariant between the two sorts of games. For example, no matter how large the total player set, two players can form a coalition. It is the invariance of the concept of the player and the role of a player in a coalition that permits the one axiomatization of the core of finite games and games with a continuum of players and finite coalitions.

Since we allow only finite coalitions, Peleg's axioms can be adapted to apply to our framework. In Peleg's axiom system, individual players and finite groups of players take aggregate outcomes to the complementary set of players as given. Similarly, in our system, whether or not the total player set is finite, individual players and finite groups of players take aggregate outcomes to the complementary player set as given.

In addition to our desire to provide an axiomatization of the core that treats individual players and coalitions of individual players consistently independent of the size of the total player set, we have several other motivations for introducing our new axiomatization. First, cooperation may take place only within small (finite) groups of players even with large total player sets. Second, any attempt to treat cooperation as the outcome of a noncooperative game or to investigate the noncooperative foundations of cooperation in situations where individual players can interact with each other would seem to demand that individual players be non-negligible relative to each other. In the context of a marriage or matching game, for example, the outcome of bargaining over the distribution of the gains from a partnership may well be influenced by the distribution of gains in other possible partnerships. The bargaining possibilities of one member of the partnership vis-à-vis the other, however, may be well-modeled as *invariant* with respect to the size of the player set. We hope that our approach, by further developing the notion of a continuum game with small (finite) effective groups, will help in establishing non-cooperative foundations of cooperation in such situations.

Within our framework three properties axiomatize the core: the reduced game property, the converse reduced game property, and a pair-wise optimality condition. These three axioms all concern the role of individual players and finite groups of players.

The reduced game property stipulates that players evaluate their power within a coalition by their options outside the coalition. This property is especially appealing for large economies where most agents have many substitutes. In a large economy a "small" group of agents can assume that aggregate outcomes are independent of their actions.

The converse reduced game property relates solutions for two-player games to solutions for the entire player set. Consider an outcome which is a solution outcome on any 2-person bargaining problem where the outside options available to any player are those induced by the given outcome. The converse reduced game property requires that such outcomes are solution outcomes for the entire game.

The pair-wise optimality condition requires that for 2-person games, the solution must yield a Pareto-optimal and individually rational outcome. More completely, the condition requires that if two individuals can do better separately than together, they must be imputed their individually rational payoffs. Otherwise, the two individuals are imputed a Pareto-optimal payoff for the 2-person game. We view the pair-wise optimality condition as a minimal condition to impose on pair-wise bargaining problems.

We show that the core is the only solution satisfying the reduced game property, the converse reduced game property, and the pair-wise optimality property. Our approach and results parallel those of Peleg (1985, 1986) and simultaneously include continuum games with finite coalitions.

The objects of our analysis are games in characteristic form. Although we do not provide any axiomatization of the competitive outcomes of economies, our results also characterize competitive payoffs in situations where the core of a game derived from an economy coincides with the competitive payoffs, for example, Aumann (1964) and Hammond, Kaneko, and Wooders (1989).

2. Measurement-consistent partitions

In continuum games with finite coalitions the connection between finite coalitions and the total player set is crucial. This connection is made through the aggregation of players in finite coalitions in a way consistent with the measure on the total player set. Two notions of measurement are involved in the aggregation, the absolute sizes given by the cardinalities of finite coalitions and the relative sizes given by the measure on the total player set. These absolute and relative measurements are made compatible by the concept of a measurement-consistent partition. A measurement-consistent partition preserves the proportions given by the measure.

Let (N, β, μ) be a measure space, where N is a measurable set, β is the σ-algebra of all measurable subsets of N with $\{i\} \in \beta$ for all $i \in N$, and μ is a measure, with $0 < \mu(N) < \infty$. When N is infinite we require that μ be nonatomic. When N is finite, μ is the counting measure.

Example. We begin with a simple example illustrating the measurement-consistency requirement. Let $N=[1,3)$ and let μ be Lebesgue measure. A measurement-consistent partition is given by

$$p=\{\{i,1+i\}:i\in[0,1)\}\cup\{\{i\}:i\in[2,3)\} .$$

This satisfies measurement-consistency because the mapping $i \to i+1$ of players i in $[0,1)$ to their partners is measure-preserving.

An example of a partition which is not measurement-consistent is given by

$$q=\{\{i,1+2i\}:i\in[0,1)\} .$$

Note that for the case of a finite number of players any isomorphism from players to their partners must be measure-preserving since the notions of cardinality and the counting measure for finite sets coincide. Since for finite sets these notions coincide, every partition of a finite set is measurement-consistent.

Let p be a partition of N into finite coalitions. For each integer k define N_k as the subset of players in k-member coalitions in p, that is, $N_k = \bigcup_{\substack{S \in p \\ |S|=k}} S$. The partition p is *measurement-consistent* if for each positive integer k,

N_k is a measurable subset of N; and
each N_k $(k=1,2,\ldots)$ has a partition into measurable sets $\{N_{kt}\}_{t=1}^{k}$, such that:

there are measure-preserving isomorphisms $\Psi_{k1},\dots,\Psi_{k2},\dots,\Psi_{kk}$ from N_{k1} to $N_{k1},\dots,N_{kk}$, respectively and $\{\Psi_{k1}(i),\dots,\Psi_{kk}(i)\}\in p$ for each $i\in N_{k1}$.[2]

Note that this condition implies that for any $S\in p$ with $|S|=k$, we have $S=\{\Psi_{k1}(i),\dots,\Psi_{kk}(i)\}$ for some i in N_{k1}. For each integer k, the set N_k consists of all the members of k-player coalitions and N_{kt} consists of the t^{th} members of these coalitions. The measure-preserving isomorphisms express the idea that coalitions of size k should have as "many" (the same measure) first members as second members, as many second members as third members, etc.

The notion of measurement-consistent partitions is basic to our axiomatization. Kaneko and Wooders (1986a, Lemma 1) show that the set of measurement-consistent partitions of a Borel subset of a complete separable metric space is nonempty. For our results, we do not need to require that N be a Borel subset of a metric space since we do not prove either existence of measurement-consistent partitions or non-emptiness of the f-core. We need, however, the following lemma. Roughly, the lemma states that the restriction of a measurement-consistent partition to a measurable subset of players induces a measurement-consistent partition on that subset. While this lemma is critical to our work, its proof is mainly technical so it is provided in an Appendix.

Lemma 1. *Let S be a measurable subset of N and let p be a measurement-consistent partition of N. Then*

$$q=\{S\cap M:\ M\in p\}$$

is a measurement-consistent partition of S.

Proof. See Appendix.

3. Games

We begin with a description of the player set. Let (Ω,β,μ) be a measure space where Ω is an infinite set, β is a σ-algebra of measurable subsets of Ω, and μ is a nonatomic measure. The set Ω can be interpreted as the universe of all possible players in any game. Any element of β is an admissible player set for a game, and μ is a nonatomic measure on the universe of players. The measure μ induces a measure on every admissible player set of positive measure.

A *characteristic function game* with side payments is a pair (N,v) where $N\subset\Omega$ can be finite or infinite, $N\in\beta$, and v assigns a real number $v(S)$ to any finite subset (coalition) of N. We will frequently identify the game with the function v.

Let Π be the set of all measurement-consistent partitions of N. For any $p\in\Pi$, we define the set of *feasible outcomes relative to p*. This is the set of payoffs that are feasible by cooperation in the coalitions in p. Let

$$H(p,v)=\{h\in L(N,\mathbb{R}):\ \sum_{j\in S} h(j)\le v(S)\text{ for all }S\in p\}$$

[2] Let A and B be sets in β. A function Ψ from A to B is called a *measure-preserving isomorphism* from A to B if *(i)* Ψ is a measure-theoretic isomorphism, i.e., Ψ is 1 to 1, onto, and measurable in both directions, and *(ii)* $\mu(C)=\mu(\Psi(C))$ for all $C\subset A$ with $C\in\beta$.

where $L(N, \mathbb{R})$ is the set of all measurable functions from N to $\mathbb{R}$, the real numbers. We call $H(p, v)$ the *outcome set relative to p*. Next we define the set of *exactly feasible outcomes*, $H(v)$, which consists of the union ot the sets of feasible outcomes relative to measurement-consistent partitions:

$$H(v) = \bigcup_{p \in \Pi} H(p, v) \ .$$

The set of *feasible outcomes* is denoted by $H^*(v)$ and defined by

$$H^*(v) = \{h \in L(N, \mathbb{R}) : h \text{ is a limit point (in measure) of some sequence } \{h^k\}_{k=1}^{\infty} \text{ in } H(v)\} \ .$$

This definition of the feasible set is consistent with the view that the continuum approximates large finite games.[3]

4. Reduced games

The reduced game property is defined as follows: given an outcome h and a subset S of the total player set N the worth of a coalition Q contained in S is the total payoff the members of Q could achieve (according to v) if they could gain the cooperation of any player i not in S by giving i his component of h, $h(i)$.

Formally, let v be a game, let h be in $H^*(v)$, and let S be a measurable subset of N. (S is not necessarily finite). The *reduced game v on S* is given by:

$$v_{h,S}(Q) = \sup_{\substack{T \subset N \setminus S \\ T \text{ finite}}} [v(Q \cup T) - \sum_{i \in T} h(i)] \quad \text{for all} \quad Q \subset S, \quad Q \text{ finite}, \quad Q \neq \emptyset$$

and

$$v_{h,S}(\emptyset) = 0 \ .$$

We say that *the reduced game is well defined* if, for all (finite) coalitions Q, $v_{h,S}(Q)$ is finite.

The notion of the reduced game captures the idea that players evaluate their power within coalitions by their options outside the coalition. The players in $Q \subset S$ expect to cooperate with some coalition $T \subset N \setminus S$, giving the members of T their payoffs according to h. The payoffs to the members of T total $h(T)$, so the members of Q expect the remaining payoff $v(Q \cup T) - h(T)$ for themselves. The "worth" $v_{h,S}(Q)$ represents, therefore, Q's most optimistic expectation relative to taking h as given for nonmembers of S.[4]

The following lemma shows that if h is feasible for the total player set N then its restriction to S is feasible for the reduced game $v_{h,S}$.

Lemma 2. *Let S be a measurable subset of N and let* $h \in H^*(v)$. *If the reduced game* $v_{h,S}$ *is well defined, then* $h|_S \in H^*(v_{h,S})$.

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[3] See Kaneko and Wooders (1986a, 1989) and Hammond, Kaneko and Wooders (1989) for arguments justifying this approximation.

[4] The definition above is used in Winter (1989) to characterize stable and semi-stable demand vectors. In contrast to Peleg (1986), since we do not restrict ourselves to one particular coalition structure for which the core payoff must be feasible, we do not distinguish between cases where $Q \subset S$ and $Q = S$.

Proof. Suppose $h \in H(p, v)$ for some measurement-consistent partition p. Then $h|_S \in H(p_S, v_{h,S})$ for some measurement-consistent partition p_S of S. To show this we take

$$p_S = \{M \cap S \colon M \in p\} \ .$$

By Lemma 1, p_S is a measurement-consistent partition of S. We next show that $h|_S$ satisfies the feasibility condition. Let $p(i)$ denote the coalition in p containing player i. From the feasibility condition for h for the game v, we have, for i in S,

$$\sum_{j \in S \cap p(i)} h(j) \leq v((S \cap p(i)) \cup T) - \sum_{j \in T} h(j) \quad \text{where} \quad T = (N \setminus S) \cap p(i)$$

$$\leq \sup_{\substack{W \subset N \setminus S \cap p(i) \\ W \text{ finite}}} [v(S \cap p(i)) \cup W) - \sum_{j \in W} h(i)] \ .$$

Let h be in $H^*(v)$. Let $\{p^k\}$ be any sequence of measurement-consistent partitions and let $\{h^k\}$ be a sequence of outcomes where, for each k, $h^k \in H(p^k)$, and $\{h^k\}$ converges (with respect to convergence in measure) to h. Since the above inequality holds for each partition p^k and any $h^k \in H(p^k, v)$, it holds for the limit h. Q.E.D.

5. Solutions

In this section we introduce the notions of the reduced game property and the converse reduced game property for solutions of games with finite and infinite player sets.

A *solution* is a correspondence ξ assigning to each game v a subset $\xi(v)$ of H^*. We say that ξ satisfies the *reduced game property* if for each v, for each $h \in \xi(v)$ and each measurable subset $S \subset N$ the reduced game $v_{h,S}$ is well defined and $h_S \in \xi(v_{h,S})$. For example, if we view the set H^* as a solution, Lemmas 1 and 2 state that H^* satisfies the reduced game property. The reduced game property is a consistency property. It asserts that any solution outcome for the whole game is also a solution outcome for every restricted set of players when the options of these players are evaluated according to the reduced game $v_{h,S}$.

The *core* of v is a solution given by:

$$\text{Core}(v) = \{h \in H^*(v) \colon \sum_{i \in S} h(i) \geq v(S) \text{ for each coalition } S\} \ .$$

(Recall that a coalition S is a finite subset of N).

Let Γ be the domain of all games with nonempty cores.[5]

Proposition 1. *The core satisfies the reduced game property on the domain Γ.*

[5] It is well-known that balanced games have nonempty cores (Bondareva 1963 and Shapley 1967). Under mild conditions, cores of continuum games with finite coalitions are nonempty (Kaneko and Wooders 1986a, 1990). Thus our axiomatization applies to finite balanced games and a reasonably broad class of "large" games.

Proof. Take $h \in \mathrm{Core}(v)$. Let S be a measurable subset of N. Then, since $h \in \mathrm{Core}(v)$, it holds that $h \in H^*(v)$ and so by Lemmas 1 and 2, we have $h_S \in H^*(v_{h,S})$. Suppose $Q \subset S$ is a coalition. Then

$$(*) \quad v_{h,S}(Q) - \sum_{i \in Q} h(i) = \sup_{\substack{T \subset N \setminus S \\ T \text{ finite}}} [v(Q \cup T) - \sum_{i \in T} h(i)] - \sum_{i \in Q} h(i) \ .$$

For any coalition T, we have

$$v(Q \cup T) - \sum_{i \in T} h(i) - \sum_{i \in Q} h(i) \leq 0 \ ,$$

since $h \in \mathrm{Core}(v)$. This shows that the right hand side of (*) is smaller or equal to zero. Thus $\sum_{i \in Q} h(i) \geq v_{h,S}(Q)$, implying that the reduced game is well defined and that h_S is in the core. Q.E.D.

We next introduce the converse reduced game property. This property asserts that a feasible outcome for a game that is a solution outcome for every 2-person reduced game is a solution outcome for the game.

A solution ξ is said to satisfy *the converse reduced game property* if the following condition is satisfied: for each $v \in \Gamma$ and each $h \in H^*(v)$ if $h_S \in \xi(v_{h,S})$ for all $S \subset N$ with $|S| = 2$, then $h \in \xi(v)$.

Proposition 2. *The core satisfies the converse reduced game property.*

Proof. Take $h \in H^*$ with $h_S \in \mathrm{Core}(v_{h,S})$ for each $S \subset N$ with $|S| = 2$. Take $T \subset N$, T finite, and $T \neq 0$. Choose $i \in T$ and $j \notin T$. Since $h_{\{i,j\}} \in \mathrm{Core}(v_{h,\{i,j\}})$ we have

$$\begin{aligned} 0 &\geq v_{h,\{i,j\}}(\{i\}) - h(i) \\ &= \sup_{\substack{Q \subset N \setminus \{i,j\} \\ Q \text{ finite}}} [v(Q \cup \{i\}) - \sum_{k \in Q} h(k)] - h(i) \geq v(T) - \sum_{k \in T} h(k) \ , \end{aligned}$$

which indicates that $h \in \mathrm{Core}(v)$. Q.E.D.

6. An axiomatization of the core

One additional axiom is required. A solution ξ is *pair-wise opimal* if for any two-player game v with player set $\{i,j\}$, it holds that

$$\text{If} \quad v(\{i\}) + v(\{j\}) \leq v(\{i,j\}) \ , \quad \text{then}$$
$$\xi(v) = \{(h(i), h(j)) : (h(i), h(j)) \geq (v(\{i\}), v(\{j\})) \text{ and } h(i) + h(j) = v(\{i,j\})\}$$

and

$$\text{If} \quad v(\{i\}) + v(\{j\}) > v(\{i,j\}) \ , \quad \text{then } \xi(v) = \{(v(\{i\}), v(\{j\}))\} \ .$$

A pair-wise optimal solution simply yields individually rational and Pareto-optimal outcomes for 2-person games. On such games the solution coincides with the core. Note that the point $(v(\{i\}), v(\{j\}))$ corresponds to the partition in which each player operates alone; if we had required superadditivity, the second case would not need to be considered.

Theorem 1. *The core is the only solution satisfying the reduced game property, the converse reduced game property, and pair-wise optimality.*

Proof. We have already shown that the core satisfies the reduced game property and the converse reduced game property. The pair-wise optimality condition follows from our observation that the core of a two-person game coincides with the Pareto-optimal and individually rational outcomes. To show that the core is the only solution satisfying the three axioms, suppose ξ denotes another solution which satisfying the three axioms. Take a game v and $h \in \xi(v)$. Since ξ satisfies the reduced game property we have $h_S \in \xi(v_{h,S})$ for all $S \subset N$ with $|S| = 2$. Since $v_{h,S}$ has 2 players, and since the core satisfies pair-wise optimality, we have $h_S \in \text{Core}(v_{h,S})$ for all $S \subset N$ with $|S| = 2$. By the converse reduced game property $h \in \text{Core}(v)$. We have now shown that $\xi(v) \subset \text{Core}(v)$. Next, take $h \in \text{Core}(v)$. Since both the core and ξ satisfy the three axioms, we can repeat the arguments to obtain $h \in \xi(v)$. We therefore have $\xi(v) = \text{Core}(v)$. Q.E.D.

7. Games without side payments

The characterization of the core by means of the reduced game property described in the previous section can also apply to situations where side payments are not necessarily permitted. In this section we describe the framework of games without side payments and provide the analogue of Theorem 1 for such games. The proofs are omitted since they use arguments similar to those of Peleg (1985) and our treatment for the transferable utility case. Moreover, Lemma 1 is totally independent of whether we have a framework with or without side payments.

A *characteristic function* V (without side payments) is a correspondence which assigns to each coalition S a subset $V(S)$ of $\mathbb{R}^S$ having the following properties:

(1) $V(S)$ is a nonempty closed subset of $\mathbb{R}^S$ for $S \neq \emptyset$, and $V(\emptyset) = 0$;
(2) for any coalition S and any $x \in V(S)$ and $y \in \mathbb{R}^S$ with $y \leq x$, we have $y \in V(S)$;
(3) If $x \in \partial V(S)$, $y \in V(S)$ and $y \geq x$, then $y = x$; and
(4) $V(S) \cap (x^S + (\mathbb{R}^S_+))$ is bounded for every x^S in $\mathbb{R}^S$ and $\inf \sup V(\{i\}) > -\infty$

where $\partial V(S)$ denotes the boundary of $V(S)$.

Conditions (1) to (4) are standard requirements in the literature of games without side payments. Condition (3) and (4) are respectively known as *comprehensiveness* and *non-levelness.*[6]

[6] The non-levelness condition (which has also been called "quasi-transferable utility") ensures a weak sort of "side-paymentness". Every point on the boundary of $V(S)$ is such that no player can increase his payoff without decreasing someone else's payoff. Any game can be approximated arbitrarily closely by one satisfying this property (Wooders 1983, Appendix).

Using the notation developed in the preceding section, we can now define feasible outcomes for each game V (without side payments), and measurement-consistent partitions $p \in \Pi$:

$$H(p, V) = \{h \in L(N, \mathbb{R}) : x_S \in V(S) \text{ for all } S \in p\} \;;$$

$$H(V) = \bigcup_{p \in \Pi} H(p, V) \;; \quad \text{and}$$

$$H^*(V) = \{h \in L(N, \mathbb{R}) : h \text{ is a limit, with respect to convergence in measure, of a sequence } \{h^k\}_{k=1}^{\infty} \text{ in } H(V)\} \;.$$

The set $H^*(V)$ is the set of *feasible outcomes*, as in the preceding section.

Given $h \in L(N, \mathbb{R})$, we say that a coalition S *can improve upon* h if there exists some $y \in V(S)$ with $y_i > h(i)$ for all i in S.

We can now define the *core of* V as

$$\text{Core}(V) = \{h \in H^*(V) : \text{no coalition } S \subset N \text{ can improve upon } h\} \;.$$

Let S be a subset of N (not necessarily a coalition), and let $h \in H^*(V)$. The *reduced game* $V_{h,S}$ is defined as follows:

$$V_{h,S}(Q) = \bigcup_{\substack{T \subset N \setminus S \\ T \text{ finite}}} \{y^Q : (h^T, y^Q) \in V(T \cup Q)\} \quad \text{for each } Q \subset S$$

where $h^T = (h(i))_{i \in T}$.

In the reduced game $V_{h,S}$ a coalition $Q \subset S$ can attain any payoff y^Q with the property that (y^Q, h^T) forms a feasible payoff for $T \cup Q$ for at least one coalition $T \subset N \setminus S$. In other words, Q can attain in $V_{h,S}(Q)$ exactly those vectors which correspond to an agreement between Q and a coalition T of players not in S in which the players in T are being paid according to the function h.

It can be shown that the reduced game of a game without side payments is indeed a game, i.e., it satisfies conditions (1) to (4). (See Peleg 1985 and our Lemma 1).

Having the definition of the reduced game now at hand the reduced game property and the converse reduced game property are defined in the same manner as in the side payments case. We omit the details.

As in the case of games with side payments, to obtain the characterization of the core, we need to specify the solution for two-player games. A solution ξ is said to be *pair-wise optimal* if for any two-player game V with player set $\{i, j\}$, letting $x_i^* = \sup\{x : x \in V(\{i\})\}$, it holds that:

If $\quad (x_i^*, x_j^*) \in V(\{i, j\})\;, \quad$ then

$$\xi(V) = \{(h(i), h(j)) : h \in \partial V(\{i, j\}) \text{ and } h \geq (x_i^*, x_j^*)\} \;; \quad \text{and}$$

If $\quad (x_i^*, x_j^*) \notin V(\{i, j\})\;, \quad$ then $\quad \xi(V) = (x_i^*, x_j^*)\;.$

With the above definitions we can now state the result parallel to Theorem 1 for games without side payments.[7]

[7] We note only that the non-leveledness condition is required to ensure strict Pareto-optimality of every core payoff – no one player can be assigned a higher payoff without another player being assigned a lower payoff.

Theorem 2. *The core is the only solution for games without side payments satisfying the reduced game property, the converse reduced game property, and pair-wise optimality.*

8. Concluding discussion

In Kaneko and Wooders (1986a, 1989) and Hammond, Kaneko, and Wooders (1989) it is claimed that the role of the individual player is the same in continuum games as it is in finite games. Kaneko and Wooders (1986a, 1990) especially stress that the distinction between finite games and continuum games with finite coalitions is the size of the total player set. Both our formulation and our axiomatization treat the individual player the same in finite and continuum games. From our axiomatization for finite and continuum games with finite coalitions we may conclude that a core of a game with finite coalitions is the same as the core of a game with a continuum of players and finite coalitions – both concepts are treated by one set of axioms. The core of a game with a continuum of players and finite coalition, called the f-core, is indeed the same concept as the core of a finite game.

Another axiomatization of the core of finite games, based on the independence of irrelevant alternatives, is due to Keiding (1986). Applications for Keiding's approach to continuum games have yet to be developed.

For a contrasting approach, we note the model introduced by Aumann (1964) and studied by Dubey and Neyman (1984). In this approach, coalitions are constrained to be of positive measure and individual players and finite groups of players can have no affect, even on their own outcomes. No two players can form a coalition, for example, and affect the payoff they receive. There is currently in the literature no one axiomatization of both finite games and games with a continuum of players and coalitions of positive measure.

Appendix

Proof of Lemma 1. Let N be a measurable set, let p be a measurement-consistent partition of N, and suppose that p is described by the measure-preserving isomorphisms Ψ_{kj}, $j \leq k$ and $k=1,\ldots$. Let S be a measurable subset of N. Let q be a partition of S defined by $M \in q$ if and only if there is some $W \in p$ with $W \cap S = M$. We will only prove the required result for all members of S in 2-member coalitions in q. The general case following by analogy.

Let $S_2 = \bigcup_{\substack{M \in q \\ |M|=2}} M$. Since S_2 can also be described as a countable union of measurable partitions, S_2 is measurable. (See the discussion after the proof). We next construct the sets S_{21} and S_{22}.

Let $S_{21} = \{i \in S\colon \text{for some } W \in p \text{ with } |W \cap S| = 2,\ W = \{\Psi_{k1}(j),\ldots,\Psi_{kk}(j)\},\ i = \Psi_{kt}(j) \text{ for some } j, \text{ and if } \Psi_{kt''}(j) = m,\ m \in S, \text{ then } t < t''\}$. The set S_{21} is simply the set of players in S_2 who have lower numbered labels than their partners in $S \cap W$; S_{21} can also be described as the union of a countable collection of measurable subsets so S_{21} is measurable.

Let $S_{22} = \{j\colon j \in S_2 \cap W,\ j \notin S_{21}\}$; since S_2 and S_{21} are measurable, it holds that S_{22} is measurable. Each member of S_{22} is the partner of some member of

S_{21} and in the isomorphisms describing p, a member of S_{22} has a higher numbered label than his partner.

We claim that there is a measure-preserving isomorphism from S_{21} to S_{22} having the required properties. To establish the claim, define $W_1^k = \{W \in p: |W| = k, |W \cap S| \in S_{21}\}$ and $W_2^k = \{W \in p: |W| = k, |W \cap S| \in S_{22}\}$. From the definition of the measurement-consistent partition p it follows that there are measure-preserving isomorphisms mod 0 from W_1^k to W_2^k for each k. Now observe that $S_{21} = \bigcup_k W_1^k$ and $S_{22} = \bigcup_k W_2^k$. Q.E.D.

To illustrate that the set S_2 is a countable union of measurable subsets, let k be fixed, and suppose that there is a pair of index numbers, j and l, such that $N_{kj} \cap S \neq \phi$, $N_{kl} \cap S \neq \phi$, and, for some subset W of N_{k1}, for each $i \in W$, $\Psi_{kj}(i) \in S$, $\Psi_{kl}(i) \in S$, and for all $m = 1, \ldots, k$, $m \neq j$, $m \neq l$, it holds that $\Psi_{km}(i) \cap S = \phi$. Then W is the subset of "1st members" of coalitions in p whose j^{th} and l^{th} members are in 2-member coalitions in m. Note that $N_{kj} \cap S$ and $N_{kl} \cap S$ are both the intersections of measurable subsets and are therefore measurable. The set S_2 is the union of all such sets $N_{kj} \cap S$ and $N_{kl} \cap S$.

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