Mathematical Social Sciences 6 (1983) 27-48
North-Holland

APPROXIMATE CORES OF REPLICA GAMES AND ECONOMIES. Part I: Replica games, externalities, and approximate cores

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Communicated by K.H. Kim
Received 3 June 1983

Sufficient conditions are demonstrated for the non-emptiness of approximate cores of sequences of replica games, i.e. for all sufficiently large replications the games have non-empty approximate cores and the approximation can be made arbitrarily 'good'. The conditions are simply that the games are superadditive and satisfy a non-restrictive 'per-capita' boundedness assumption (these properties are satisfied by games derived from well-known models of replica economies). It is argued that the results can be applied to a broad class of games derived from economic models, including ones with external economies and diseconomies, indivisibilities, and non-convexities. To support this claim, in Part I applications to an economy with local public goods are provided, and in Part II, to a general model of a coalition production economy with few restrictions on production technology sets and with (possibly) indivisibilities in consumption. Additional examples in Part I illustrate the generality of the result.

Key words: Approximate cores; replica games; replica economies; local public goods.

1. Introduction

In this paper we show that under simple and natural conditions, games with many players and relatively few types of players have non-empty approximate cores. The conditions are that the games are superadditive and satisfy a weak boundedness property. We indicate how these results can be applied to games derived from a broad class of economic models including ones derived from standard Arrow-Debreu models of economies with private goods and coalition production economies, ones with local public goods, and ones with a mix of external economies and diseconomies.

It is well known that if there are non-convexities the core of a private goods economy may be empty, but, if the economy is sufficiently large, approximate cores may well be non-empty. Although a large amount of research has been devoted to the study of approximate cores of private goods economies with non-convexities,

0165-4896/83/$3.00

relatively few results similar to those for private goods economies have been obtained for economies where there may be externalities in production and/or consumption of some goods. Moreover, most of the work on cores of economies with externalities has been devoted to the case where the externalities are only beneficial, i.e. to external economies rather than diseconomies (cf. Shapley and Shubik, 1969; Foley, 1970); for these economies, Pareto-optimal states can in general be realized only by the coalition consisting of all the agents in the economy. Counterexamples to the non-emptiness of the core of an economy where there are external diseconomies have been obtained (cf. Shapley and Shubik, 1969).

For economies where there is a mix of external economies and diseconomies, and where optimality requires that agents be appropriately partitioned into groups for the purposes of joint consumption and/or production within the groups, negative results are in general obtained – the core may well be empty (cf. Greenberg, 1978; Drèze and Greenberg, 1980; and the examples in Parts I and II of this paper).

Both economies with coalition production and ones with local public goods display a mix of external economies and diseconomies. When there is coalition production (i.e. the technology set available to a coalition of agents may be different for different coalitions), up to some point adding an additional member to a coalition may increase the average productivity of the members of a coalition but eventually average productivity may decrease as the size of the coalition increases. It is easy to imagine a similar phenomenon occurring in economies with local public goods. Consider, for example, the sharing of a swimming pool by the members of a club. For such clubs with small memberships, as the size of the club increases it may be possible for all members of the club to become better off because of the reduced per-capita costs of the swimming pool and increased possibilities for social interaction – we have a situation where there are external economies. However, eventually the pool can become crowded and then over-crowded as the membership is increased – there are external diseconomies.

Under some special assumptions, non-emptiness of the core of a coalition production economy has been demonstrated (when there are no externalities in consumption, only in production). These assumptions take two forms: there are non-decreasing returns to coalition size so the firm containing all agents in the economy is 'optimal' (cf. Hildenbrand, 1968; Sondermann, 1974) or the technologies satisfy a 'balancedness' condition, for which there has been little intuitive justification (cf. Ichiishi, 1977).

Another class of economies with a mix of external economies and diseconomies for which non-emptiness of cores and approximate cores has been demonstrated is those with local public goods in Wooders (1978, 1980a, 1981). The results and the techniques used in these papers, however, depend on a special restriction on preferences and production technologies. This restriction is that only the number of agents with whom an agent consumes and produces the local public goods, and not any characteristics of these agents, is relevant. (The results in the current paper can be used to show non-emptiness of approximate cores of economies with local public goods without this special restriction; see Wooders, 1954.)

In summary we stress that: (1) although cores of private goods economies have been intensively studied, beyond some counterexamples to general existence, relatively little is known about cores of economies with externalities, especially when there is a mix of external economies and diseconomies; and (2) the results currently in the literature cover only a limited set of cases.

The relative sparsity of results for economies with a mix of external economies and diseconomies may suggest that techniques used for the analysis of economies without such externalities do not easily extend to cover a variety of more complex situations. There are, in the literature, suggestions that the basic feature of convexity ensures the existence of competitive-like equilibria and the core (cf. Arrow, 1970). In the appendix we provide an example of an economy with a local public good where agents are assumed to be 'divisible' (i.e. part-time membership in a jurisdiction[1] is allowed), preferences are convex, and production-possibility sets are convex yet the core may well be empty. We argue that mixes of external economies and diseconomies create problems that are not amenable to usual methods of convexification and, indeed, our results are not based on convexifying effects of large numbers.[2]

Rather than specifying an economic model in terms of preferences, production possibilities, etc. we use the framework of a game in characteristic function form without side payments.[3] This framework is sufficiently general to accommodate games derived from a broad class of economies. We next discuss what is meant by a game derived from an economy, a sequence of replica games, and then a class of economies to which our results can be applied.

Given the data of an economy – endowments; preferences; production possibilities; the defining properties of the goods, such as whether the goods are private goods, public goods, or local public goods; and a specification of what is feasible for a coalition of agents to consume and produce given the endowments of the members of the coalition – suppose each agent's preferences are given by a utility function. For each coalition S of agents we define $V(S)$ as a set of utility vectors in $\mathbb{R}^n$, where n is the total number of agents in the economy. A member of $V(S)$, say $u=(u^1,\dots,u^n)$, must have the property that there is a feasible state of the economy restricted to the members of S such that this state yields agent i the utility u^i for each i in S (coordinates of u not associated with members of S are

[1] A jurisdiction is a subset of agents who jointly produce and consume the local public good. A jurisdiction structure is a partition of the set of agents.

[2] They may more appropriately be viewed as based on a 'balancifying' effect of large numbers.

[3] By 'without side payments', we do not mean to exclude the case of a game with side payments. Instead, a game with side payments is a special case of one without.

unrestricted). Let A denote the set of agents in the economy. Then the ordered pair (A, V) is the game derived from the economy.

In the case where there are external economies or diseconomies, we define $V(S)$ so that members of $V(S)$ represent utility vectors attainable by the membership of S (again, ignoring coordinates not associated with members of S) regardless of the actions of the complimentary coalition.[4] A utility vector u is in the core of the game (A, V) if u is in $V(A)$ (i.e. u is feasible) and no coalition of agents S can improve on u by itself.[5]

We consider sequences of games $(A_r, V_r)_{r=1}^{\infty}$, where A_r is the set of players of the rth game, consisting of r players of each of T types, and V_r is a correspondence from subsets of A_r to $\mathbb{R}^{rT}$. We assume that $A_r \subset A_{r+1}$ for all r. The sequence is then said to be a sequence of replica games if (a) all players of the same type are substitutes for each other, and (b) $V_r(S)$ does not 'decrease' as r increases; i.e. if $S \subset A_r$ and $S \subset A_{r'}$, where $r \leq r'$, then the projection of $V_r(S)$ on the subspace associated with the members of S is contained in that of $V_{r'}(S)$.

For our main theorem we require two assumptions on the sequence of replica games. First, it is required that the games are superadditive – for any r and any two disjoint subsets of A_r, say S and S', we have $V_r(S) \cap V_r(S') \subset V_r(S \cup S')$. Informally, the superadditivity property is that a larger coalition $S \cup S'$ can 'do at least as well by' all its members as the two coalitions S and S' can do independently. The second condition, per-capita boundedness, is that equal-treatment utility vectors in $V_r(A_r)$ are bounded independently of r; i.e. there is a constant real number K, independent of r, such that if $u \in V_r(A_r)$, where $u^i = u^j$ for all players i and j of the same type for each type, then $u^i \leq K$ for all players i in A_r.[6] Under these assumptions it is shown that certain approximate cores are non-empty and have desirable convergence properties as r grows large. We remark that the approximate core concept introduced in this paper is less restrictive than the one in Wooders (1983) and the conditions required for our theorem are also less restrictive.

We now address the question of the class of replica economies to which our results can be applied. We view the per-capita boundedness requirement as a sufficiently weak and relatively natural economic assumption to not require further discussion or justification. The superadditivity property arises naturally in games derived from

[4] This is a somewhat strong assumption about externalities related to the concept of a c-game (Shapley and Shubik, 1973) and the modeling difficulties encountered using the characteristic function. It is simple to calculate for exchange economies, but requires special modeling when diseconomies are involved.

[5] Formally, u is not in the interior of $V(S)$ for any non-empty subset S of A.

[6] This property is satisfied by games derived from various models of replication economies in the literature, such as Debreu and Scarf (1963), Shapley and Shubik (1966), Boehm (1974), and Wooders (1980a).

economies in which one of the options open to a coalition S is to subdivide into smaller, disjoint coalitions.

To apply our results to sequences of games derived from sequences of replica economies we require, in addition, that what the members of a coalition can ensure for themselves is independent of the replication number of the economy, or at least does not shrink as the economy is replicated. We remark that this property is satisfied by games derived from the replication economies in Debreu and Scarf (1963), Shapley and Shubik (1966), Boehm (1974), and Wooders (1980a). It is also satisfied directly by more game-theoretic models such as Shubik's bridge-game (1976), and assignment games, as in Shapley and Shubik (1972). It is not satisfied by examples in the literature of economies with a pure public good since initial endowments and/or production technologies of the membership of a coalition change as the economy is replicated[7] (cf. Milleron, 1972, pp. 459–463).

We present two examples of applications of our results to an economy with a local public good and endogenous jurisdiction structures (i.e. coalitions of agents who jointly consume and produce the local public good).[8]

In the first example we consider cases where jurisdictions associated with states of the economy in the core may be heterogeneous. This possibility of heterogeneity arises because preferences and/or production possibilities may depend on the numbers of agents of each type in a jurisdiction rather than simply upon the total number of agents in the jurisdiction (as in Wooders, 1978, 1980a, 1981). The second example serves to highlight that with local public goods there are both external economies and diseconomies.

A game-theoretic example is presented to illustrate non-emptiness of approximate cores of sequences of replica games without side payments. Another example illustrates the emptiness of approximate cores of games with many players when the conditions of our theorem are not satisfied.

In Part II, we present an application of our results to a general model of an economy in which indivisibilities and non-convexities are allowed in production. Moreover, the production technology available to a coalition may depend on the number of agents of each type in the coalition. In addition, we allow indivisibilities in the sets of possible consumptions. Besides noting that our theorems apply to the derived games, we show that for large replications of the economy there are states of the economy which cannot significantly be improved upon by any coalition and which are approximately feasible.

Recently, there has been a concern for the development of the theoretical foundations of Perfect Competition (cf. *J.E.T.*, 2, 1980, and especially the article therein

[7] That is, it is the manner of replication, not necessarily the presence of public goods, that prevents the games generated by the economies from satisfying our definition of replica games. Consider, also, the problems associated with the 'replication' of voting systems – see the example in Section 4.3.

[8] To keep these examples relatively simple, they are constructed so that utility is 'transferable'.

by Mas-Collel, 1980). For example, Novshek and Sonnenschein (1978) have developed a replication model of a closed economy with the number of producing firms determined endogenously and which is consistent, in the 'limit', with the Walrasian model. They, and others, consider games in strategic form and investigate non-cooperative equilibria.

One of the first developments in the foundations of the hypothesis of Perfect Competition, however, was suggested by Edgeworth (1881) who provided an example of the convergence of the contract curve, which can be interpreted as the core, to the set of equilibrium allocations. This idea has been successfully generalized, first by Debreu and Scarf (1963), and subsequently further generalized by many authors. Part of our intent in this work is to initiate the study of the cooperative game core approach to more general economic situations. We utilize the apparatus of games in coalition form and obtain general results concerning the existence and convergence properties of approximate cores.

Part I is divided as follows. Section 2 introduces notation. Section 3 consists of a formal statement of our game-theoretic model and results. The examples are provided in Section 4. Proofs of the results are contained in Section 5. Further interpretive remarks are given in Part II.

2. Notation

The following notation is used: $\mathbb{R}^n$ = Euclidean N-dimensional space; $\mathbb{R}^N_+$ = the non-negative orthant of $\mathbb{R}^N$; given $K \subset \mathbb{R}^N$, int$K$ denotes the interior of the set K; given a finite set S, $|S|$ denotes the cardinal number of S and $\mathbb{R}^S$ is the Euclidean $|S|$-dimensional space. Define $1 = (1, 1, \ldots, 1) \in \mathbb{R}^N$. Given $x \in \mathbb{R}^N$, we denote the (sup) norm of x by $\|x\|$, where $\|x\| = \max_i \text{ab}(x_i)$ and ab(x_i) denotes the absolute value of $x_i \in \mathbb{R}^1$.

Given x and y in $\mathbb{R}^N$, we write $x \geq y$ if $x_i \geq y_i$ for all i; $x > y$ if $x \geq y$ and $x \neq y$; and $x \gg y$ if $x_i > y_i$ for all i.

3. The model and the results

A *game without side payments*, or simply a *game*, is an ordered pair (A, V), where A, called the set of *players*, is a finite set and V is a correspondence from the set of non-empty subsets of A into subsets of $\mathbb{R}^A$ such that:

(i) for every non-empty $S \subset A$, $V(S)$ is a non-empty, proper, closed subset of $\mathbb{R}^A$ containing some member, say x, where $x \gg 0$;

(ii) if $x \in V(S)$ and $y \in \mathbb{R}^A$, with $x^i = y^i$ for all $i \in S$, then $y \in V(S)$;

(iii) $V(S)$ is bounded relative to $\mathbb{R}^S_+$, i.e. for each S there is a vector $k(S) \in \mathbb{R}^A$ where, for all $x \in V(S)$, $x^i \leq k^i(S)$ for all $i \in S$.

Let (A, V) be a game. A vector $x \in \mathbb{R}^A$, where the coordinates of x are superscripted by the members of A, is called a *payoff* for the game. A payoff x is *feasible* if $x \in V(A)$. Given a payoff x and players i and j, let $\sigma[x; i, j]$ denote the payoff formed from x by permuting the values of the coordinates associated with i and j. Players i and j are *substitutes* if: for all $S \subset A$, where $i \notin S$ and $j \notin S$, given any $x \in V(S \cup \{i\})$, we have $\sigma[x; i, j] \in V(S \cup \{j\})$; and, for all $S \subset A$, where $i \in S$ and $j \in S$, given any $x \in V(S)$, we have $\sigma[x; i, j] \in V(S)$. The game is *superadditive* if, whenever S and S' are disjoint, non-empty subsets of A, we have $V(S) \cap V(S') \subset V(S \cup S')$. It is *comprehensive*[9] if for all non-empty subsets S of A, if $x \in V(S)$ and $y \leq x$, then $y \in V(S)$.

Given a game (A, V) and $\varepsilon \geq 0$, a payoff x is in the *ε-core* of (A, V) if x is feasible and if, for all non-empty subsets S of A, there does not exist an $x' \in V(S)$ such that $x' \gg x + \varepsilon I$. When $\varepsilon = 0$, we call the ε-core simply the *core*.

We review the concept of balancedness. Let (A, V) be a game. Consider a family β of subsets of A and let $\beta_i = \{S \in \beta : i \in S\}$. A family β of subsets of A is *balanced* if there exists positive 'balanced weights' w_S for S in β with $\sum_{S \in \beta_i} w_S = 1$ for all $i \in A$. Let $\mathbb{B}(A)$ denote the collection of all balanced families of subsets of A. Define $\tilde{V}(A) = \bigcup_{\beta \in \mathbb{B}(A)} \bigcap_{S \in \beta} V(S)$. Define $\tilde{V}(S) = V(S)$ for all $S \subset A$ with $S \neq A$. Then $\tilde{V}$ maps subsets of A into $\mathbb{R}^A$ and is called the *balanced cover of* V. The game $(A, \tilde{V})$ is called the *balanced cover of* (A, V). If the game (A, V) has the property that $V(A) = \tilde{V}(A)$, the game (A, V) is *balanced*, and from Scarf's theorem (1967), the core of the game is non-empty.

Given a game (A, V), define $V^P(S) = \{x \in \mathbb{R}^S$: for some $x' \in V(S)$, x is the projection of x' on $\mathbb{R}^S\}$, where $\mathbb{R}^S$ is the subspace of $\mathbb{R}^A$ associated with the members of S. (Note that if (A, V) and (A, V') are two games where $V^P(S) = V'^P(S)$ for all $S \subset A$, then the correspondences V and V' are identical.)

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of games where, for each r, $A_r \subset A_{r+1}$ and $A_r = \{(t, q) : t \in \{1, \dots, T\},\ q \in \{1, \dots, r\}\}$. Write $x = (x_1, \dots, x_q, \dots, x_r) \in \mathbb{R}^{A_r}$ for a payoff for the rth game where $x_q = (x^{1q}, \dots, x^{tq}, \dots, x^{Tq}) \in \mathbb{R}^T$ and x^{tq} is the component of the payoff associated with the (t, q)th player. Given r and t, define $[t]_r = \{(t, q) \in A_r : q \in \{1, \dots, r\}\}$; the set $[t]_r$ consists of *the players of type t of the rth game*. The sequence $(A_r, V_r)_{r=1}^{\infty}$ is a *sequence of replica games* if:

(i) for each r and each $t = 1, \dots, T$, all players of type t of the rth game are substitutes for each other;

(ii) for any r' and r'', where $r' < r''$ and any $S \subset A_{r'}$, we have $V_{r'}^P(S) \subset V_{r''}^P(S)$ (i.e. the set of utility vectors achievable by the coalition S does not decrease as r increases).

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games. A payoff x for the game (A_r, V_r) is said to have the *equal-treatment property* if, for each t, we have $x^{tq'} = x^{tq''}$ for all q' and q''; players of the same type are allocated the same amount. The sequence

[9] We remark that it is often assumed that comprehensiveness is a property of a game merely for technical convenience. Since our results are intended to apply to games derived from economies, including ones with indivisibilities, for our purposes the comprehensiveness property is actually restrictive.

of games is *superadditive* if (A_r, V_r) is a superadditive game for all r. The sequence is *per-capita bounded* if there is a constant K such that for all r and for all equal-treatment payoffs x in $V_r(A_r)$ we have $x^{tq} \leq K$.

Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of replica games. We say that the sequence has a non-empty *strong approximate core* if, given any $\varepsilon > 0$, there is an r^* sufficiently large so that for all $r \geq r^*$ the ε-core of (A_r, V_r) is non-empty. Define $\tilde{V}_r^c$ to be the *comprehensive cover* of $\tilde{V}_r$, i.e. $\tilde{V}_r^c(S) = \{x \in \mathbb{R}^{A_r} : \text{for some } x' \in \tilde{V}_r(S), \text{ we have } x \leq x'\}$. The sequence has a non-empty *weak approximate core* if, given any $\varepsilon > 0$ and any $\lambda > 0$, there is an r^* such that for all $r \geq r^*$ for some payoff $\tilde{x}$ in the ε-core of $(A_r, \tilde{V}_r^c)$ and some x in $V_r(A_r)$ we have $|\{(t,q) \in A_r : \tilde{x}^{tq} \neq x^{tq}\}| < \lambda |A_r|$.

We remark that it is immediate that if the weak approximate core of the sequence is non-empty, then given any $\varepsilon > 0$ and any $\lambda > 0$ there is an r^* such that for all $r \geq r^*$, for some $\tilde{x} \in \mathbb{R}^{A_r}$, and some $x \in V_r(A_r)$, we have:

(i) $|\{(t,q) \in A_r : x^{tq} \neq \tilde{x}^{tq}\}| < \lambda |A_r|$, and

(ii) $\tilde{x}$ cannot be ε-improved upon by any coalition S; i.e. there does not exist an $S \subset A_r$ and an $x' \in V_r(S)$ such that $x' \gg \tilde{x} + \varepsilon I$.[10]

Therefore we could have defined the weak approximate core without reference to the balanced cover games.

In Wooders (1983) it is shown that if a sequence of replica games $(A_r, V_r)_{r=1}^{\infty}$ is superadditive and per-capita bounded and $V_r(A_r)$ is convex for all r, then the strong approximate core is non-empty. Our theorem concerning the non-emptiness of weak approximate cores cannot be regarded simply as a result of the convexifying effect of large numbers. Indeed, since the dimensionality of the payoff sets $V_r(A_r)$ is increasing as r becomes large, and $V_r(A_r)$ is in 'utility space', methods used to show the existence of equilibria and the non-emptiness of cores of private goods economies are not applicable.[11]

Our first theorem states simple conditions under which sequences of replica games have non-empty weak approximate cores.

Theorem 1. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive, per-capita bounded replica games. Then the weak approximate core is non-empty.*

A stronger result can be obtained for a subsequence of the sequence.

Theorem 2. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive, per-capita bounded replica games. Then some subsequence of the sequence has a non-empty strong approximate core.*

We remark that in the proof of Theorem 2, when the games are comprehensive,

[10] Similar notions of approximate cores have been used in economic models (cf. Henry, 1972, and also Part II of this paper).

[11] It may be possible that if the problem is embedded in some appropriate space, convexity in this space yields the results. This, however, is only a conjecture.

we actually show the existence of equal-treatment payoffs in the ε-cores of the games in the subsequence. More specifically, we show that, given $\varepsilon > 0$, there is an r^* and an $\bar{x} \in \mathbb{R}^T$ such that $\prod_{i=1}^{lr^*} \bar{x}$ is in the ε-core of (A_{lr^*}, V_{lr^*}) for all positive integers l, where (A_{lr^*}, V_{lr^*}) is the game (A_r, V_r) with $r = lr^*$.

4. Examples

4.1. Preliminaries

In the examples, we require the following definitions and notations.

Given r, *the set of agents in the* rth *economy* is $A_r = \{(t,q) : t = 1, \dots, T$ and $q = 1, \dots, r\}$. Given a non-empty subset S of A_r, let $s = (s_1, \dots, s_t, \dots, s_T)$ be the vector defined by its coordinates $s_t = |S \cap \{(t,q) : q = 1, \dots, r\}|$. Define $\varrho(S) = s$ so $\varrho(S)$ is a vector lising the numbers of agents of each type in S. We call $\varrho(S)$ the *profile* of S. Let I_r denote the set of profiles of subsets of A_r and let $I = \bigcup_{r=1}^{\infty} I_r$. Define $I(t) = \{s \in I : s_t \neq 0\}$.

Except for the final example, all the examples are of games with side payments, i.e. given (A, V), for each non-empty subset S of A there is a real number, say $v(S)$, such that $V(S) = \{x \in \mathbb{R}^A : \sum_{i \in S} x^i \le v(S)\}$. These games are completely determined by the function v, called the characteristic function, mapping non-empty subsets of A into $\mathbb{R}$. Therefore we let (A, v) represent the game (A, V) and, for convenience, define $v(\emptyset) = 0$.

For a game (A, V) with side payments, there are real numbers $\tilde{v}(S)$ for each non-empty subset S in A such that $\tilde{V}(S) = \{x \in \mathbb{R}^A : \sum_{i \in S} x^i \le \tilde{v}(S)\}$. Therefore the game $(A, \tilde{V})$ is determined by the game $(A, \tilde{v})$. We remark that it can be shown that $\tilde{v}(A) = \max_{\beta \in \mathrm{B}(A)} \sum_{S \in \beta} w_S v(S)$, where w_S are the weights for $S \in \beta$ (see Hildenbrand and Kirman, 1976, p. 88).

It is not difficult to show that for a sequence of superadditive, per-capita bounded replica games with side payments, $(A_r, v_r)_{r=1}^{\infty}$, the strong approximate core is non-empty if and only if the weak approximate core is non-empty. Therefore, for sequences of games with side payments, we refer simply to the approximate core. It can easily be shown that the approximate core of $(A_r, v_r)_{r=1}^{\infty}$ is non-empty if and only if, given $\varepsilon > 0$, there is an r^* such that for all $r \ge r^*$, we have[12]

$$\frac{\tilde{v}_r(A_r)}{|A_r|} - \frac{v_r(A_r)}{|A_r|} > \varepsilon.$$

4.2. An economy with a local public good

The economy has one private good and one local public good. The local public

[12] This is carried out in Wooders (1980b).

good has the property that any coalition of agents can produce the good for the consumption of the members of that coalition exclusively. A *jurisdiction* is a subset of agents who jointly consume and produce the local public good. A *jurisdiction structure* is a partition of the set of agents into jurisdictions.

A jurisdiction S with profile s has access to the production function $x+b(s)z=0$, where x is the output of the local public good, z is the input of the private good, and $b(\cdot)$ is a function from I to $\mathbb{R}_{++}$, the positive real numbers.

Each agent (t,q) has a positive initial endowment of w^{tq} units of the private good and all agents of the same type have the same initial endowment. Let $\bar{w}^t = w^{tq}$ for each type t and let $\bar{w}=(\bar{w}^1,\dots,\bar{w}^T)$.

The utility function of an agent of type t, u^{tq}, maps $I(t)\times\mathbb{R}^2_+$ into $\mathbb{R}$. The utility function is

$$u^{tq}(s,x,y)=xy+c^t(s),$$

where x and y are the levels of consumption of the public and private good, respectively, and $c^t(\cdot)$ is a function from $I(t)$ into $\mathbb{R}$. All agents of the same type have the same utility function. Let u^t denote the utility funtion of an arbitrary agent of type t so $u^t=u^{tq}$ for any q.

We construct a mapping v of I into $\mathbb{R}$, where $v(s)$ is the maximal sum of utilities achievable by a coalition S with profile s given that all members of S are in one jurisdiction. From v we then construct another function $\bar{v}$ where $\bar{v}(S)$ is the maximal sum of utilities achievable by a coalition S given that the agents in S can be partitioned into jurisdictions.

Given a profile s, using Langrangian techniques it can easily be verified that x^* and y^{*t} maximize $\sum_{t=1}^{T} s_t u^t(s,x,y^t)$ subject to the constraints[13] that

$$x+b(s)z=0, \tag{1}$$

$$x,y^t\geq 0, \tag{2}$$

$$\sum_{t=1}^{T} s_t(y^t-\bar{w}^t)=z, \tag{3}$$

if and only if

$$\sum_{t=1}^{T} s_t u^t(s,x^*,y^{*t})=\frac{b(s)}{4}(s\cdot w)^2+s\cdot\bar{c}(s),$$

where $\bar{c}(s)\in\mathbb{R}^T$ whose tth coordinate equals $c^t(s)$ if $s_t\neq 0$ and equals zero otherwise. Given s, define $v(s)=(b(s)/4)(s\cdot w)^2+s\cdot\bar{c}(s)$ for each $s\in I$. Given any jurisdiction S with profile s, define $v(S)=v(s)$ so $v(S)$ is the maximal sum of the utilities achievable by the membership of S using only their own resources. Let $\mathbb{P}(S)$ denote the collection of all partitions of S into non-empty subsets. Define $\bar{v}(S)=$

[13] For simplicity, we have explicitly only considered the case where all agents of the same type have the same private good allocation. Since utility is transferable, there is no loss in generality.

$\max_{\mathcal{P} \in \mathcal{P}(S)} \sum_{S' \in \mathcal{P}} v(S')$. Given S, $\bar{v}(S)$ is the maximum sum of the utilities achievable by the members of S using their own resources when the jurisdiction structure of S is variable.

Given any r, the pair $(A_r, \bar{v})$ is a game with side payments (where $\bar{v}$ is restricted to subsets of A_r) and the sequence $(A_r, \bar{v})_{r=1}^{\infty}$ is a sequence of replica games with side payments. The sequence of games is clearly superadditive. Since the set $\{\bar{u} \in \mathbb{R}^{A_r} : \sum_{(t,q) \in A_r} \bar{u}^{tq} \leq \bar{v}(A_r)\}$ is convex for all r and utility is transferable, from both the results of Wooders (1983) and this paper, if the sequence is per-capita bounded then it has a non-empty approximate core.

Per-capita boundedness of the sequence $(A_r, \bar{v})_{r=1}^{\infty}$ is a reasonable economic assumption; mathematically, we need to impose some conditions on $b(\cdot)$ and $c^t(\cdot)$ for each t. We assume that there is a constant K such that for all s in I:

$$\frac{b(s)(s \cdot w)^2}{4|s|} + \frac{s \cdot \hat{c}(s)}{|s|} \leq K,$$

where $|s| = s \cdot \mathbf{1}$. Informally, this assumption ensures that the benefits from sharing costs of the local public good are eventually outweighed by adverse crowding effects.

We now provide a simple example where optimal jurisdictions are heterogeneous. Assume there are two types of agents. The utility function of a representative agent of type 1 is $u^1(s, x, y) = xy - s_1^2$ and that of a representative agent of type is $u^2(s, x, y) = xy - s_2^2$. Informally, agents have an aversion to other agents of the same type. Each agent has an initial endowment of 4 units of the private good (the public good is not initially endowed). For simplicity we assume all jurisdictions have access to the same production function $x + z = 0$, where x is the output of the public good and z is the input of the private good.

In homogeneous jurisdictions the maximum feasible utility level of a representative agent when all agents are treated equally (and only resources owned by the members of the jurisdiction are distributed among consumers and firms) is four 'utils' and is obtained in two-person jurisdictions with $y = 2$ for both agents in the jurisdiction and $x = 4$.

In a jurisdiction consisting of one agent of each type, by allocating 2 units of the private good to each agent and letting $x = 4$, which is feasible for the jurisdiction, each agent can have 7 utils.

It can easily be verified that if the number of agents of each type is r, then a state of the economy in the core has associated heterogeneous jurisdictions consisting of one agent of each type, and each jurisdiction produces 4 units of the public good. For this case, for $r > 2$, only equal treatment payoffs will be in the core of the game.[14] Let $\bar{u}$ be in $\mathbb{R}^2$; then $\prod_{i=1}^{r} \bar{u}$ is in the core of the rth game if and only if $\bar{u}_1 + \bar{u}_2 = 14$, $\bar{u}_1 \geq 4$, and $\bar{u}_2 \geq 4$. In the case where there are more agents of, say, type 1 than of type 2 (using 'type' informally to mean a set of agents of all of whom are

[14] This is easy to prove and also follows from Theorem 3 stated in the next section.

substitutes for each other),[15] then again payoffs in the core have the equal-treatment property and if $\bar{u} \in \mathbb{R}^{A_r}$ is in the core of the rth game, $\bar{u}^{tq} = 4$ for all agents (t, q) of type 1 and $\bar{u}^{tq} = 10$ for all agents (t, q) of type 2.

We remark that the cores of some games in the sequence may be empty if A_1 does not have an even number of players of each type. Suppose, for example, A_1 contains 3 players with utility functions u^1 and 2 players with utility functions u^2. Now $\bar{v}(a_1) = 31$. Each set of players consisting of one player with utility function u^1 and one with utility function u^2 must realize a total utility of 14 at any state in the core. We can form a balanced family of subsets of A_1, say β, consisting of six distinct subsets containing one player of each type, say $S_1, \ldots, S_6$, with weights 1/3 each, and three distinct subsets each containing two players of type one, say S_7, S_8, S_9, with weights 1/6 each. Then

$$\sum_{k=1}^{9} w_k v(S_k) = 6 \cdot \tfrac{1}{3} \cdot 14 + 3 \cdot \tfrac{1}{6} \cdot 8 = 32 > \bar{v}(A_1)$$

so the game is not balanced and has an empty core.[16] Also, it is easy to see that for any even r, given A_1 as described, the core of $(A_r, \bar{v})$ is non-empty.

Regardless of the number of agents in A_1 who are substitutes for each other, if the sequence is per-capita bounded it has a non-empty approximate core.

To clearly illustrate that economies with local public goods have a 'mix' of external economies and diseconomies, we consider another specification of parameter values for our example. Suppose there is only one type of agent, $b(s) = 1$ for all s in I, and $c(s) = -s^2$. Let $w^{tq} = 8$ be the initial endowment of each agent. In this case when the number of agents is greater than 8, the core of the economy is non-empty if and only if the agents can be partitioned into jurisdictions each containing 8 members. Increasing the size of a jurisdiction has a negative effect: utilities of the members of the jurisdiction decrease because of 'crowding' - each agent is a 'local public bad'. However, when jurisdictions are 'small', the diseconomy caused by an additional agent can be more than compensated for by decreased per-capita costs of the local public good - a public good effect.

4.3. Simple games

This example is of a non-replica sequence of games where the number of players grows large but given any 'small' $\varepsilon > 0$ there is a term in the sequence such that all larger games have empty ε-cores. Each game in the sequence is superadditive and the sequence of games is per-capita bounded. Moreover, the players of each game are all substitutes for each other. The sequence is not, however, a sequence of

[15] Formally we should specify that there are, for example, n 'types' of agents and all agents of types 1 to $n-1$ have the utility functions u^1 and are thus substitutes for each other.

[16] For games with side payments, the core is non-empty if and *only if* the game is balanced (see Shapley, 1967).

replica games since the payoff to any given coalition eventually decreases as the games grow large. The games considered are sequences of 'simple games' with side payments.

Formally, let $(A_r, v_r)_{r=1}^{\infty}$ denote a sequence of games with side payments where for each r we have $|A_r| = 2r-1$ and for all subsets S of A_r, we have

$$v_r(S) = \begin{cases} 0 & \text{if } |S| < \dfrac{2r-1}{2}, \\ 2r-1 & \text{if } |S| > \dfrac{2r-1}{2}. \end{cases}$$

We note that, for each r, $v_r(A_r)/|A_r| = 1$ so the sequence is per-capita bounded. Also, given any r the game (A_r, v_r) is superadditive. However, given any r and any coalition S contained in A_r, even if $v_r(S) = 2r-1$ there is an $r' > r$ such that $v_{r'}(S) = 0$; the total utility achievable by a coalition eventually decreases as the number of players of the game is increased so the sequence is not a sequence of replica games.

It is easy to see that for all $r \geq 2$ the core of the game (A_r, v_r) is empty.

To determine $\tilde{v}_r(A_r)$, the value of the characteristic function of the balanced cover game for A_r, observe that if u is a feasible payoff in the core of $(A_r, \tilde{v}_r)$, we must have $\sum_{i \in S} u^i \geq 2r-1$ for all coalitions S with $|S| = r$. Therefore the average payoff to the members of S must be at least $(2r-1)/r$. Since this is true for any subset S with $|S| = r$, and since the balanced cover $\tilde{v}_r$ is the smallest valued function such that $\tilde{v}_r(S) \geq v_r(S)$ for all S and $(A_r, \tilde{v}_r)$ has a non-empty core, it follows that

$$\tilde{v}_r(A_r) = |A_r| \frac{2r-1}{r} = \frac{(2r-1)^2}{r}.$$

Recall that the game (A_r, v_r) has a non-empty ε-core if and only if $\tilde{v}_r(A_r) - v_r(A_r) < \varepsilon |A_r|$. Since $\tilde{v}_r(A_r) = (2r-1)^2/r$ and $v_r(A_r) = 2r-1$, it follows that given any ε with $0 \leq \varepsilon < 1$, for all sufficiently large r the ε-core of (A_r, v_r) is empty.

As we stated in the introduction our definition of a sequence of replica games is satisfied by games derived from a diverse set of economic models. This example, however, resembles examples of sequences of economies with public goods, such as Milleron's, in that the utility achievable by the members of a given coalition decreases as the number of players increases (although in Milleron's example each economy has a non-empty core).

4.4. The weak approximate core

The following game-theoretic example illustrates a sequence of replica games with a non-empty weak approximate core and an empty strong approximate core. While a particularly simple case is considered, the example provides some insight into the theorems.

We assume $T=1$ so all players of each game are substitutes for each other. For simplicity, we denote the players of the rth game by $q=1,\dots,r$.

For any r and any player $q\leq r$, define

$$V_r(S)=\{x\in\mathbb{R}^{A_r}: x^q\leq 1\},\quad \text{where } S=\{q\}.$$

For any $S=\{q,q':q\neq q'\}$, define $\hat{V}_r(S)$ as those sets of vectors x in $\mathbb{R}^r_+$ where $(x^q+1)(x^{q'}+1)=9$, $x^q\geq 0$, and $x^{q'}\geq 0$ and let $V_r(S)$ be the comprehensive cover of $\hat{V}_r(S)$, i.e. $V_r(S)=\{x\in\mathbb{R}^{A_r}: \text{for some } x'\in\hat{V}_r(S),\ x\leq x'\}$. The projection on $\mathbb{R}^S$ of the set $V_r(S)$, where $|S|=2$, is depicted in Fig. 1.

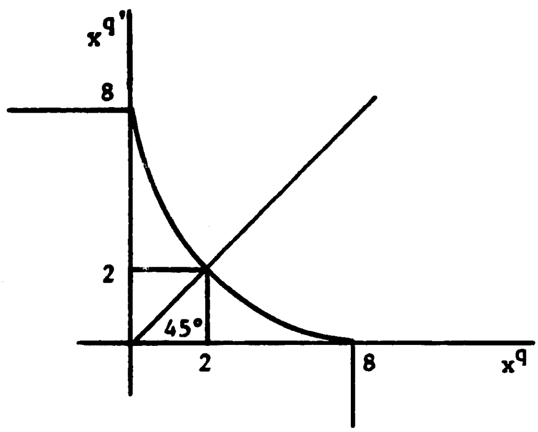

Fig. 1.

For any S in A_r, define $\mathbb{P}^*(S)$ as the set of all partitions of S into subsets where each subset contains either one or two players. Then define $V_r(S)=\bigcup_{P\in\mathbb{P}^*(S)}\bigcap_{S'\in P}V_r(S')$. From Theorem 3 in the following section, if the core of (A_r, V_r) is non-empty, then it contains a member with the equal-treatment property. Therefore, it is easy to see that if r is odd (and greater than 1) the core of the game is empty. If r is even, the core is non-empty and $2I=(2,\dots,2,\dots,2)\in\mathbb{R}^r$ is in the core.

Informally, the reason why the core is empty when r is odd is that the two-player coalitions are more beneficial per person than a one-player coalition, but players cannot be partitioned into two-player coalitions; there is a 'left-over' player. The strong approximate core is empty for the same reason and because (unlike the case where the set of feasible payoffs is convex) regardless of the magnitude of r, if r is odd there does not exist a feasible payoff x with the property that x^q is 'close' to 2 for all q.[17]

For each r it can be shown that $2I\in\mathbb{R}^r$ is in the core of the balanced cover game. Since for any r there is an x in $V_r(A_r)$ with

$$|\{q\in A_r: x^q=2\}|\geq r-1,$$

it is obvious that the conclusion of Theorem 1 holds.

[17] Convexity and comprehensiviness of $V_r(A_r)$ for all r, along with our other assumptions, leads to the result that the equal treatment payoffs for the game converge to those of the balanced cover game (see Lemma 6 in Wooders, 1983).

5. Proofs of the theorems

Throughout this section we let $(A_r, V_r)_{r=1}^{\infty}$ denote a sequence of superadditive replica games with T types of players. We continue to let $\mathbf{1}$ denote the vector of ones and the reader is to infer from the context the dimension of the space in which $\mathbf{1}$ is contained. Given r and a positive integer n, we write (A_{nr}, V_{nr}) for the game $(A_{r'}, V_{r'})$ where $r' = nr$.

Given a payoff x for the rth game, (A_r, V_r), when we write $y = \prod_{i=1}^{n} x$ it is to be understood that the coordinates of y are superscripted so that y is a payoff for the nrth game.

Throughout the following, given any $S \subset A_r$, it is to be understood that S is non-empty.

The following lemmas, proven in Wooders (1983), will be used in the proofs of the theorems.

Lemma 1. *Given any r and any positive integer n, we have* $\prod_{i=1}^{n} V_r(A_r) \subset V_{nr}(A_{nr})$.

Lemma 2. *Given any r, there is a positive integer n such that if* $x \in \tilde{V}_r(A_r)$, *then* $\prod_{i=1}^{n} x \in V_{nr}(A_{nr})$.

Define $E(r)$ and $\tilde{E}(r)$ to be subsets of $\mathbb{R}^T$ representing the equal-treatment payoffs in $V_r(A_r)$ and $\tilde{V}_r(A_r)$, respectively, i.e.

$$E(r) = \left\{ x \in \mathbb{R}^T : \prod_{i=1}^{r} x \in V_r(A_r) \right\}$$

and

$$\tilde{E}(r) = \left\{ x \in \mathbb{R}^T : \prod_{i=1}^{r} x \in \tilde{V}_r(A_r) \right\}$$

From the preceding lemma we immediately have the result that, given any r, there is a positive integer n such that if $x \in \tilde{E}(r)$, then $x \in E(nr)$.

Lemma 3. *For all r we have* $\tilde{E}(r) \subset \tilde{E}(r+1)$.

In what follows we use the concept of the closed limit of a sequence of sets. A definition of this concept and some properties can be found in Hildenbrand (1974, pp. 15–18). We remark that a sequence of subsets (F_n) of a compact metric space converges to a subset F with respect to the Hausdorff distance if and only if the closed limit of the sequence exists and equals F (see Hildenbrand, 1974, p. 17). We denote the Hausdorff distance between two sets, F and G, with respect to the metric $\|\cdot\|$, by $\|F, G\|$. The closed limit of the sequence $(\tilde{E}(r))$ is denoted by $L(\tilde{E})$.

Most of the following lemma appears as Lemma 8 in Wooders (1983). Given that $\tilde{E}(r) \subset \tilde{E}(r+1)$ for all r and from the per-capita boundedness assumption, it is also easily verified.

Lemma 4. *The closed limit, $L(\tilde{E})$, of $((\tilde{E}(r))$ exists and $\|\tilde{E}(r), L(\tilde{E})\| \to 0$ as $r \to \infty$.*

Recall that given a subset S in A_r for some r, the profile of S, denoted by $\varrho(S)$, is the vector $s \in \mathbb{R}^T$ defined by its coordinates $s_t = |S \cap [t]_r|$ for each $t \in \{1, \ldots, T\}$.

We require the following additional definitions.

A sequence of replica games $(A_r, V_r)_{r=1}^{\infty}$ is said to satisfy the assumption of *minimum efficient scale* (*MES*) (for coalitions) if there is an r^* such that for all $r > r^*$, given $x \in \tilde{V}(A_r)$, there is a balanced collection β of subsets of A_r with the properties that (1) $\varrho(S) \leq \varrho(A_{r^*})$ for all $S \in \beta$ and (2) $x \in \bigcap_{S \in \beta} V_r(S)$. We call r^* an *MES bound.*

Given r, r', and $S \subset A_r$, let $\mathbb{P}(S; r')$ denote the collection of all partitions of A_r into non-empty subsets where if $\mathscr{P} \in \mathbb{P}(S; r')$ and $S' \in \mathscr{P}$, then $\varrho(S') \leq \varrho(A_{r'})$ Given a sequence of replica games $(A_r, V_r)_{r=1}^{\infty}$ and given r', we define the r'th *truncation of* V_r by the correspondence $V_r(\cdot\,; r')$, where, for each non-empty subset S of A_r, we have $V_r(S; r') = \bigcup_{\mathscr{P} \in \mathbb{P}(S; r')} \bigcap_{S' \in \mathscr{P}} V_r(S')$. It is easily verified that the sequence $(A_r, V_r(\cdot\,; r'))_{r=1}^{\infty}$ is a sequence of superadditive replica games satisfying the assumption of MES with MES bound r'. Let $\tilde{V}_r(\cdot\,; r')$ denote the balanced cover of $V_r(\cdot\,; r')$. Define

$$E(r; r') = \left\{ x \in \mathbb{R}^T : \prod_{t=1}^{r} x \in V_r(A_r; r') \right\}$$

and

$$\tilde{E}(r; r') = \left\{ x \in \mathbb{R}^T : \prod_{t=1}^{r} x \in \tilde{V}_r(A_r; r') \right\}$$

A game (A, V) satisfies the assumption of *quasi-transferable utility*[18] (QTU) if, given any S in A, if $x \gg 0$ and x is in the boundary of $V^P(S)$, then $V^P(S) \cap \{x' \in \mathbb{R}^S : x' \geq x\} = x$. A sequence of games $(A_r, V_r)_{r=1}^{\infty}$ satisfies the assumption of QTU if each game in the sequence does.

Given a game (A, V) and $\delta > 0$, a game (A, V^δ) is a *δ-QTU cover of* (A, V) if (A, V^δ) is a game with the QTU property and if, for all $S \subset A$, we have $V(S) \subset V^\delta(S)$ and $\| V(S), V^\delta(S)\| < \delta$. In Wooders (1983) it is shown that given any $\delta > 0$, every comprehensive game has a comprehensive δ-QTU cover.

The following theorem is proven in Wooders (1983).

Theorem 3. *Let $(A_r, V_r)_{r=1}^{\infty}$ be a sequence of superadditive replica games satisfying the assumptions of QTU and MES with MES bound r^*. For any $r > r^*$, the core of the game $(A_r, \tilde{V}_r)$ is non-empty and if x is a payoff in the core, then x has the equal treatment property.*

[18] An analogous concept for preferences of agents in economic models is discussed in Rader (1972, pp. 69 and 74).

The non-emptiness of the core in Theorem 3 is Scarf's theorem (1967).

Before proving the theorems, we require one final lemma.

Lemma 5. *Given $\delta>0$, let (A, V^δ) be a comprehensive, δ-QTU cover of a comprehensive game (A, V). Then $\|\tilde{V}^\delta(A), \tilde{V}(A)\| \le \delta$, where $(A, \tilde{V}^\delta)$ is the balanced cover of (A, V^δ).*

Proof. Let $x \in \tilde{V}^\delta(A)$. Then for some balanced family β of subsets of A we have $x \in \bigcap_{S\in\beta} V^\delta(S)$. Since (A, V^δ) is a comprehensive, δ-QTU cover of (A, V), we have $V^\delta(S) \subset V(S)+\delta\{I\}$ for all S in β. Therefore $x \in \bigcap_{S\in\beta}(V(S)+\delta\{I\}) \subset \bigcap_{S\in\beta} V(S)+\delta\{I\} \subset \tilde{V}(A)+\delta\{I\}$ and we have $\tilde{V}^\delta(A) \subset \tilde{V}(A)+\delta\{I\}$. It can be shown easily that $\tilde{V}(A) \subset \tilde{V}^\delta(A)$ since $V(S) \subset V^\delta(S)$ for all $S \subset A$. $\square$

5.1. Sketch of the proof of Theorem 1

To prove Theorem 1, we take covers of covers until we can apply Theorem 3 to a sequence of games and then 'uncover' appropriate payoffs in the underlying sequences of games.

More precisely:

(1) We first consider the sequence of games formed by taking the balanced cover of a comprehensive δ-QTU cover of the r-1th truncation of the comprehensive cover of (A_r, V_r) for each r. Each game in the sequence of games constructed has the MES and QTU properties and therefore an equal-treatment payoff in its core. From per-capita boundedness, a sequence of equal-treatment payoffs (represented as vectors in $\mathbb{R}^T$) in the cores of the constructed games has a convergent subsequence.

(2) Given the limit payoff of some convergent subsequence, we 'uncover' feasible equal-treatment payoffs near this limit payoff in the ε-cores of the balanced covers of the comprehensive covers of the games for all sufficiently large r.

(3) We then use Lemmas 1 and 2 to show that for some subsequence of the sequence of comprehensive cover games, the equal-treatment payoffs in the ε-cores of the balanced covers of the comprehensive covers are feasible for the comprehensive cover games.

(4) We then uncover feasible payoffs for the underlying games and their balanced covers which satisfy the desired properties.

5.2. Proof of Theorem 1

5.2.1. Preliminaries

We introduce here notation for the covers used in the proof.

Let (A_r, V_r^c) denote the comprehensive cover of (A_r, V_r) for each r and let $(A_r, \tilde{V}_r^c)$ denote the balanced cover of (A_r, V_r^c). Let

$$\tilde{E}^c(r) = \left\{x \in \mathbb{R}^T : \prod_{t=1}^{r} x \in \tilde{V}_r^c(A_r)\right\}$$

and let $L(\tilde{E}^c)$ denote the closed limit of the sequence $(\tilde{E}^c(r))$; from Lemma 4, this limit exists. Similarly, let $(A_r, V_r^c(\cdot; r-1))$ denote the comprehensive cover of $(A_r, V_r(\cdot; r-1))$, let $(A_r, \tilde{V}_r^c(\cdot; r-1))$ denote the balanced cover of $(A_r, V_r^c(\cdot; r-1))$, and let

$$\tilde{E}^c(r; r-1) = \left\{x \in \mathbb{R}^T : \prod_{i=1}^{r} x \in \tilde{V}_r^c(A_r; r-1)\right\}.$$

It follows from Lemma 4 that the closed limit of the sequence $(\tilde{E}^c(r; r-1))$ exists and it is easily verified that the limit is $L(\tilde{E}^c)$.

Select a positive real number $\delta < \varepsilon/3$. Let $(A_r, V_r^\delta(\cdot; r-1))$ denote a comprehensive δ-QTU cover of $(A_r, V_r^c(\cdot; r-1))$ and let $(A_r, \tilde{V}_r^\delta(\cdot; r-1))$ denote the balanced cover of $(A_r, V_r^\delta(\cdot; r-1))$. Let

$$\tilde{E}^\delta(r; r-1) = \left\{x \in \mathbb{R}^T : \prod_{i=1}^{r} x \in \tilde{V}_r^\delta(A_r; r-1)\right\}$$

Let $L(\tilde{E}^\delta)$ denote the closed limit of $(\tilde{E}^\delta(r; r-1))$.

5.2.2. The proof

Given r, let $x^r \in \tilde{E}^\delta(r; r-1)$ be such that $\prod_{i=1}^r x^r$ is in the core of $(A_r, \tilde{V}_r^\delta(\cdot; r-1))$; from Theorem 3 this is possible. Since $V_r^P(\{t,q)\})$ contains a member greater than zero, we have $x^r > 0$ for each r. From the per-capita boundedness assumption, (x^r) has a convergent subsequence. Suppose (x^{r_j}) converges to $\bar{x}^*$. Since $\|\tilde{V}_r^\delta(A_r; r-1), \tilde{V}_r^c(A_r; r-1)\| < \varepsilon/3$ from Lemma 5, we have $\|\tilde{E}^\delta(r; r-1), \tilde{E}^c(r; r-1)\| < \varepsilon/3$ for all r. Therefore $\|L(\tilde{E}^\delta), L(\tilde{E}^c)\| < \varepsilon/3$ and there is an $x^* \in L(\tilde{E}^c)$ such that $\|\bar{x}^* - x^*\| < \varepsilon/3$ (this completes (1) of the sketch and begins (2)).

For each r, define $y_r = \prod_{i=1}^r (x^* - (\varepsilon/3)I)$. We claim that given $\varepsilon > 0$ there is an r'' such that for all $r \geq r''$ we have y_r in the ε-core of $(A_r, \tilde{V}_r^c)$. First, let r' be such that for all $r \geq r'$ we have $L(\tilde{E}^c) \subset \tilde{E}^c(r) + (\varepsilon/3)\{I\}$. Since $x^* \in L(\tilde{E}^c)$ and $y_r = \prod_{i=1}^r (x^* - (\varepsilon/3)I)$, we have y_r in $\tilde{V}_r^c(A_r)$ for all $r \geq r'$. Now let $r'' \geq r'$ be sufficiently large so that for all $r_j \geq r''$ we have $\|\bar{x}^* - x^{r_j}\| < \varepsilon/3$ and, consequently, $\|x^* - x^{r_j}\| < 2\varepsilon/3$. Suppose $r \geq r''$ and for some $S \subset A_r$ we have $y_r \in \operatorname{int} \tilde{V}_r^c(S) - \varepsilon\{I\}$. It follows that $\prod_{i=1}^r x^* \in \operatorname{int} \tilde{V}_r^c(S) - \frac{2}{3}\varepsilon\{I\}$. From (ii) of the definition of a sequence of replica games and since for any $r_j > r$ we have $\|x^* - x^{r_j}\| < 2\varepsilon/3$, it follows that $\prod_{i=1}^r x^{r_j} \in \operatorname{int} \tilde{V}_{r_j}^c(S)$ (since $\varrho(S) \leq \varrho(A_r)$ and $r_j > r$, $\tilde{V}_{r_j}^c(S) = \tilde{V}_{r_j}^c(S; r_j - 1)$). From the fact that $\tilde{V}_{r_j}^c(S) \subset \tilde{V}_{r_j}^\delta(S)$, we have a contradiction to the fact that $\prod_{i=1}^{r_j} x^{r_j}$ is in the core of $(A_{r_j}, \tilde{V}_{r_j}^\delta(\cdot; r_j - 1))$. Therefore y_r is in the ε-core of $(A_r, \tilde{V}_r^c)$ for all $r \geq r''$ (this completes (2) of the sketch).

Given r'' as determined above, let n^0 be a positive integer such that if $y \in \tilde{V}_{r''}^c(A_{r''})$, then $\prod_{i=1}^{n^0} y \in V_{n^0 r''}^c(A_{n^0 r''})$; from Lemma 2 this is possible. Let $r^0 = n^0 r''$. Let r^* be such that for all $r \geq r^*$, we have $r^0/r < \lambda$. Throughout the remainder of the proof, given $r \geq r^*$, write $r = lr^0 + j$, where l is a positive integer and $j \in \{1, \ldots, r^0\}$. From Lemma 2 and the choice of n^0, we have $y_{lr^0} \in V_{lr^0}^c(A_{lr^0})$ for all positive integers l

where, as above, $y_{lr^0}=\prod_{t=1}^{lr^0}(x^*-(\varepsilon/3)I)$. This completes (3) of the sketch.)

For each $r\geq r^*$ select $y_r^*\in V_r(A_r)$ such that the projection of y_r^* on A_{lr^0} is greater than or equal to y_{lr^0}; from the definition of V_r^c such a selection is possible. Also, for each $r\geq r^*$ select $\tilde{y}_r\in \tilde{V}_r^c(A_r)$ such that $\tilde{y}_r\geq \prod_{t=1}^{r}(x^*-(\varepsilon/3)I)$ and such that the projection of $\tilde{y}_r$ on A_{lr^0} equals that of y_r^*; this is possible from the definition of the balanced cover and comprehensiveness. Since $\tilde{y}_r\geq y_r$ for each r and since $y_r\in \tilde{V}_r^c(A_r)$, from the fact that y_r is in the ε-core of $(A_r, \tilde{V}_r^c)$, it follows that $\tilde{y}_r$ is in the ε-core of $(A_r, \tilde{V}_r^c)$. Also, we have

$$\begin{aligned}|\{(t,q)\in A_r: y_r^{*tq}\neq \tilde{y}_r^{tq}\}| &= |\{(t,q)\in A_r: q>lr^0\}|\\ &= jT\leq r^0T\leq \lambda|A_r|. \qquad \square\end{aligned}$$

5.3. Proof of Theorem 2

Given $\varepsilon>0$ and r^0 as in the preceding proof, let y_{lr^0} be as defined in the proof of the preceding theorem for each positive integer l. It is easy to see that in the case where (A_r, V_r) is comprehensive for each r, the payoff y_{lr^0} is in the ε-core of (A_{lr^0}, V_{lr^0}) for each positive integer l. When (A_r, V_r) is not necessarily comprehensive, there is a $y_{lr^0}^*\geq y_{lr^0}$ such that $y_{lr^0}^*$ is in the ε-core of (A_{lr^0}, V_{lr^0}) for each l. $\square$

Acknowledgments

This research was partially supported by Contract N00014-77-C0518 issued by the Office of Naval Research under Contract Authority NR-047-006 and by National Science Foundation Grant SES-8006836. The authors are indebted to Thomas Muench, Herbert Scarf, and Mark Walker for helpful suggestions. This research was undertaken when the second author was a Visiting Fellow at the Cowles Foundation and this support is gratefully acknowledged.

Appendix

The following example is to illustrate that the 'components' of an economic system, i.e. the preferences and production technologies, may be convex but the derived game does not necessarily have a non-empty core.

Consider again the local public goods example developed in Section 4, where there is only one type of agent, $w^{tq}=\bar{w}$ for all (t,q), $b(n)=1$ for all n, and $c(s)=-n^2$. Assume, however, that agents can be 'part-time' members in jurisdictions so the domain of $u^i(n,x,y^i)$ is a subset of $\mathbb{R}^3_+$.[19] In this case, for $r>\bar{w}^2/8$ the core of the derived superadditive cover game is non-empty if and only if r is an integer multiple of $\bar{w}^2/8$.

The problem of the possible emptiness of the core in this example is clearly not one of indivisibilities of agents. Moreover, it is not one of a lack of convexity in the usual sense since, given any $u^0 \geq 0$, the upper contour sets $\{(n, x, y) : u(n, x, y) \geq u^0\}$ are convex and obviously the production technology is convex.

To show that $\{(n, x, y) : u(n, x, y) \geq u^0\}$ is convex, we show that $u = xy - n^2$ is quasi-concave for all non-negative x, y, and n such that $xy - n^2$ is non-negative. We consider the determinants of the principal minors of the bordered Hessian.

The bordered Hessian is:

$$\begin{bmatrix} 0 & u_x & u_y & u_n \\ u_x & u_{xx} & u_{xy} & u_{xn} \\ u_y & u_{yx} & u_{yy} & u_{yn} \\ u_n & u_{nx} & u_{ny} & u_{nn} \end{bmatrix} = \begin{bmatrix} 0 & y & x & -2n \\ y & 0 & 1 & 0 \\ x & 1 & 0 & 0 \\ -2n & 0 & 0 & -2 \end{bmatrix}.$$

The determinant of the bordered Hessian is $4n^2 - 4xy$ which is non-positive since $xy - n^2$ is non-negative.

The determinant of the principal minor

$$\begin{bmatrix} 0 & y & x \\ y & 0 & 1 \\ x & 1 & 0 \end{bmatrix} \quad \text{is } 2xy \geq 0.$$

Therefore u is quasi-concave.

The above example suggests that standard methods exploiting the convexifying effect of large numbers are not immediately applicable to economies with a mix of external economies and diseconomies since the economy in the example is already 'convex' - but not necessarily balanced. It may be the case that if preferences and production possibilities are represented in some 'appropriate' spaces, non-balancedness and non-convexity are equivalent. However, to our knowledge, this has not been demonstrated.

[19] This is an analogous to assuming the set of players is a continuum. Allowing part-time membership in a jurisdiction is not equivalent to allowing part-time jurisdictions, as, for example, in Littlechild (1975). Allowing part-time jurisdictions balances the game whereas part-time membership does not necessarily do so.

We have not formally defined the core for the class of games where players can be part-time members in coalitions (or even such games). Informally, a payoff (function) is in the core if it is feasible and if no set of players can use part of their time to form a coalition and improve their payoffs.

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