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Continuum Economies with Finite Coalitions: Core, Equilibria, and Widespread Externalities

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We develop a new model of a continuum economy with coalitions consisting of only finite numbers of agents. The core, called the f-core, is the set of allocations that are stable against improvement by finite coalitions and feasible by trade within finite coalitions. Even with widespread externalities—preferences depend on own consumptions and also on the entire allocation up to the null set—we obtain the result that the f-core coincides with the Walrasian allocations. Without widespread externalities, the f-core, the Aumann core, and the Walrasian allocations all coincide; however, with widespread externalities there is no obvious natural definition of the Aumann core. *Journal of Economic Literature* Classification Numbers: 021, 022, 026. © 1989 Academic Press, Inc.

1. INTRODUCTION

1.1. *Motivation and Results*

The purpose of this paper is to give a new picture of perfect competition with recontracting. A perfectly competitive economy is one with a very large number of participants, each of whom actively pursues his own interests and each of whom has negligible influence on economic aggregates. Recontracting involves individuals meeting face to face and

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making mutually advantageous agreements. This suggests that in our picture, individuals should be effective in pursuit of their own interests through recontracting and simultaneously be ineffective in influencing any broad economic totals.

One innovative formulation of perfect competition with recontracting was given by Aumann [2]. There, the set of agents is a continuum—a nonatomic measure space—and coalitions of recontracting agents are subsets of positive measure. In this formulation each individual agent is negligible relative to the total economy and thus one aspect of the notion of perfect competition in exchange economies is elegantly captured. However, each agent is also negligible relative to any recontracting group; i.e., coalitions are of positive measure. This implies that Aumann's formulation does not capture the aspect of the individual as effective in recontracting. If we interpret Aumann's model in any way which makes the individual nonnegligible or significant in recontracting, then the individual must also be interpreted as nonnegligible relative to the entire continuum.[1] Thus Aumann's formulation does capture the aspect of the individual as negligible relative to the total but not the aspect of the individual as effective in recontracting.

In this paper we formulate our picture as a model with a nonatomic measure space of agents and with finite coalitions, i.e., ones consisting of finite numbers of agents. In this formulation each individual agent is negligible relative to the total, as in Aumann's formulation, but is nonnegligible in pursuit of his own interests through recontracting. Thus our formulation captures the two features of perfect competition with recontracting. The subtle aspects of the treatment of finite coalitions in the continuum will be discussed in the next subsection.

Before discussing the basic problems of modeling finite coalitions in the continuum, we describe the model and its major economic aspects and results. The solution concept adopted in this paper is the core. An allocation is in the core of the economy, called the f-core, if it satisfies two properties: First, it cannot be improved upon by any finite coalition and second, it is achievable by trades only in finite coalitions in some partition of the player set into finite coalitions. The second requirement is necessary to make the outcomes of cooperation for the entire economy consistent with the requirement that only finite coalitions can form. (We will also require that the partitions of the player set into finite coalitions be consis-

[1] Aumann himself interprets the continuum *not* as a continuum of agents but as a large, finite set of agents. He writes "one should not think of the points t of T (the agent set) as individual players: rather, one should think of an individual player as an infinitesimal subset dt of T" (—a subset of small, but nevertheless, positive measure) (Aumann [4], p. 614). Since the total agent set has a finite measure, it follows that one should think of the total player set as large but finite, and thus of the individual agent as nonnegligible relative to the total agent set. See also Aumann [2, p. 41] and Aumann and Shapley [5, pp. 176–178].

tent with the proportions given by the measure.) An aggregate outcome of recontracting is an allocation in the f-core.[2]

Our framework allows us to show the equivalence of the f-core and the Walrasian allocations in the context of an economy with widespread externalities where the Walrasian allocations may be Pareto-inefficient. By widespread externalities we mean that the preference of an individual depends on his own consumption of goods and also on the standards or averages of the consumptions in the economy; more precisely, on the entire allocation, up to null sets. For the special case of no widespread externalities, the equivalence of the f-core, the Aumann core, and the Walrasian allocations is obtained. When there are widespread externalities, the difficulty with the Aumann core is in defining the concept of "can improve upon," prior to any consideration of equivalence. This will be discussed after our theorems.

A general form of externalities could be that preferences depend on the entire allocation of goods in the economy. In this case, the externalities fall into two types—local externalities and global, or widespread, externalities. In this paper, we exclude local externalities by restricting preferences to depend on the entire allocation, but only up to null sets (and, of course, on own consumptions). Here, by local externalities, we mean situations where the preferences of individuals depend on the consumptions of sets of other individuals of measure zero—the situation ruled out by the term "widespread." With widespread externalities we can capture the idea that preferences depend on the standards or averages of the consumptions in the economy, and not on the consumptions of other particular individuals. For example, preferences could depend on the prevailing fashions, or how much clothing and housing is consumed on average. With widespread externalities, even though an individual is affected, in a sense, by the total environment, he need not explicitly take it into account in his individual decision making. From the viewpoint of the individual, the average consumptions are unchangeable. This is quite different from local externalities which are noticeable because they may differ in different localities and the individual may well benefit by taking them into account. By excluding local externalities we rule out situations where preferences depend on, e.g., the taste for loud music of one's next-door neighbors.

With widespread externalities, as in perfectly competitive economies, the individual takes the entire allocation as given. The idea of perfect competition is that the individual is negligible relative to the total, and thus it is

[2] The concepts of measurement-consistent partitions and the f-core were introduced and are further discussed in Kaneko and Wooders [10, 11] in a game-theoretic framework, sufficiently general to accommodate the model herein and other economic situations such as a continuum version of an assignment game of Shapley and Shubik [17].

natural to incorporate widespread externalities into the perfectly competitive model. More specifically, the equivalence of the f-core and the Walrasian allocations is obtained with widespread externalities. In spite of our equivalence result, with local externalities, where preferences may depend on the consumptions of specific individuals, it is commonly recognized that we cannot expect the equivalence; the local externalities pose an obstruction to the notion of perfect competition. When preferences depend on the entire allocation in some way, widespread externalities may well be the only case consistent with the notion of perfect competition.

1.2. *Finite Coalitions, the Continuum, and Measurement Consistency*

As indicated above, our model has a continuum of agents and finite coalitions. At first glance, these aspects of our model may appear inconsistent.[3] Therefore, we will first discuss the meaning of the measure and our treatment of finite sets in a nonatomic measure space, and then we consider a procedure of aggregating finite coalitions into the total economy.

In our model, the nonatomic measure on the agent set describes proportional magnitudes of aspects of the economy while, as in finite models, the cardinal numbers describe absolute magnitudes of aspects of finite coalitions. In a finite world, the proportional magnitudes described by the counting measure and the absolute magnitudes described by the cardinal numbers are mutually replaceable. However, in a continuum world these two notions are not mutually replaceable; the nonatomic measure cannot capture (it simply ignores) aspects of zero proportional magnitudes and the cardinal numbers cannot capture the proportional aspects of the total continuum. Thus, for the description of our picture of perfect competition with recontracting, formulated as a model with finite coalitions and a continuum of agents, we need to use both notions of measurement.

When we focus on activities of finite coalitions, measurement by cardinal numbers is sufficient. For example, a trade in a finite coalition is described by finite sums, as in finite models. When we consider only aspects of economic variables for sets of positive measure, we need only measure-theoretic measurements. These distinct notions of measurement meet when we aggregate activities, more specifically, trades within finite coalitions, to the total economy. Then these notions of measurement must be consistently connected.

The aggregation of trades within finite coalitions is made through the definition of feasible allocations for the total agent set. To be feasible, an allocation must be achievable by trades within the finite coalitions in some "measurement-consistent" partition of the total player set into such coali-

[3] Indeed, we cannot find this kind of mixture of a continuum with finite subsets in the literature of mathematics.

tions. A measurement-consistent partition is defined in a fashion consistent with both absolute magnitudes for finite coalitions and proportional magnitudes for economic totals. Roughly speaking, the measurement-consistency requirement is that the "integration" of the cardinal numbers of members of finite coalitions in a partition "recovers" the measure on the total agent set. This notion of measurement-consistent partitions enables us to interpret finite sums and integrals meaningfully in the same model.

In addition to logical consistency of the model itself, an important question to be asked of the continuum model is how it is to be interpreted in a finite world. This is the subject of Hammond [6], where the model is related to extant models without externalities, and of Kaneko and Wooders [12], where convergence in solution and structures is studied in the presence of widespread externalities. In these papers the restriction of coalitions in the continuum economy is approximated by the restriction of (permissible) coalitions to relatively small sets of agents in the finite economies. In the presence of widespread externalities (Kaneko and Wooders [12]), the permissible coalitions must be further restricted in that the squares of the bounds on permissible coalition sizes are required to become small relative to the total economy. In this manner, the effectiveness of the individual agent in coalitions is the same in the continuum as in the finite economies while, relative to the total economy, the individual agent becomes negligible. Thus, the prominent features of the continuum model with finite coalitions are reflected in large finite economies with relatively small coalitions.

This paper is organized as follows. In Section 2, we formulate our mathematical model, and then give our two equivalence theorems. The meaning of our model and results is carefully discussed after the statement of the theorem. Proofs are in Section 3.

2. The Model and the Theorems

2.1 *Agents and Measurement-Consistent Partitions of Agents*

Let $(A, \mathscr{A}, \mu)$ be a measure space, where A is a Borel subset of a complete separable metric space; $\mathscr{A}$, the σ-algebra of all Borel subsets of A; and μ, a nonatomic measure with $0 < \mu(A) < +\infty$.[4] Each element in A is called an *agent*. The measure μ expresses proportional magnitudes of the measurable components of the economy. The σ-algebra $\mathscr{A}$ is necessary for measurability arguments but does not play any important game-theoretic role.

[4] Under our assumptions, $(A, \mathscr{A})$ is measure-theoretically isomorphic to $([0, 1], C)$, where C is the σ-algebra of all Borel subsets of the interval $[0, 1]$ (see Parthasarathy [15], pp. 12–14).

Let $\mathscr{F}$ be the set of all finite subsets of A. Each element S in $\mathscr{F}$ is called a *finite coalition* or simply a *coalition*. As mentioned in Section 1, only finite subsets of agents can form coalitions in our model. Singleton sets are closed in A, so every coalition is measurable. Finite coalitions are negligible from the viewpoint of the measure (i.e., contain zero proportion of the economy); this means that finite coalitions are ignored by the measure. Their behavior will, however, be constrained and described by absolute magnitudes.

Since only finite subsets of agents are allowed to form coalitions, allocations arising from recontracting are required to be attained by trades within the finite coalitions of a partition of the agent set A into finite coalitions. Partitions of agents into finite coalitions involve measurements both in absolute magnitudes and in proportional magnitudes. Partitions which reconcile these measurements are called measurement-consistent. We first define measurement-consistent partitions and then motivate these conditions via an example.

A partition p of A is *measurement-consistent* iff for any positive integer k

> $A_k^p \equiv \bigcup_{S \in p, |S|=k} S$ is a measurable subset of A, and each A_k^p $(k = 1, 2, ...)$ has a partition $\{A_{kt}^p\}_{t=1}^k$, where each A_{kt}^p is measurable, with the following property: there are measure-preserving isomorphisms[5] $\psi_{k1}^p, \psi_{k2}^p, ..., \psi_{kk}^p$ from A_{k1}^p to $A_{k1}^p, ..., A_{kk}^p$, respectively, such that $\{\psi_{k1}^p(a), ..., \psi_{kk}^p(a)\} \in p$ for all $a \in A_{k1}^p$.[6] (2.1)

Note that (2.1) implies that for any $S \in p$ with $|S| = k$, we have $S = \{\psi_{k1}^p(a), ..., \psi_{kk}^p(a)\}$ for some $a \in A_{k1}^p$. Therefore, for each integer k, A_k^p consists of all the members of k-agent coalitions and A_{kt}^p consists of the tth members of these coalitions. The requirements that all the sets A_{kt}^p have equal measure and that the isomorphisms be measure-preserving then capture the idea that coalitions of size k should have as "many" (i.e. the same measure) first members as second members, as many second members as third members, etc.

Let Π denote *the set of measurement-consistent partitions.*

To exemplify the need for measurement-consistent partitions, we consider an economy where the set of agents is $[0, 3)$ with Lebesgue measure. The agents indexed by real numbers in $[0, 1)$ each own a right-hand

[5] Recall that a function ψ from a set B in a to a set C in a is a *measure-preserving isomorphism* from B to C iff (i) ψ is a measure-theoretic isomorphism, i.e., ψ is 1 to 1, onto, and measurable in both directions, and (ii) $\mu(T) = \mu(\psi(T))$ for all $T \subset A$ with $T \in \mathscr{A}$.

[6] In Kaneko and Wooders [11, Lemma A.1] a simple example of the construction of a measurement-consistent partition is provided. The example considers the case of $k = 2$ but the extension of the general case is clear.

glove (RHG) and those in $[1, 3)$, a left-hand glove (LHG). Clearly $\{\{a, 1+2a\}: a \in [0, 1)\}$ is a partition of $[0, 3)$ which pairs every agent owning a LHG with an agent owning a RHG. However, this partition violates the definition of a measurement-consistent partition since it matches, in a one-to-one way, one-third of the agents to two-thirds of the agents. Were such partitions allowed we could not consider the idea of relative scarcities of commodities reflected by the measure in economic models. In this economy, partition $p = \{\{a, 1+a\}: a \in [0, 1)\} \cup \{\{a\}: a \in [2, 3)\}$, where the agents in $[0, 1)$ are arranged in pairs with the agents in $[1, 2)$ and the others in $[2, 3)$ remain single, is measurement-consistent.[7]

2.2 *The Economy and Allocations*

Let Z_+ be the set of nonnegative integers and let $\Omega = Z_+^I \times R_+^D$ be the *consumption set* were I is a finite index set for indivisible commodities and D is a finite index set for divisible commodities with $|D|$, the cardinality of D, greater than zero. Let $M = I \cup D$. We use the following conventional symbols for the orderings of elements of Ω: $\gg, > \geqslant$. We define the vector

$$1_M = (\overbrace{1, 1, ..., 1}^{|M|}),$$

and define 1_I and 1_D analogously. Finally, we will denote the space of measurable functions from the agent set A to Ω by $L(A, \Omega)$.[8]

We model widespread externalities by having preferences depend on the allocation space $L(A, \Omega)$ in addition to the commodity space Ω. Thus a preference relation $\succ$ is defined to be a subset of $(\Omega \times L(A, \Omega)) \times (\Omega \times L(A, \Omega))$, and we typically write

$$[x, f] \succ [x', f']$$

to denote the preference between two members of the extended consumption set $\Omega \times L(A, \Omega)$. We assume

(A.0) if $f(a) = g(a) = h(a)$ a.e. in A, then $[x, f] \succ [y, g] \Leftrightarrow [x, h] \succ [y, h]$.

This means that a preference relation is not affected by a change in the allocation on a null set of agents. (No other assumptions are required on the widespread externalities.)

[7] The reader may notice from this example that a continuum version of an assignment game of Shapley and Shubik [17] is well formulated in our framework. For more detailed discussions of this, see Kaneko and Wooders [10, 11].

[8] Note that this is *not* to be taken as the space of equivalence classes of functions which are equal almost everywhere.

For each fixed $f \in L(A, \Omega)$ and preference relation $\succ$, we define the *conditional preference relative to f*, $\succ(f)$, of $\succ$ as the relation on Ω satisfying

$$x \succ(f)\, y \Leftrightarrow [x, f] \succ [y, f].$$

We impose mathematical structure on preferences via conditions on conditional preferences. The space of possible conditional preference relations is taken to be

$$\mathrm{PC} = \{\succ^c : \succ^c \text{ is an irreflexive, transitive, and open binary relation on } \Omega \times \Omega\}.$$

In particular, continuity and transitivity of conditional preferences is assumed, but not of $\succ$ itself. The space PC is then given the topology induced by closed convergence on the class $\{\Omega \times \Omega \backslash \succ^c : \succ^c \in \mathrm{PC}\}$ of closed subsets of $\Omega \times \Omega$ as in Hildenbrand [8, pp. 96–98].

The *space of preference relations P* is defined by

$$P = \{\succ : \succ \text{ satisfies (A.0) and } \succ(f) \in \mathrm{PC} \text{ for all } f \in L(A, \Omega)\}.$$

An *economy E* is a mapping $E(a) = (\succ_a, e(a))$ from A to $P \times \Omega$ such that for any $f \in L(A, \Omega)$ the mapping $E_f(a) := (\succ_a(f), e(a))$ is measurable—i.e., for any Borel set T of $PC \times \Omega$, we have $E_f^{-1}(T) \in \mathscr{A}$. The economy E assigns to each agent a in A a preference relation $\succ_a$ and an initial endowment $e(a)$. For each f in $L(A, \Omega)$, the mapping from agents to their conditional preferences and their initial endowments is measurable. We assume that $\int_A e$ is finite and strictly positive.

We will now define feasible allocations for the economy E. A function f in $L(A, \Omega)$ is called an *allocation with respect to a* (measurement-consistent) *partition p* ($p \in \Pi$) of A into finite subsets iff

$$\sum_{a \in S} f(a) \leqslant \sum_{a \in S} e(a) \qquad \text{for all} \quad S \in p. \tag{2.2}$$

An allocation f with respect to p can be attained by trading commodities only within coalitions in p. Define the sets $F(p)$ ($p \in \Pi$), F, and F^* by

$$F(p) = \{f \in L(A, \Omega) : f \text{ is an allocation with respect to } p\}; \tag{2.3}$$

$$F = \bigcup_{p \in \Pi} F(p); \tag{2.4}$$

$$F^* = \{f \in L(A, \Omega) : \text{for some sequence } \{f^n\} \text{ in } F, \; \{f^n\} \text{ converges in measure to } f\}. \tag{2.5}$$

Note that $F(p) \subset F \subset F^*$ for all $p \in \Pi$ and F^* is the closure of F with respect to convergence in measure. Also, since the endowment e satisfies (2.2), $F(p)$ is not empty. Recall that "$\{f^n\}$ converges in measure to f" means that for any $\varepsilon > 0, \mu(\{a \in A: d(f^n(a), f(a)) > \varepsilon\}) \to 0$ as $n \to \infty$.[9] In interpretation we consider an element of F^* as approximately feasible in the sense that it is in the closure of F and not necessarily in F itself.[10] The end of this subsection includes an example illustrating that F may not be closed.

We say that a coalition S in *can improve upon* a function f in $L(A, \Omega)$ iff there is a vector $(x^a)_{a \in S}$ such that

$$x^a \in \Omega \qquad \text{for all} \quad a \in S; \tag{2.6}$$

$$\sum_{a \in S} x^a \leqslant \sum_{a \in S} e(a); \tag{2.7}$$

$$x^a \succ_a(f) f(a) \qquad \text{for all} \quad a \in S. \tag{2.8}$$

Of course, if the members of S change their part of the allocation f to $(x^a)_{a \in S}$ then the resulting allocation agrees with f only on $A \backslash S$. However, from the assumption (A.0) that preferences are unaffected by a change in the allocation on a null set, the change in f on the finite set S can be ignored in condition (2.8).

The *f-core of the economy E* is defined to be

$C_f = \{f \in F^*$: there is a full[11] set $\tilde{A} \subset A$ such that no finite coalition in $\tilde{A}$ can improve upon $f\}$.

An allocation f in the f-core C_f is stable, roughly speaking, in the sense that no finite coalition can improve upon f. It is also approximately feasible in the sense that $f \in F^*$; i.e., f can be approximated by a sequence of exactly feasible allocations. The exact definition takes a full subset $\tilde{A}$ so that no finite coalition in $\tilde{A}$ can improve upon f. This intermediate step of taking a full set $\tilde{A}$ is made for simplicity of the statements of our results. Indeed, we could adopt a definition of the f-core requiring that no finite coalition could improve upon an allocation in the f-core. However, this complicates the statements of our results without adding any substance to the meaning.

The following example illustrates that F may not be closed with respect to convergence in measure so it may be the case that $F \neq F^*$.

[9] If $\{f^n\}$ converges in measure to f, then $\{f^n\}$ has a subsequence which converges pointwise to f a.e. and, conversely, if $\{f^n\}$ converges pointwise to f a.e., then $\{f^n\}$ itself converges in measure to f.

[10] In Kaneko and Wooders [12] we discuss the appropriateness of F^* as the idealized feasible set for sequences of finite economies.

[11] A full set of agents is one whose complement is null.

EXAMPLE 2.2.1: F is not closed. Suppose that $A=[0,1)$, that $\Omega = R^2_+$, μ is Lesbesgue measure, and

$$e(a)=\begin{cases}(1,0) & (0 \leqslant a < \alpha)\\ (0,1) & (\alpha \leqslant a < 1)\end{cases}$$

for some irrational number α. Consider the allocation $f(a)=(\alpha, 1-\alpha)$ (all $a \in A$) which satisfies $\int_A (f-e)=0$. Each finite coalition consisting of m agents with endowment $(1,0)$ and of n agents with endowment $(0,1)$ has an aggregate endowment equal to (m,n). To achieve the allocation $(\alpha, 1-\alpha)$ for each of the $m+n$ agents, one needs $(m+n)(\alpha, 1-\alpha) \leqslant (m,n)$. This implies that $(m+n)\alpha \leqslant m$ and $m \leqslant (m+n)\alpha$, so $m=(m+n)\alpha$ which is impossible because α is irrational. Therefore $f \notin F$. One can verify that $f \in F^*$. (One can also verify by appropriate choice of preferences that there may be no Pareto-efficient allocations in F. For example, choose preferences given by the utility function $u(x_1, x_2)=\min\{(1-\alpha)x_1, \alpha x_2\}$.)

2.3. *Equivalence of the f-Core and the Walrasian Allocations*

This subsection presents conditions on E under which the f-core and the set of Walrasian allocations W coincide. We first define the Walrasian allocations and then state the required additional conditions.

A function f in $L(A,\Omega)$ is called a *Walrasian allocation* iff for some $p \in R^M_+$,

$$p \cdot f(a) \leqslant p \cdot e(a) \qquad \text{a.e. in } A; \tag{2.10}$$

$$\text{a.e. in } A,\ x \not\succ_a(f) f(a) \qquad \text{for all} \quad x \in \Omega \text{ with } p \cdot x \leqslant p \cdot e(a); \tag{2.11}$$

$$\int_A f \leqslant \int_A e. \tag{2.12}$$

Let W denote the set of Walrasian allocations.

The first step in showing equivalence of the f-core and the Walrasian allocations is to demonstrate that $f \in F^*$ iff f satisfies the mean excess demand condition (2.12). This is proved[12] in Kaneko and Wooders [11, Lemma 3.1]. Therefore we have

$$F^* = \{f \in L(A,\Omega) \colon \int_A f \leqslant \int_A e\}. \tag{2.13}$$

In particular, this implies that $W \subset F^*$.

[12] Under slightly less restrictive assumptions on Ω, in particular, the number of divisible goods may be zero.

Additional conditions are required on the conditional preferences $\succ_a(f)$ in the economy E: For all $a \in A$ and for any $f \in L(A, \Omega)$,

(A.1) $\succ_a(f)$ is strictly monotone on R^D_+ (the divisible goods);

(A.2) For all $(x_I, x_D) \in \Omega = Z^I_+ \times R^D_+$, there is a $y_D \in R^D_+$ such that $(0_I, y_D) \succ_a(f)(x_I, x_D)$; and

(A.3) $e(a) \succ_a(f)(x_I, 0_D)$ for all $x_I \in Z^I_+$.

Assumption (A.1) is standard. Assumption (A.2) is simply that for any commodity bundle (x_I, x_D) there is a commodity bundle $(0_I, y_D)$, with divisible goods only, such that the second bundle is preferred—"enough" of the divisible goods is better than the given bundle. Our third assumption (A.3) is that the initial endowment is preferred to any commodity bundle with only indivisible goods.

THEOREM 1. *Under assumptions* (A.0), (A.1), (A.2), *and* (A.3), *the f-core C_f coincides with the set of Walrasian allocations W.*

Next we will consider the case of *no widespread externalities*; for any $a \in A$,

(A.0′) $\succ_a(f) = \succ_a(g)$ for all $f, g \in L(A, \Omega)$

In this case, we write simply $\succ_a$. The following standard definition of the A- (Aumann) core now applies. The *A-core* of the economy is the set of allocations

$$C_A = \{f \in L(A, \Omega): \int_A f \leqslant \int_A e \text{ and no subset of } A \text{ of positive measure can improve upon } f\}, \tag{2.9}$$

where a subset S of positive measure is said to be able to *improve upon f* iff for some $g \in L(S, \Omega)$, we have $\int_S g \leqslant \int_S e$ and

$$g(a) \succ_a f(a) \qquad \text{for all } a \text{ in } S.$$

THEOREM 2. *Under assumptions* (A.0′), (A.1), A.2), *and* (A.3), *we have* $C_f = W = C_A$.

Theorem 2 demonstrates the equivalence of the f-core, the Walrasian allocations, and the A-core without any widespread externalities.[13] On the other hand, Theorem 1 demonstrates the equivalence only of the f-core and

[13] For an exchange economy with indivisible goods but without externalities, Mas-Colell [14] gave a counterexample to the existence of equilibrium but with a nonempty Aumann core, implying the nonequivalence. Our assumption (A.3) excludes this example (see also Remark 2.4.1), and, together with (A.1) and (A.2), ensures the equivalence.

the Walrasian allocations with widespread externalities. The problem with widespread externalities is not the nonequivalence of the f-core and the A-core but the definition of the A-core, more precisely, of "can improve upon" with coalitions of positive measure. The problem in the definition is maintaining feasibility for the complementary agent set when a coalition of positive measure changes its own part of an allocation. To see the difficulty in defining "can improve upon" with widespread externalities, suppose that a coalition S of positive measure changes that part of an allocation under its control. Then the feasibility of the allocation received by the complementary agent set is violated almost necessarily. Therefore we need to consider the response of the complementary agent set to the change brought about by S. We cannot simply take their part of the allocation as fixed. In the case of widespread externalities, the effects of the behavior of S are not limited to the change in its own part of the allocation. Through the external effects, the coalition S incurs further changes in utilities through the violation in feasibility and resultant change in behavior of the complementary agent set. With no widespread externalities, S is unaffected by any outcome for the complementary agent set.

The definition of the f-core is free from these problems of effects of the complementary agent set. Essentially, since a finite coalition S is of measure zero a change in its behavior will not affect the feasibility of the allocation for the complementary agent set. For a more precise argument, recall that an f-core allocation f is the limit of a sequence of exactly feasible allocations, say $\{f^n\}$, with associated measurement-consistent partitions $\{p^n\}$. In other words, our feasibility concept is approximate. To discuss precisely the question of feasibility, we need to return to the sequences $\{f^n\}$ and $\{p^n\}$. In any exactly feasible allocation $f^n \in F(p^n)$, if the finite coalition S forms and changes its part of the allocation then feasibility is violated for the coalitions in p^n intersecting S. However, the number of these coalitions is finite and each contains only a finite number of agents, so feasibility is violated for only a finite number of players in total. Since these players constitute a set of measure zero, by assumption (A.0), the violation in feasibility and the resultant responses can be ignored.[14]

To define the A-core from the above argument we must make some hypotheses about the behavior of the complementary agent set in response to a change in the behavior of a coalition of positive measure. So far the two approaches most frequently used to define a core are the minimax criterion (more precisely, the α-core, β-core due to Aumann [3]) and the strong equilibrium (also due to Aumann [1]). For the reasons discussed above concerning the feasibility of a new allocation made by a coalition

[14] For another, more detailed explanation in the context of convergence of finite economies to the continuum, see Kaneko and Wooders [12].

and the complementary coalition, the strong equilibrium concept cannot be naturally applied. As is well known, the minimax criterion is not particularly persuasive except for two-person zero sum games.

For completeness, the minimax criterion is defined below for the case of preferences monotonically increasing on the externalities, i.e., for external economies. The definition according to the minimax criterion of "can improve upon" would be to assume that the complementary coalition disposes of all its private goods. An allocation $f \in F^*$ is in the A-core of E, denoted by C_A, if there is no subset S of A with positive measure and no allocation $g \in L(A, \Omega)$ where

$$\int_S g \leqslant \int_S e$$

$$g(a) = 0 \qquad \text{a.e. in } A \backslash S$$

and

$$[g(a), g] \succ_a [f(a), f] \qquad \text{for all} \quad a \in S.$$

We observe that an allocation in the A-core is Pareto-efficient whereas those in the f-core (and Walrasian allocations) are not necessarily so. Thus, the A-core and the Walrasian allocations do not coincide except in very special situations including, of course, the no-widespread-externalities case. A counterexample to equivalence is given in Kaneko and Wooders [11]. Even under our simple monotonicity assumptions on the externalities, the minimax criterion does not lead to a natural definition of the A-core in the sense that we cannot justify the resulting assumption that a complementary coalition disposes of all its private goods.

With no widespread externalities, our results are suggestive of those due to Schmeidler [16] and Mas-Colell [14]. Schmeidler considers a continuum economy where only coalitions of small positive measure are permitted, while Mas-Colell uses the framework of a finite economy where only coalitions bounded in size (membership) are permitted. Schmeidler obtains Aumann core–equilibrium equivalence, while Mas-Colell obtains approximate equivalence. These papers are similar to ours in that core–equilibrium equivalence is obtained when only "small" coalitions are allowed. However, neither author restricts allocations arising from recontracting to be achievable by cooperation only within small coalitions. Also neither author considers an actual "limit model." Although Mas-Colell does not impose the feasibility for small coalitions constraint, his model is quite close to a large finite analogue of ours.[15]

[15] For a more detailed discussion of the finite analogue and the convergence of finite economies with small coalitions to continuum economies with finite coalitions in an exchange economy with widespread externalities, see Kaneko and Wooders [12], where convergence of solutions *and* game structure is shown.

2.4 *Remarks*

Remark 2.4.1. Our assumptions are similar to those of Khan and Yamazaki [9] and Yamazaki [18]. (In fact, if the cardinality $|D|$ of D, the set of divisible goods, is equal to one, our assumptions imply those of Khan and Yamazaki. If the cardinality of D is possibly greater than one, Yamazaki [18] uses a stronger assumption than our (A.2) and also uses other assumptions for the existence of Walrasian equilibrium.) Khan and Yamazaki show the equivalence of the A-core to the set of "weakly competitive allocations," and the non-emptiness of the A-core; they employ an example due to Mas-Colell [13] to illustrate that the set of Walrasian allocations may be empty. Therefore, with their assumptions the A-core and the set of Walrasian equilibrium allocations might not coincide. Our assumption (A.3) rules out Mas-Colell's example.

Without widespread externalities, in fact, we can prove, under assumptions (A.0), (A.1), (A.2), and (A.3), that

$$C_f = W = C_A \neq \varnothing$$

by modifying our equivalence in Theorem 2 and using Khan and Yamazaki's [9] existence of weakly competitive allocations. The modifications will be indicated in Section 3, where the proofs of Theorems 1 and 2 are provided.

Remark 2.4.2. With widespread externalities, the existence of Walrasian equilibrium and the f-core can be obtained by imposing only a slight restriction on the externalities. Specifically, we assume that there is a given finite family of subsets of positive measure of A, say $T_1, ..., T_n$, such that the externalities depend on an allocation only via the value of its integral over each of the subsets.[16] That is, we assume

(A.0*) $[x, f] \succ_a [y, g] \Leftrightarrow [x, \int_{T_1} f, ..., \int_{T_n} f] \succ_a [y, \int_{T_1} g, ..., \int_{T_n} g]$
for all $[x, f], [y, g] \in \Omega \times L(A, \Omega)$.

Assume also that preferences are open in $(\Omega \times R_+^{|M|n}) \times (\Omega \times R_+^{|M|n})$. Then essentially the same proof (as the one herein) of the existence of equilibrium can be applied by using Liapunov's theorem to convexify over the externalities.

Remark 2.4.3. Although we have shown that without widespread externalities the Aumann core coincides with the f-core, this does not mean that these two *concepts* are equivalent nor that Aumann's theorem [2] is equivalent in interpretation to ours. The Aumann core permits only coalitions containing positive *proportions* of the total agent set to improve upon

[16] This enables us to work in a finite dimensional "commodity" space.

allocations; thus individual agents and finite coalitions are unable to have any affect on the outcome of cooperation. *Both* the Aumann core and the competitive equilibrium build in completely negligible agents. In contrast, in our approach individual agents are nonnegligible relative to themselves and to any finite group. We obtain, as a conclusion, that individual agents and finite coalitions can do no better than taking prices as given. Even though finite coalitions are negligible relative to the economy, we have f-core–Walrasian equilibrium equivalence.

Remark 2.4.4. A different type of consideration rather than those discussed in the introduction was the original source of one author's interest in finite coalitions. This concerns the power of finite coalitions to manipulate resource allocation mechanisms in continuum economies. When one considers manipulation the size of permissible coalitions—finite, or of positive measure—is critical. See Hammond [7] and also Hammond [6], where another core concept, which "combines" finite coalitions and coalitions of positive measure, is introduced.

3. Proofs

In this section, we prove $C_f \subset W$ in the case of *no* widespread externalities. The proof that $C_A \subset W$ is a simple modification of the proof of $C_f \subset W$. The proofs that $C_f \supset W$ and that $C_A \supset W$ are straightforward and standard, so we omit them. Thus we will have $C_f = W = C_A$ in the case of no widespread externalities.

In the case of widespread externalities, we can obtain $C_f = W$ by applying the equivalence result $C_f = W$ in the case of no widespread externalities as follows. Let g be a function in $L(A, \Omega)$ and let E be an economy with widespread externalities. Then we have an economy E^g: $a \to (\succ_a(g), e(a))$, without widespread externalities, induced from E by fixing g in each preference relation. Then the above equivalence result without widespread externalities states that $C_f^g = W^g$, where C_f^g and W^g are respectively the f-core and the set of Walrasian allocations of the economy E^g. The conditions for g to belong to C_f and W of the economy E can be written as

$$g \in C_f^g \qquad \text{if and only if} \quad g \in C_f;$$

and

$$g \in W^g \qquad \text{if and only if} \quad g \in W.$$

Since $C_f^g = W^g$ for all g, these conditions are identical and so $C_f = W$.

In the following analysis of an economy without widespread externalities, individual preferences are regarded as a subset of $\Omega \times \Omega$ instead of $(\Omega \times L(A, \Omega))^2$.

Proof that $C_f \subset W$

Let f be an allocation in the f-core. Define the set $\psi(a)$ for each agent $a \in A$ by

$$\psi(a) = \{x \in Z^I \times R^D : x + e(a) \succ_a f(a)\} \cup \{0\}$$

(which is equivalent to Hildenbrand's definition of $\psi(a)$ in [8, Theorem 1, p. 133]). Then the following lemma holds.

LEMMA 1. *There is no* $w \in \int \psi$ *such that* $w \ll 0$.

Since $\int \psi$ is convex by Liapunov's theorem, from this lemma it follows that the set $\int \psi$ can be separated from the strictly negative orthant of R^M by a hyperplane. That is, there is a price vector p in R^M, $p > 0$, such that

$$0 \leqslant p \cdot w \qquad \text{for all} \quad w \in \int \psi \tag{3.1}$$

Proof of Lemma 1. Suppose there exists a vector $w \in \int \psi$ such that $w \ll 0$. Then there exists a measurable function $t: A \to Z^I \times R^D$ such that $t(a) \in \psi(a)$ a.e. in A and $\int_A t = w \ll 0$.

Define $S = \{a \in A: t(a) + e(a) \succ_a f(a)\}$. Then $\mu(S) > 0$ and $x = \int_S t \ll 0$ and the following claim holds.

Claim. There is a finite partition $(S_0, S_1, ..., S_m)$ of S and a simple function $\bar{t}: S \to Z^I \times R^D$ such that

$$\mu(S_1) = \mu(S_2) = \cdots = \mu(S_m) > 0; \tag{3.2}$$

$$\bar{t}(a) = \bar{t}(a') \qquad \text{if} \quad a, a' \in S_j (j = 1, ..., m); \tag{3.3}$$

$$\bar{t}(a) = 0 \qquad \text{for all} \quad a \in S_0; \tag{3.4}$$

$$\bar{t}_D(a) \gg t_D(a) \qquad \text{and} \qquad \bar{t}_I(a) = t_I(a) \qquad \text{for all} \quad a \in \bigcup_{j=1}^{m} S_j;$$

$$\int_S \bar{t}(a) \ll 0. \tag{3.5}$$

Proof of Claim. We begin by partitioning a set consisting of "most" agents into subsets so that if a and a' are in the same subset, then $t(a)$ is approximately equal to $t(a')$. For any positive integer n, define

$$Z_n = \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1\}$$

$$Z_n^I = \overbrace{Z_n \times \cdots \times Z_n}^{|I|};$$

$$K_n = \left\{-n, -n+\frac{1}{2^n}, \ldots, -\frac{1}{2^n}, 0, \frac{1}{2^n}, \ldots, n-\frac{1}{2^{n-1}}, n-\frac{1}{2^n}\right\};$$

$$K_n^D = \overbrace{K_n \times \cdots \times K_n}^{|D|}$$

Define a simple function $t^n = (t_I^n, t_D^n)\colon S \to Z^I \times R^D$ by

$$t_I^n(a) = \begin{cases} t_I(a) & \text{if } -1_M \leqslant t(a) \ll n1_M \\ 0_I & \text{otherwise,} \end{cases} \tag{3.6}$$

$$t_D^n(a) = \begin{cases} k_D + \frac{1}{2^n} 1_D & \text{if } \quad -n1_I \leqslant t_I(a) \ll n1_I \text{ and} \\ & t_D(a) \in \prod_{d \in D} \left[k_d, k_d + \frac{1}{2^n}\right) \text{ for some } k_D \in K_n^D \\ 0_D & \text{otherwise.} \end{cases} \tag{3.7}$$

Then, for each $a \in A$, there is an $\bar{n}$ such $-\bar{n}1_M \leqslant t(a) \ll \bar{n}1_M$. Therefore the sequence $\{t^n(a)\}$ is nonincreasing for all $n \geqslant \bar{n}$ and converges to $t(a)$. This implies that the sequence $\{\int_S t^n\}$ converges to $\int_S t \ll 0$. Therefore there is an integer n_0 such that

$$\int_S t^{n_0} \ll 0. \tag{3.8}$$

For $(z_I, k_D) \in Z_{n_0}^I \times K_{n_0}^D$, denote the set $\{a \in A\colon t_I(a) = z_I \text{ and } k_D \leqslant t_D(a) \ll k_D + (1/2^{n_0}) I_D\}$ by $T(z_I, k_D)$. Then it follows from (3.6), (3.7), and (3.8) that

$$\sum_{(z_I, k_D) \in Z_{n_0}^I \times K_{n_0}^D} \left(z_I, k_D + \frac{1}{2^{n_0}} 1_D\right) \mu(T(z_I, k_D)) = \int_S t^{n_0} \ll 0. \tag{3.9}$$

From each $T(z_I, k_D)$ $((z_I, k_D) \in Z_{n_0}^I \times K_{n_0}^D)$, we can choose a subset $S(z_I, k_D)$ so that

$$\mu(S(z_I, k_D)) \text{ is a rational number;} \tag{3.10}$$

and

$$\sum_{(z_I, k_D) \in Z_{n_0}^I \times K_{n_0}^D} \left(z_I, k_D + \frac{1}{2^{n_0}} 1_D\right) \mu(S(z_I, k_D)) \ll 0. \tag{3.11}$$

Since each $\mu(S(z_I, k_D))$ is a rational number (possibly zero), we can find a subpartition $(S_1, ..., S_m)$ of the finite partition $\{S(z_I, k_D)\colon (z_I, k_D) \in Z^I_{n_0} \times K^D_{n_0}$ and $\mu(S(z_I, k_D)) > 0\}$ such that $\mu(S_1) = \cdots = \mu(S_m) > 0$. Define S_0 and $\bar{t}$ by

$$S_0 = S - \bigcup_{j=1}^{m} S_j;$$

$$\bar{t}(a) = \begin{cases} t^{n_0}(a) & \text{if } a \in \bigcup_{j=1}^{m} S_j \\ 0 & \text{otherwise.} \end{cases}$$

It immediately follows that $(S_0, S_1, ..., S_m)$ and $\bar{t}$ satisfy conditions (3.2), (3.3), and (3.4). From (3.11) we have

$$\int_S \bar{t} = \sum_{(z_I, k_D) \in Z^I_{n_0} \times K^D_{n_0}} \left(z_I, k_D + \frac{1}{2^{n_0}} 1_D\right) \mu(S(z_I, k_D)) \ll 0.$$

This completes the proof of the claim.

Now select one agent $a_j \in S_j$ for each $j = 1, ..., m$, and write $C = \{a_1, ..., a_m\}$.[17] Since $\bar{t}_D(a_j) \gg t_D(a_j)$ and $\bar{t}_I(a_j) = t_I(a_j)$ for all $j = 1, ..., m$ by (3.4), we have, by Assumption (A.1),

$$\bar{t}(a_j) + e(a_j) \succ_{a_j} t(a_j) + e(a_j) \succ_{a_j} f(a_j) \qquad \text{for all} \quad j = 1, ..., m.$$

The feasibility of the allocation $(\bar{t}(a_j) + e(a_j))_{a_j \in C}$ follows from conditions (3.2) and (3.5); indeed

$$\begin{aligned} \sum_{a_j \in C} (\bar{t}(a_j) + e(a_j)) &= \frac{1}{\mu(S_1)} \sum_{j=1}^{m} \bar{t}(a_j)\, \mu(S_j) + \sum_{a_j \in C} e(a_j) \\ &= \frac{1}{\mu(S_1)} \int_S \bar{t} + \sum_{a_j \in C} e(a_j) \ll \sum_{a_j \in C} e(a_j). \end{aligned}$$

Therefore the finite coalition C can improve upon f. For any full subset $\tilde{A}$ of A we can also find a finite coalition C in $\tilde{A}$ so that C consists of only one agent in S_j for each $j = 1, ..., m$. Then, C can improve upon f. This is a contradiction to the supposition that f is in C_f. This completes the proof of Lemma 1. ∎

[17] To modify the proof of this theorem to obtain $C_A \subset W$, replace C by $\bigcup_{j=1}^{m} S_j$.

Recall that from Lemma 1 it follows that there is a price vector $p \in R^M$ ($p > 0$) satisfying (3.1). We will now show that p is an equilibrium price vector.

It follows from (3.1) and Hildenbrand [8, p. 63, Proposition 6] that

$$\inf_{w \in \int \psi} p \cdot w = \int_{x \in \psi(\cdot)} \inf p \cdot x.$$

It also follows from (3.1) that $0 \leqslant \int_A \inf p \cdot \psi$. Since $\psi(a)$ contains 0, we have $\inf p \cdot \psi(a) \leqslant 0$. Hence we have $\int_A \inf p \cdot \psi = 0$, which implies that $\inf p \cdot \psi(a) = 0$ a.e. in A. Then this implies that a.e. in A,

$$x \succ_a f(a) \Rightarrow p \cdot e(a) \leqslant p \cdot x, \tag{3.12}$$

or equivalently,

$$p \cdot e(a) > p \cdot x \Rightarrow x \not\succ_a f(a).$$

LEMMA 2. $p \cdot e(a) = p \cdot f(a)$ a.e. *in* A *(budget constraints are satisfied).*

Proof. For all $a \in A$, by Assumption (A.1) (strict monotonicity of preferences for divisible goods) we can choose a sequence $\{y^n\}$ such that $\{y^n\}$ converges to $f(a)$ and $y^n \succ_a f(a)$ for all n. Then we have, from (3.12), $p \cdot e(a) \leqslant p \cdot y^n$ for all n a.e. in A. This implies $p \cdot e(a) \leqslant p \cdot f(a)$ a.e. in A. If $p \cdot e(a) < p \cdot f(a)$ for a set of agents with positive measure, then we obtain $p \cdot \int_A e < p \cdot \int_A f$. Since f is in the f-core, $\int_A e \geqslant \int_A f$ by (2.13) and we have a contradiction. ∎

Remark 3.3.1. We have now shown that f is a "weak Walrasian equilibrium allocation," i.e., it satisfies (3.12) and budget constraints. In the remainder of our arguments, we show that f is a Walrasian equilibrium allocation without using directly the fact that f is in the f-core. Therefore our arguments also show that a weak equilibrium allocation is a Walrasian allocation. Consequently, Khan and Yamazaki's proof of the existence of the weak equilibrium [9, pp. 223–224, Proof of Proposition 2] proves the existence of Walrasian equilibrium in our framework.

LEMMA 3. $p_D \gg 0_D$.

Proof. First we prove that $p_D > 0_D$. Suppose $p_D = 0_D$. Since $\int_A e \gg 0$ and $p > 0$ there is a subset S of A such that $p \cdot e(a) > 0$ for all $a \in S$ and $\mu(S) > 0$. Choose an agent a from S for whom (3.12) holds and $p \cdot f(a) = p \cdot e(a) > 0$. By Assumption (A.2), there is a vector $y = (0_I, y_D) \in \Omega$ such that $y \succ_a f(a)$. Since $p_D = 0_D$, we have $0 = p \cdot y < p \cdot f(a) = p \cdot e(a)$, which contradicts (3.12). This shows that $p_D > 0_D$.

Now let us show that $p_D \gg 0_D$. Since $\int_A e \geqslant \int_A f$ by (2.13) and $p \cdot \int_A e = p \cdot \int_A f$ by Lemma 2, we have, for all $k \in I \cup D$, the usual "rule of free goods,"

$$p_k > 0 \Rightarrow \int_A e_k = \int_A f_k. \tag{3.13}$$

Since $p_D > 0_D$ and $\int_A e_D \gg 0_D$, it follows from (3.13) that $p_D \cdot \int_A f_D = p_D \cdot \int_A e_D > 0$. Therefore there is a subset S of A with positive measure such that $p_D \cdot e_D(a) = p_D \cdot f_D(a) > 0$ and (3.12) holds for all $a \in S$.

Consider an arbitrary agent a in S. We shall prove that

$$x \succ_a f(a) \quad \text{and} \quad x_I = f_I(a) \Rightarrow p \cdot x > p \cdot f(a). \tag{3.14}$$

This implies $p_D \gg 0_D$; indeed, if $p_d = 0$ for some $d \in D$, then x, given by $x_d = f_d(a) + 1$ and $x_k = f_k(a)$ for all $k \neq d$, violates (3.14) because of Assumption (A.1). Therefore let us now prove (3.14). Suppose on the contrary that $x \succ_a f(a)$, $x_I = f_I(a)$, and $p \cdot x \leqslant p \cdot f(a)$ for some x. Because $\succ_a$ is continuous, there exists $\lambda \in (0, 1)$ near 0 so that $\lambda(x_I, 0) + (1 - \lambda)x \succ_a f(a)$ and $p \cdot [\lambda(x_I, 0) + (1 - \lambda)x] = \lambda p_I \cdot x_I + (1 - \lambda) p \cdot x \leqslant \lambda p \cdot f_I(a) + (1 - \lambda) p \cdot f(a) = p \cdot f(a) - \lambda p_D \cdot f_D(a) < p \cdot f(a)$ where the last inequality follows because $a \in S$, as defined above. This is a contradiction to (3.12). So $p_D \gg 0_D$. ∎

LEMMA 4. *A.e. in A,* $p \cdot x \leqslant p \cdot e(a) \Rightarrow x \not\succ_a f(a)$ *(utility maximization on the budget sets).*

Proof. It follows from Assumption (A.3) that $e_D(a) > 0_D$ for all $a \in A$. This together with Lemma 3 implies that $p \cdot e(a) > 0$ for all $a \in A$.

Consider an arbitrary agent a for whom (3.12) ($p \cdot x < p \cdot e(a) \Rightarrow x \not\succ_a f(a)$) holds. It suffices to show that, for these agents, $p \cdot x = p \cdot e(a) \Rightarrow x \not\succ_a f(a)$. Suppose that $p \cdot x = p \cdot e(a)$, $x_D > 0_D$, and $x \succ_a f(a)$ for some x. By continuity of $\succ_a$ and Lemma 3 we can find an x' such that $p \cdot x' < p \cdot e(a)$ and $x' \succ_a f(a)$, which is a contradiction. Finally suppose that $p \cdot x = p \cdot e(a)$, $x_D = 0_D$, and $x \succ_a f(a)$ for some x. Then Assumption (A.3) implies that $e(a) \succ_a x \succ_a f(a)$, so $e(a) \succ_a f(a)$ by transitivity of $\succ_a$. (This contradicts our assumption that f is in the f-core, so we have shown Lemma 4. So that we can apply our proofs to the existence of a Walrasian equilibrium we do not use this argument.) Since $e_D(a) > 0_D$, by continuity of $\succ_a$ we can find an x' such that

$$x'_D < e_D(a), x'_I = e_I(a) \qquad \text{and} \qquad x' \succ_a f(a).$$

Since $p_D \gg 0_D$ by Lemma 3, we have $p \cdot x' < p \cdot e(a)$, which contradicts (3.12). ∎

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