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Equivalence of Lindahl equilibrium with participation prices and the core*

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Summary. In a model of an economy with multiple public goods and differentiated crowding, it is shown that asymptotically the core has the equal treatment property and coincides with the equilibrium outcomes. It follows that all individuals of the same type in the same jurisdiction must pay the same Lindahl taxes and, with strict convexity of preferences, the same Lindahl prices. With only one private good, for sufficiently large economies we show (a) the equivalence of the core and the set of equilibrium outcomes and (b) the nonemptiness of approximate cores and their equivalence to the set of approximate equilibrium outcomes.

1. Introduction

Perhaps the most intriguing idea in the theory of local public finance has been the Tiebout Hypothesis – the idea that when public goods are subject to crowding or congestion and are provided by multiple jurisdictions, then equilibrium outcomes are "market-like". The current paper contributes to a literature characterizing cores and equilibrium outcomes of economies with crowded public goods and demonstrating that, when small groups are effective – that is, when all or almost all gains to collective activities can be realized by groups bounded in size or by relatively small groups – the core converges to the set of equilibrium outcomes.

The possibility that crowding or congestion may ensure the equivalence of the core and equilibrium outcomes with public goods was hypothesized by Roberts (1974). The study of core convergence and the equivalence of the core to price-taking equilibrium outcomes in economies with congested public goods was initiated in Wooders (1978, 1980). In these papers, preferences and production possibilities depend on the number of consumers with whom the public goods are jointly consumed and produced and not on the characteristics of these consumers – crowding is *nondifferentiated*. There have been a number of further contributions to

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this literature, and we refer the reader to Ellickson (1973) for an illuminating discussion of the difficulties caused by crowding in public goods economies and to Barham and Wooders (1994) for a recent survey.

In the current paper, we return to the study of the core and equilibria in models with *differentiated crowding*, where preferences and/or production possibilities depend not only on the numbers of participants consuming the goods but also on the characteristics of those participants. In a model of comparable generality to that of Debreu and Scarf (1963) but with local public goods, introduced in Wooders (1981, 1989)[1], we show that the core converges to the set of Lindahl equilibrium outcomes. A very closely related convergence theorem first appeared in Wooders (1981) and in fact most of the proof our current result is from that paper. The equilibrium has Lindahl prices for the public goods provided by jurisdictions and participation prices – fees required for membership within a jurisdiction. We also show that asymptotically (a) the core and equilibrium have the equal-treatment property, (b) all individuals in the same jurisdiction of the same type pay the same Lindahl taxes and (c) with strict convexity of preferences, all individuals of the same type in the same jurisdiction must pay the same Lindahl prices. The equality of Lindahl taxes for all individuals of the same type means that asymptotically the identities of the individual participants are irrelevant to their Lindahl taxes – in equilibrium, Lindahl taxes depend on the type of the participant and not on his name.

In the one-private-good case, we are able to provide further results. For all sufficiently large (but finite) economies (a) the core coincides with the set of Lindahl equilibrium outcomes, (b) all states of the economy in the core, and thus all equilibrium states, have the equal treatment property and (c) with strict convexity of preferences, all participants of the same type in the same jurisdiction pay the same Lindahl prices. Moreover, for all sufficiently large economies, the set of certain approximate equilibrium outcomes coincides with an approximate core.[2]

In this paper we bound admissible jurisdiction sizes. This feature, that small groups of participants can realize all gains to sharing public goods, makes our model (and previous versions of the model where small groups could realize *almost* all gains to collective activities) stand in stark contrast to extant works on the equivalence of the core and valuation equilibrium outcomes. See, for example, Mas-Colell and Silvestre (1989), and Weber and Wiesmeth (1991). In those papers, the equivalence of the core and the equilibrium outcomes holds even when there are further gains to replicating the economy. Moreover, the equal treatment property of the core does not hold – if two participants are identical it does not necessarily hold that they obtain the same utility levels in equilibrium. The equal treatment property of our model rests on the feature that small groups are effective.[3] Such a condition is not imposed on the economic models of the above referenced authors. The current paper

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[1] See also Berglas (1976), Scotchmer and Wooders (1986), Scotchmer (1994), and Wooders (1988, 1993).

[2] The approximate core notion used is from Shubik and Wooders (1983), where a small percentage of participants can be ignored. Our approximate Lindahl equilibrium uses a corresponding notion of approximation.

[3] This also holds for the equal-treatment property of large games generally – see Wooders (1983, Theorem 3)) or Wooders (1994).

suggests that the equal treatment property, ruling out outcomes that discriminate between identical participants, is a key aspect of a competitive economy and also of the competitive nature of economies with congestable public goods.

2. Economies with public goods

Notation. The following notation and terminology will be used: $\mathbf{R}^n_+$ – the non-negative orthant of Euclidean n-dimensional space $\mathbf{R}^n$; $\mathbf{R}^n_{++}$ – the positive orthant of Euclidean n-dimensional space $\mathbf{R}^n$; I^T – the T-fold Cartesian product of the non-negative integers; $|S|$–the cardinal number of a set S; $\|x\|$ – the sup norm of $x \in \mathbf{R}^n$, $\|x\| = \max_{i=1}^{x} |x_i|$ where $|x_i|$ denotes the absolute value.

Participants. Let $m = (m_1, \ldots, m_T)$ be an arbitrary given vector of integers. For each positive integer r *the set of participants of the r^{th} replica economy* is denoted by $N_r = \{(t, q): t = 1, \ldots, T \text{ and } q = 1, \ldots, rm_t\}$. The pair (t, q) denotes the *qth participant of type t*. All participants of the same type are identical in terms of their endowments and preferences but it is not required that participants of different types are different.

Given N_r and $t \in \{1, \ldots, T\}$, let $[t]_r$ denote the set of participants of type t in the r^{th} replica economy. Let S be a nonempty subset of N_r, called a *jurisdiction*. A jurisdiction describes a group of participants who collectively consume the same public goods. Let s be a vector in I^T with t^{th} component given by $s_t = |S \cap [t]_r|$. The vector s, called the *profile* of S, lists the numbers of participants of each type in S. Let $I[t]$ be the subset of elements of I^T whose t^{th} component is nonzero; the set $I[t]$ consists of all profiles of all jurisdictions containing participants of type t.

Commodities. The economy has L private goods and M public goods. A vector of the public goods is denoted by $x = (x_1, \ldots, x_m, \ldots, x_M) \in \mathbf{R}^M_+$ and a vector of private goods by $y = (y_1, \ldots, y_\ell, \ldots, y_L) \in \mathbf{R}^L_+$.

Endowments and preferences. It is assumed that each participant has a positive endowment of each private good and that there are no endowments of public goods. Let $w^{tq} \in \mathbf{R}^L_+$ be the *endowment of the $(t, q)^{th}$ participant of private goods.* We assume that $w^{tq} = w^{t'q'}$ whenever $t = t'$; participants of the same type have the same endowments.

The preferences of a participant of type t, say (t, q), are described by a complete preordering $\succsim_{tq}$ on $X^t \times I[t]$, where X^t is a subset of $\mathbf{R}^M_+ \times \mathbf{R}^L$. The set X^t is called the *(commodities) consumption set* (for participants of type t) and a pair (x, y) in X^t is called a *consumption*. The dependence of preferences on profiles of jurisdictions reflects the feature that *crowding is differentiated in consumption* – participants are affected by the composition of the group with whom the public goods are shared.[4] The set $X^t \times I[t]$ is called the *total consumption set* (for participants of type t).

It is assumed that for each profile s in $I[t]$ the preference ordering has the usual properties of monotonicity, continuity, convexity, and interiority of the endowment

[4] Our terminology is motivated by the differentiated crowding literature, cf., Mas-Colell (1975). Differentiated crowding has also been called "nonanonymous" but this may suggest that identities or names of participants are relevant. For a model with nonanonymous crowding see, for example, Manning (1992). For further discussion of models with crowding, see Conley and Wooders (1994).

and that all participants of the same type have the same preferences. In addition, for our equal-treatment results it is required that there is at least one private good which is essential for survival.[5] For our convergence Theorem, we also require an assumption ensuring substitutability of commodities for changes in jurisdiction. This assumption allows us, in the proof, to define a finite worth (or cost) to a consumer of a change in jurisdiction. Specifically, for any profile s in $I[t]$, preferences satisfy:

Monotonicity: Given any (x, y) in X^t, if (x', y') is in $\mathbf{R}^{M+L}$ and $(x, y) < (x', y')$ then (x', y') is in X^t and $(x, y; s) \prec_{tq} (x', y'; s)$.

Continuity: For any (x', y') in X^t the sets $\{(x, y):(x, y; s) \preceq_{tq} (x', y'; s)\}$ and $\{(x, y):(x, y; s) \succeq_{tq} (x', y'; s)\}$ are closed.

Convexity: Let (x', y') and (x^*, y^*) be two consumptions for (t, q) where $(x', y'; s) \succeq_{tq} (x^*, y^*; s)$. Let λ be an arbitrary real number in the open interval $(0, 1)$ and let $(x, y) = \lambda(x', y') + (1 - \lambda)(x^*, y^*)$. Then $(x, y; s) \succeq_{tq} (x^*, y^*; s)$.

Interiority: The endowment $(0, w^{tq}) \in \mathbf{R}^{M+L}$ is in X^t and w^{tq} is in the interior of the projection of X^t onto the private goods consumption set (the projection of X^t onto its last L components).

Essentiality: There is at least one private good ℓ^* with the property that for all participants $(t, q) \in N_r$ it holds that $w^{tq}_{\ell^*} > 0$ and the projection of X^t on the coordinate space of the ℓ^* commodity is R_{++}.

Substitution: For any (x, y) in X^t and any pair of profiles s and s' in $I[t]$ there is a consumption (x', y') in X^t satisfying the property that $(x, y; s) \sim_{tq} (x', y'; s')$.

Admissible jurisdictions and jurisdiction structures. We will assume that there is a bound B such that admissible jurisdictions contain no more than B members and, for each type t, the number m_t of players of type t in N_1 is greater than B. This assumption is a strong form of small group effectiveness – all gains to collective activities can be realized by groups bounded in absolute size – and, when there is only one private good, enables us to obtain equivalence results for finite economies.[6]

Let S be a nonempty subset of N_r. A *jurisdiction structure of* S is a partition of S, denoted by $\mathscr{S} = \{S_1, \ldots, S_k, \ldots, S_K\}$ where $|S_k| < B$ for each $S_k \in \mathscr{S}$. A jurisdiction structure of N_r, called simply a *jurisdiction structure*, is denoted by $\mathscr{N}_r = \{J_1, \ldots, J_g, \ldots, J_G\}$.

Given a jurisdiction structure $\mathscr{S}$ of S and S_k in $\mathscr{S}$, let (x, y) be a commodity bundle. Define $(x, y; \mathscr{S}) = (x, y; S_k)$.

Allocations. An allocation of commodities to consumers is required to satisfy the properties that all consumers in the same jurisdiction are allocated the same amounts of the public goods. Let S be a nonempty subset of N_r and let $\mathscr{S}$ be a jurisdiction structure of S. An *allocation for S relative to* $\mathscr{S}$, or simply *an*

[5] This ensures that participants who may be "left out" of desirable jurisdictions containing members of the same type have the means to attempt to buy their way in, and, with bounded effective coalition sizes, also ensures the equal treatment property of the core.

[6] See Wooders (1993) for a version of the model where a less restrictive assumption of boundedness of per capita payoffs is used. See Conley and Wooders (1994) for discussion of small group effectiveness in the context of economies with local public goods.

allocation for $\mathscr{S}$, is a consumption for each consumer in S, $(x^{\mathscr{S}}, y^{\mathscr{S}})$, where $x^{\mathscr{S}} = \{x^{tq} \in \mathbf{R}_+^M : (t,q) \in S\}$ and $y^{\mathscr{S}} = \{y^{tq} \in \mathbf{R}_+^L : (t,q) \in S\}$, and for each S_k in $\mathscr{S}$, $x^{tq} = x^{t'q'}$ for all (t,q), (t',q') in S_k.

Given (a) two jurisdictions S and S', (b) a participant (t,q) in both S and S', and (c) consumptions (x,y) and (x',y'), in X^t, we write $(x,y;S) \succsim_{tq} (x',y';S')$ if and only if $(x,y;s) \succsim_{tq} (x',y';s')$ where s is the profile of S and s' is the profile of S'. Given two subsets of participants S and S' and jurisdiction structures $\mathscr{S}$ and $\mathscr{S}'$ of these subsets, (b) and (c), we write $(x,y;\mathscr{S}) \succsim_{tq} (x',y';\mathscr{S}')$ if and only if $(x,y;S) \succsim_{tq} (x',y';S')$.

Production. The production possibility set available to a jurisdiction depends on the profile of that jurisdiction – *crowding is differentiated in production.* A correspondence Y_0 from the set of profiles I^T to $\mathbf{R}^M \times -\mathbf{R}_+^L$, called the *public goods production correspondence*, specifies the production set available to each jurisdiction. For each profile s, an element of $Y_0[s]$ is denoted by (x,z), where $x \in \mathbf{R}_+^M$ denotes *outputs of public goods* and $z \in -\mathbf{R}_+^L$ denotes *inputs of private goods.* It is assumed that $Y_0[s]$ is a closed convex cone with vertex the origin. For each nonempty subset S of N_r with profile s define $Y_0[S] = Y_0[s]$.

The *production possibility set for private goods* is denoted by Y_1 and an element of Y_1 is denoted by $z \in \mathbf{R}^L$. We assume that $Y_1 \subset \mathbf{R}^L$ is a closed, convex cone with vertex zero satisfying the usual conditions of irreversibility ($Y_1 \cap (-Y_1) = \{0\}$), no free production ($Y_1 \cap \mathbf{R}_+^L = \{0\}$), and free disposal ($Y_1 \supset (-\mathbf{R}_+^L)$).

Let S be a nonempty subset of N_r and let $\mathscr{S} = \{S_1, \ldots, S_k\}$ denote a jurisdiction structure of S. A production plan for S relative to $\mathscr{S}$ is denoted by $(\{(x_k, z_k) : k = 1, \ldots, K\}, z)$ where $(x_k, z_k) \in Y_0[S_k]$. The set $\{(x_k, z_k) : k = 1, \ldots, K\}$ is called *a production plan for public goods* and the vector $z \in \mathbf{R}^L$ is called *a production plan for private goods.*

States of the economy. Let S be a nonempty subset of N and let $\mathscr{S} = \{S_1, \ldots, S_k, \ldots, S_K\}$ be a jurisdiction structure of S. A *state of the economy for S relative to* $\mathscr{S}$ is an ordered pair $\Psi(\mathscr{S}) = ((x^{\mathscr{S}}, y^{\mathscr{S}}), (\{(x_k, z_k) : k = 1, \ldots, K\}, z))$ where $(x^{\mathscr{S}}, y^{\mathscr{S}})$ is an allocation for S relative to $\mathscr{S}$ and $(\{(x_k, z_k) : k = 1, \ldots, K\}, z)$ is a production plan for S relative to $\mathscr{S}$ such that for each $S_k \in \mathscr{S}$ and each $(t,q) \in S_k$ it holds that $x^{tq} = x_k$; a state of the economy for S relative to $\mathscr{S}$ satisfies the condition that the consumption of public goods of each consumer in each jurisdiction equals the specified production of public goods for that jurisdiction.

A state of the economy $\Psi(\mathscr{S}) = ((x^{\mathscr{S}}, y^{\mathscr{S}}), (\{(x_k, z_k) : k = 1, \ldots, K\}, z))$ for S relative to $\mathscr{S}$ is *feasible* if $\sum_{tq \in S} (y^{tq} - w^{tq}) \le z + \sum_{S_k \in \mathscr{S}} z_k$. The state of the economy has the *equal treatment property* if, for each type t, and for all (t,q) and (t,q') in S it holds that $(x^{tq}, y^{tq}, \mathscr{S}) \sim_{tq} (x^{tq'}, y^{tq'}, \mathscr{S})$.

3. The Lindahl equilibrium and the core

Let $\Psi(\mathscr{N}_r) = ((x^{\mathscr{N}_r}, y^{\mathscr{N}_r}), (\{(x_g, z_g) : g = 1, \ldots, G\}, z))$ be a state of the economy relative to the jurisdiction structure $\mathscr{N}_r = \{J_1, \ldots, J_G\}$ of N_r. A coalition S can *improve upon* the state $\Psi(\mathscr{N}_r)$ if there is a jurisdiction structure $\mathscr{S} = \{S_1, \ldots, S_K\}$ of S and a feasible state of the economy $\Psi(\mathscr{S}) = ((x'^{\mathscr{S}}, y'^{\mathscr{S}}), (\{(x'_k, z'_k) : k = 1, \ldots, K\}, z'))$ such that for all consumers $(t,q)\in S$ it holds that $(x'^{tq}, y'^{yq}; \mathscr{S}) \succ_{tq} (x^{tq}, y^{tq}; \mathscr{N}_r)$. A state of the economy $\Psi(\mathscr{N}_r)$ is in the *core of the r^{th} economy* if it cannot be improved upon by any coalition S.[7] We say that the *core has the equal treatment property* if all states of the economy in the core have the equal treatment property.

A *price system for private goods* is a vector $p\in\mathbf{R}^L_+$. A *complete Lindahl price system* is a set $\Gamma = \{\gamma^{tq}(S)\in\mathbf{R}^M_+ : S\subset N_r \text{ and } (t,q)\in S\subset N_r\}$; a complete Lindahl price system states a price for each public good for each consumer in each possible jurisdiction.[8] A *participation price system* is a set $\Pi = \{\pi^{tq}(S)\in\mathbf{R} : S\subset N_r \text{ and } (t,q)\in S\}$, stating a participation price, positive, negative, or zero, for each participant in each jurisdiction. Participation prices may be positive or negative since a participant may impose a net cost or benefit on the other members of the jurisdiction. Hence, the participation prices may be positive or negative. Participants who are "scarce" relative to demand for their types will receive positive payments for participating while those who are in abundant supply will pay for the participation of others. The Lindahl prices and participation prices are both "personalized", in the sense that they depend on the "names" of consumers, rather than just their types. As we will show in the next Section, however, in large economies with small effective groups the Lindahl taxes (or, with strict convexity, Lindahl prices) for participants of the same type in the same jurisdiction must be the same.[9]

A *Lindahl equilibrium* (for an economy with local public goods) is an ordered quadruple $(\Psi(\mathscr{N}_r), p, \Pi, \Gamma)$ consisting of a state of the economy $\Psi(\mathscr{N}_r) = ((x^{\mathscr{N}_r}, y^{\mathscr{N}_r}), (\{(x_g, z_g) : g = 1,\ldots,G\}, z))$, a price system p for private goods, a participation price system Π, and a Lindahl price system Γ such that:

(i) $\sum_{tq\in N_r}(y^{tq} - w^{tq}) \le \sum_g z_g + z$ (Γ is feasible);

(ii) $p\cdot z \ge p\cdot z'$ for all $z'\in Y_1$ (profit maximization in private goods production);

(iii) for each jurisdiction $S\subset N_r$, for all (x', z') in $Y_0[S]$, $\sum_{tq\in S}\gamma^{tq}(S)\cdot x' - \sum_{tq\in S}\pi^{tq}(S) + p\cdot z' \le 0$ and for each jurisdiction J_g in $\mathscr{N}_r$, and for all (x', z') in $Y_0[J_g]$, it holds that $\sum_{tq\in J_g}\gamma^{tq}(J_g)\cdot x' - \sum_{tq\in J_g}\pi^{tq}(J_g) + p\cdot z' \le \sum_{tq\in J_g}\gamma^{tq}(J_g)\cdot x_g - \sum_{tq\in J_g}\pi^{tq}(J_g) + p\cdot z_g = 0$ (profit maximization in public goods production) and;

(iv) for each $J_g\in\mathscr{N}_r$ and each $(t,q)\in J_g$, $p\cdot(y^{tq} - w^{tq}) + \gamma^{tq}(J_g)\cdot x^{tq} = \pi^{tq}(J_g)$ and for any consumer (t,q) and any jurisdiction S containing (t,q), if $(x', y'; S) \succ_{tq} (x^{tq}, y^{tq}; J_g)$ then $p\cdot(y' - w^{tq}) + \gamma^{tq}(S)\cdot x' > \pi^{tq}(S)$ (given their budget constraints and all prices, consumers optimize in their choice of consumption and jurisdiction).

Note that from condition (iii), since $Y_0[S]$ contains the origin, for each jurisdiction S it holds that $\sum\pi^{tq}(S)\ge 0$. From constant returns to scale and profit maximiza-

[7] From essentiality and interiority we could equivalently define the core so that only one participant need be better off in an improving coalition.

[8] For a discussion of price systems in economies with local public goods, see Conley and Wooders (1994).

[9] We could equally well have defined the Lindahl prices to depend only on the type of a consumer rather than his name, as in Scotchmer and Wooders (1986). But the equal-treatment property of the Lindahl equilibrium would then hold by definition (and Lindahl prices would still not be anonymous since the taste-type of a participant would still need to be known). See Conley and Wooders (1994) for further discussion of anonymity of pricing in economies with public goods.

tion it follows that $\sum_{tq \in J_g} \pi^{tq}(J_g) = 0$. A further noteworthy aspect of the equilibrium is that the choice sets of consumers include choices in jurisdictions not in the equilibrium jurisdiction structure. Such choices are required for a complete price system. For an equilibrium with Pareto-optimal outcomes and with endogeneity of the jurisdiction structure, there must be sufficient choice of jurisdictions, even if in equilibrium all consumers reside in one jurisdiction.

A state of the economy $\Psi(\mathcal{N}_r)$ is a *Lindahl equilibrium state of the economy* if there is a price system p for private goods, a complete Lindahl price system Γ and a participation price system Π such that $(\Psi(\mathcal{N}_r), p, \Pi, \Gamma)$ is a Lindahl equilibrium. The proof of the following sort of result is standard; see Wooders (1993) for details.

Theorem 1. A Lindahl equilibrium state of the economy is in the core.

Corollary 1. Let $\Psi(\mathcal{N}_r)$ be a Lindahl equilibrium. Then $\Psi(\mathcal{N}_r)$ is Pareto-optimal.

This is an immediate consequence of Theorem 1 since the state of the economy $\Psi(\mathcal{N}_r)$ cannot be improved upon by the total participant set $\mathcal{N}_r$.

4. Equivalence of the core and the equilibrium states of the economy

For the general case of many private and public goods, we provide an Edgeworth-type equivalence theorem. To replicate a state of the economy, in addition to replicating the set of participants we replicate the jurisdiction structure, consumptions, and production plans, so that all replicas of an individual participant are in jurisdictions with identical profiles, and are allocated identical consumptions. We restrict our definition to replications of the economy N_1. Since we can always renumber participants this involves no loss of generality.

Let $\mathcal{N} = \{J_1, \dots, J_g, \dots, J_G\}$ be a jurisdiction structure of N_1 and let r be a positive integer. Let $\mathcal{N}_r$ be a jurisdiction structure of N_r containing rG jurisdictions and denoted by $\mathcal{N}_r = \{J_{gj}: j = 1, \dots, r \text{ and } g = 1, \dots, G\}$. Suppose that for each $j = 1, \dots, r$ and each $g = 1, \dots, G$ the profile of J_{gj} equals the profile of J_g. Then $\mathcal{N}_r$ is the *r^{th} replication of $\mathcal{N}$*.

Let $\Psi(\mathcal{N}) = ((x^{\mathcal{N}}, y^{\mathcal{N}}), (\{(x_g, z_g): g = 1, \dots, G\}, z))$ be a state of the economy. Given a positive integer r, let $\mathcal{N}_r$ be the r^{th} replication of $\mathcal{N}$. A state of the economy $\Psi(\mathcal{N}_r) = ((x^{\mathcal{N}_r}, y^{\mathcal{N}_r}), (\{(x_{gj}, z_{gj}): g = 1, \dots, G, j = 1, \dots, r\}, z'))$ is an *r^{th} replication of* $\Psi(\mathcal{N})$ if (a) for each $g = 1, \dots, G$ and each $j = 1, \dots, r$, $(x_{gj}, z_{gj}) = (x_g, z_g)$; (b) $z' = rz$; and (c) for each participant (t, q) where $q \le m_t$ there are r participants (t, q') such that $(x^{tq'}, y^{tq'}; \mathcal{N}) = (x^{tq}, y^{tq}; \mathcal{N}_r)$; each of r participants in the r^{th} replication have the same allocation as a participant of the same type in the original economy.

A state of the economy $\Psi(\mathcal{N})$ is in the *core for all replications* if, for each positive integer r it holds that an r^{th} replication of $\Psi(N)$ is in the core of the r^{th} economy.

Theorem 2. Suppose that a state of the economy $\Psi(\mathcal{N})$ is in the core of the replicated economy for all replications. Then $\Psi(\mathcal{N})$ is a Lindahl equilibrium state of the economy. Moreover, $\Psi(\mathcal{N})$ has the equal treatment property and if preferences are strictly convex, all members of the same type in the same jurisdiction pay the same Lindahl prices.

5. Existence and equivalence with one private good

When there is only one private good the core has the equal treatment property and all participants of the same type in the same jurisdiction pay the same Lindahl taxes (Lindahl prices times quantities). The following result is virtually immediate from similar results for large games (see Kaneko and Wooders (1982) and Wooders (1983)) and the fact that when there is only one private good there are no possible gains to trade between jurisdictions.

Theorem 3. Assume that there is only one private good. Then there is an integer r_0 such that (a) for all positive integers ℓ the ℓr_0^{th} economy has a nonempty core and all states of the economy in the core have the equal treatment property, (b) every state of the ℓr_0^{th} economy in the core is an equal-treatment Lindahl equilibrium state and (c) if $\Psi(\mathcal{N}_{r_0})$ is in the core of the r_0^{th} economy, then $\Psi(\mathcal{N}_{r_0})$ is in the core for all replications of the r_0^{th} economy.

The following Corollaries are immediately clear.[10]

Corollary 2. Assume that there is only one private good. Then there is a sequence of replica economies with the property that the Lindahl equilibrium exists and the set of Lindahl equilibrium states of the economy coincides with the core.

Corollary 3. Let $(\Psi(\mathcal{N}_r), p, \Pi, \Gamma)$ be a Lindahl equilibrium. Assume that there is only one private good and that preferences are strictly convex.[11] Then (a) $\Psi(\mathcal{N}_r)$ has the equal treatment property and (b) for each jurisdiction J_g and for all participants (t, q) and (t, q') in J_g it holds that $\gamma^{tq}(J_g) = \gamma^{tq'}(J_g)$.

Using notions of approximate equilibrium analogous to the notion of approximate cores in Shubik and Wooders (1983), where a "small" set of participants can be ignored, it is immediate that approximate equilibria exist for all sufficiently large economies.[12] Specifically, let $\Psi(\mathcal{N}_r)$ be a feasible state of the economy. Then $\Psi(\mathcal{N}_r)$ is in the (Shubik-Wooders) *weak ε-core* if there is a subset of participants $N_0 \subset N_r$ such that $\Psi(\mathcal{N}_r)$ cannot be improved upon by any coalition $S \subset N_0$ and $\left|\frac{N_0}{N_r}\right| > 1 - \varepsilon$. Similarly, we will say that a state of the economy $\Psi(\mathcal{N}_r)$ is an *Lindahl ε-equilibrium* if $\Psi(\mathcal{N}_r)$ is feasible and if there is a subset of participants $N_0 \subset N_r$ such that the state of the economy restricted to N_0 is a Lindahl equilibrium state and $\left|\frac{N_0}{N_r}\right| > 1 - \varepsilon$.

The following Theorem states that for all sufficiently large economies the ε-core and the set of ε-equilibrium outcomes coincide. When small groups are strictly

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[10] This result holds in general for large games with effective groups bounded in size; see Kaneko and Wooders (1982). It was presented for the case with one private good and one public good in Scotchmer and Wooders (1986). See also Shubik and Wooders (1983) and Wooders (1988).

[11] That is, for any participant (t, q), any two consumptions (x', y') and (x^*, y^*) where $(x', y'; s) \succsim_{tq} (x^*, y^*; s)$ and any $\lambda \in (0, 1)$, $(x, y; s) \succ_{tq} (x^*, y^*; s)$ where $(x, y) := \lambda(x', y') + (1 - \lambda)(x^*, y^*)$.

[12] Nonemptiness of this approximate core and other sorts of approximate cores of economies with local public goods is shown in Wooders (1988).

effective and there is only one private good then the presence of "left-over participants" who cannot be accommodated in core jurisdictions that prevents existence of equilibrium and nonemptiness of the core. But these left-overs must be small in number.

Theorem 4. Assume that there is only one private good and one public good. Given $\varepsilon > 0$ there is an integer η such that if $|N_r| > \eta$ then (a) the weak ε-core of the economy is nonempty and (b) a state of the economy restricted to a set of participants $N_0 \subset N_r$ where $\left|\frac{N_0}{N_r}\right| > 1 - \varepsilon$ is in the core of the economy with the set of participants N_0 if and only if the state of the economy restricted to the set of participants N_0 is a Lindahl ε-equilibrium state.

The results with one private good are especially interesting in that they show that if all gains to collective activities can be realized by subgroups of the population not containing all players of any type then the equivalence of the core and equilibrium holds. Note that this is even true in the many public good case.

6. Concluding remarks on pricing crowded public goods

Rather than charge a participation price for entry into a jurisdiction and a per unit price for the public goods, an alternative approach to equilibrium is to require one total admission price for each jurisdiction for each type of participant represented in the jurisdiction for each level of public goods. Such an approach appears in Scotchmer and Wooders (1986), where it was shown that in a model with one private good, one public good, and quasi-linear utilities, the admission equilibrium outcomes, the Lindahl equilibrium outcomes and the core are equivalent.[13] This is easily shown also for the model herein and can also be shown more generally – admission prices coincide with Lindahl taxes plus participation prices (see Wooders (1993)).

Participation prices do not appear in neither Vasilev,' Weber, and Wiesmeth (1992) nor Conley (1994). In these papers the only admissible jurisdiction structure is one where all consumers are in the same jurisdiction. In Wooders (1993) an example is provided showing that participation prices may be required even in this situation for a first best equilibrium – thus, with crowding, even if it is optimal for all participants to be in one jurisdiction, without participation prices a Lindahl equilibrium may not have a complete price system.

We conclude by noting that, in this paper and in prior papers on core convergence in economies with differentiated crowding, prices for public goods are defined to depend on the taste types of participants. Recent research shows that core convergence also holds when the prices for public goods depend only on the crowding types of agents – their direct effects on others – and not on their tastes; see Conley and Wooders (1994).

[13] Closely related results are presented in Scotchmer and Wooders (1986) and Scotchmer (1994). Conley and Wooders (1994) discuss, however, that the equivalence of admission pricing and Lindahl pricing depends on the nonanonymity of prices.

7. Appendix: Proofs

As noted in the Introduction, the proof of convergence of this paper is a continuation of the proof of convergence of Wooders (1981, 1989). Thus, we merely sketch the parts of the argument already in Wooders (1989).

Proof of Theorem 2: For ease in notation, we prove the Theorem for the original (un-replicated) economy. This entails no loss of generality since the set of consumers in N_r can be renumbered so that there is only one consumer of each type. We will denote a member $(t, 1)$ of N_1 simply by t. To prove equal treatment we revert to the notation introduced earlier.

Let $\{S_1, \ldots, S_k, \ldots, S_K\}$ denote the set of all jurisdictions in N_1, and, for each k, let s_k denote the profile of S_k.

Let $\Psi(\mathcal{N}) = ((x^{*\mathcal{N}}, y^{*\mathcal{N}}), (\{(x_g^*, z_g^*): g = 1, \ldots, G\}, z^*))$ be a state of the economy, with $\mathcal{N} = \{J_1, \ldots, J_g, \ldots, J_G\}$. Assume that $\Psi(\mathcal{N})$ is in the core for all replications of the economy.

Preliminaries: We first extend the commodity space for public goods, similar to Foley's (1970) extension of the commodity space for pure public goods. The number of commodities in the extended commodity space is the number of public goods M times the number of player types T times the number of jurisdictions K.

Let $a = (a_1, \ldots, a_k, \ldots, a_K)$ be a vector where, for each k, $a_k = (a_k^1, \ldots, a_k^t, \ldots, a_k^T)$ and for each t, $a_k^t \in \mathbf{R}^M$. The vector a_k^t denotes an allocation of public goods for the t^{th} consumer for the jurisdiction S_k. We will require that a_k^t equal zero if t is not a member of the jurisdiction S_k. Let A_k be the set of elements in $\mathbf{R}^{MTK}$ defined by $A_k = \{a \in R^{MTK}: a_{k'}^t = 0 \text{ if } k \neq k' \text{ or if } t \notin S_k\}$. An element of A_k represents consumptions of public goods by the members of the jurisdiction S_k. Note that we have not required that $a_k^t = a_k^{t'}$ for all consumers t and t' in S_k; such a restriction will be imposed on the production side of the economy in the extended commodity space.

Let $a_k = (a_k^1, \ldots, a_k^t, \ldots, a_k^T)$ denote an element of $\mathbf{R}^{MT}$, with $a_k^t \in \mathbf{R}^M$. For each k define $Y_0'[S_k]$as the set of elements (a_k, z) having the properties that: (i) $a_k^t = a_k^{t'}$ for all consumers t and t' in S_k; (ii) $a_k^t = 0$ for all t not in S_k; and (iii) for (any) t in S_k, (a_k^t, z) is in $Y_0[S_k]$; the same amounts of public goods are produced for all consumers in the same jurisdiction S_k and no public goods are produced for consumers not in the jurisdiction.

Define the aggregate production set Y as the set of elements $(a, b) \in \mathbf{R}^{MTK+L}$ such that, for some $(a_k, z_k) \in Y_0'[S_k]$ for each k and some $z \in Y_1$ it holds that $a = (a_1, \ldots, a_k, \ldots, a_K)$ and $b = \sum_{k=1}^{K} z_k + z$. The set Y is the aggregate production set in the extended commodity space. Since $Y_0[S_k]$ is a closed convex cone with vertex zero, Y is a closed convex cone with vertex zero.

Rather than considering a set of preferred net trades for each individual consumer, as Debreu and Scarf, we consider preferred net consumptions for jurisdictions.

Step 1: The sets of preferred allocations Ω_k. Let Ω_k denote the set of members of (a, b) in $A_k \times \mathbf{R}^L$ with the properties that, for each $t \in S_k$ there is a $y^t \in \mathbf{R}^L$ such that

$(a_k^t, y^t; S_k) \succ_t (x^{*t}, y^{*t}; N)$ and $b = \sum_{t \in S_k}(y^t - w^t) - c'(S_k)$. The set Ω_k is a subset of $\mathbf{R}^{MTK+L}$. It is a standard argument to show that Ω_k is convex.

Let (a, b) be in Ω_k and suppose that $\{\lambda^n\}$ is a sequence of positive numbers with $\lambda^n \leq 1$ for each n. Suppose that λ^n converges to one as n goes to infinity. From arguments similar to those in Debreu and Scarf (1963) it follows that for all n sufficiently large, $\lambda^n(a, b)$ is in Ω_k. We next show that for all n sufficiently large, $\lambda^n(a, b)$ is in Ω_k.

Step 2: The preferred set Ω. Let Ω denote the convex hull of the union of the sets Ω_k, $k = 1, \ldots, K$. Since Ω_k is convex for each k, Ω is the set of all vectors which can be written $\sum_{k=1}^{K} \lambda_k(a^k, b^k)$ with $\lambda_k \geq 0$, $\sum_{k=1}^{K} \lambda_k = 1$, and $(a^k, b^k) \in \Omega_k$ for each k.

It can now be shown by a modification of Debreu and Scarf (1963) that $\Omega \cap Y = \phi$.

Step 3: Prices. From the Minkowski Separating Hyperplane Theorem, there is a hyperplane with normal $(\tilde{\gamma}, p)$, where $\tilde{\gamma}$ is in the extended public goods space and p is in the private goods price space, such that, for some constant C, $\tilde{\gamma} \cdot a + p \cdot b \geq C$ for all $(a, b) \in \Omega$ and $\tilde{\gamma} \cdot a + p \cdot b \leq C$ for all $(a, b) \in Y$. Since Y is a closed convex cone with vertex zero, it follows that we can choose $C = 0$.

For each jurisdiction J_g in $\mathcal{N}$ we represent the consumptions of public goods of members of J_g in the extended commodity space and then add these to obtain a representation of the consumptions of public goods of all consumers. Let $\tilde{x}_g = (\tilde{x}_{g1}, \ldots, \tilde{x}_{gk}, \ldots, \tilde{x}_{gK})$ denote the vector in $\mathbf{R}^{MTK}$ where $\tilde{x}_{gk} = (\tilde{x}_{gk}^1, \ldots, \tilde{x}_{gk}^t, \ldots, \tilde{x}_{gK}^T) \in \mathbf{R}^{MT}$, $\tilde{x}_{gk}^t = 0$ if $S_k \neq J_g$ or if t is not in J_g, and $\tilde{x}_{gk}^t = x^{*t}$ otherwise. The vector $\tilde{x}_g$ consists of zeros except for the components associated with members of the jurisdiction J_g. The components of $\tilde{x}_g$ associated with members of J_g all equal x_g^*. Let $\tilde{x} = \sum_{g=1}^{G} \tilde{x}_g$. The vector $\tilde{x}$ represents the consumptions x^{*N} in the extended public goods space. Since $\Psi(\mathcal{N})$ is feasible, $(\tilde{x}, \sum_{g=1}^{G} z_g^* + z^*) \geq (\tilde{x}, \sum_{t \in N}(y^{*t} - w^t))$. Since, for each g, $(\tilde{x}_g, \sum_{t \in J_g}(y^{*t} - w^t))$ is in the closure of Ω, it holds that $\tilde{\gamma} \cdot \tilde{x}_g + p \cdot z_g^* \geq 0$. However, since $(\tilde{x}_g, z_g^*)$ is in Y it holds that $\tilde{\gamma} \cdot \tilde{x}_g + p \cdot z_g^* \leq 0$ for each g. Since $(0, z^*)$ is in Y it holds that $p \cdot z^* \leq 0$. It follows that for each g, $\tilde{\gamma} \cdot \tilde{x}_g + p \cdot z_g^* = 0$, $\tilde{\gamma} \cdot \tilde{x}_g + p \cdot \sum_{t \in J_g}(y^{*t} - w^t) = 0$, and $p \cdot z^* = 0$.

From the separating hyperplane theorem it follows that z^* maximizes profit, $p \cdot z^*$, in the production of private goods.

For each S_k and each $t \in S_k$ define $\gamma^t(S_k) = \tilde{\gamma}_k^t$. Define $\Gamma = \{\gamma^t(S_k) : k = 1, \ldots, K$ and $t \in S_k\}$. Observe that for each $g = 1, \ldots, G$ it follows that $\sum_{t \in Jg} \gamma^t(J_g) \cdot x^{*t} + p \cdot z_g^* = 0$ from the fact that $\tilde{\gamma} \cdot \tilde{x}_g + p \cdot z_g^* = 0$.

As a first step towards showing profit maximization in the production of public goods, assume that for some S_k we have $(x', z') \in Y_0[S_k]$ and $\sum_{t \in S_k} \gamma^t(S_k) \cdot x' + p \cdot z' + p \cdot c'(S_k) > 0$. (This ignores the possibility of hiring other players at their participation prices, to be discussed below.) As in the proof of profit maximization in the production of private goods, this leads to a contradiction to the fact that $(\tilde{\gamma}, p) \cdot (a, b) \leq 0$ for all (a, b) in Y.

From monotonicity of preferences it follows that $p \gg 0$ and that $\gamma^t(S_k) \gg 0$ for all S_k and all t.

Step 4: Participation prices. For each J_g and each t in J_g define $\pi^t(J_g) = p\cdot(y^{*t} - w^t) + \gamma^t(J_g)\cdot x^{*t}$. Note that, from the relationship $\tilde{\gamma}\cdot\tilde{x}_g + p\cdot\sum_{t\in J_g}(y^{*t} - w^t) = 0$ derived in the preceding Section, it follows that $\sum_{t\in J_g}\pi^t(J_g) = 0$ for each J_g.

To complete the proof it remains to determine participation prices for consumers in jurisdictions S_k not in $\mathcal{N}$ and to show that both the profit maximization in public goods production and the individual consumer optimization conditions of the equilibrium are satisfied.

For each jurisdiction S_k not in $\mathcal{N}$ and each consumer t in S define $\pi^t(S_k)$ as the infimum of amounts of monetary transfers ρ^t such that, given ρ^t there is a consumption (x', y') in X^t satisfying $\gamma^t(S_k)\cdot x' + p\cdot(y' - w^t) \le \rho^t$ and $(x', y'; S_k) \succ_t (x^{*t}, y^{*t}; \mathcal{N})$. From the substitution assumption on preferences, (e) of Section 2.3, there is such a number $\pi^t(S_k)$. The number $\pi^t(S_k)$ can be interpreted as the amount of monetary income that the consumer t in J_g would require to be indifferent between living in J_g with income $p\cdot w^t + \pi^t(J_g)$ and living in S_k with income $p\cdot w^t + \pi^t(S_k)$. Note that $\pi^t(S_k)$ may be positive, negative, or zero. Also note that $\sum_t \pi^t(S_k) \ge 0$; otherwise we have a contradiction to the separating hyperplane property of $(\tilde{\gamma}, p)$. Along with the participation prices for participants in the jurisdictions J_g in N, we have now defined a participation price for each member t of each jurisdiction S_k.

We have already shown that for each k and all (x', z') in $Y_0[S_k]$ it holds that $\sum_{t\in S_k}\gamma^t(S_k)\cdot x' + p\cdot z' \le 0$. Since $\sum_{t\in S_k}\pi^t(S_k) \ge 0$ for each jurisdiction S_k not in N and since $\sum_{t\in Jg}\pi^t(J_g) \ge 0$, it holds that $\sum_{t\in S_k}\gamma^t(S_k)\cdot x' + p\cdot z' - \sum_{t\in S_k}\pi^t(S_k) + p\cdot c'(S_k) \le 0$ for all jurisdictions S_k; thus profit maximization in public goods production is satisfied.

Step 5: Individual Consumer Optimization. We claim that $(\Psi(\mathcal{N}_r), p, \Pi, \Omega)$ is a Lindahl equilibrium. It remains only to show that given the participation prices Π individual consumers are optimizing, condition (iv) of the definition of an equilibrium. This follows from the construction of the participation prices and arguments similar to those in Debreu and Scarf (1963). (Again, details are provided in Wooders (1993)).

From strict convexity of preferences, it is immediate that all participants of the same type in the same jurisdiction must pay the same Lindahl taxes. If they did not then they would of necessity have different after tax incomes and would consume different amounts of some private goods. A convex combination would be preferred, contradicting the fact that the state of the economy is in the core. ■

Proof of Theorem 3. We first show the equal treatment property of the core. Since admissible group sizes are bounded by B we can suppose that there are more than B participants of each type. Thus, for each type t it must hold that $\mathcal{N}_r$ contains at least two jurisdictions with members of that type. Then the argument that the state has the equal treatment property is standard; see, for example, the proof of Wooders (1983, Theorem 3). Essentiality of the private good implies the quasi-transferability of utility of Wooders (1983, Theorem 3) and, along with small group effectiveness, implies the equal treatment property for all sufficiently large economies. ■

Proof of Theorem 4. Theorem 4 follows familiar techniques and is an easy consequence of the previous results (see especially the lemmas in Wooders (1983)), so we leave the proof to the reader. ■

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