Chapter XII

Large Games and Economies with Effective Small Groups

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1 Introduction

Our social and economic life is carried out within groups — firms, families, market-places, and clubs, for example. Individuals cooperate within groups to benefit from increasing returns to group size and coordination of activities. Individuals compete within groups for shares of the surplus generated by the activities of the group. There is competition between groups for scarce resources and for group members. Collective activities of groups of individuals are widespread in social and economic interaction.

This paper discusses research on large games and economies with effective small groups. A large game is one with the property that most players have many substitutes. Small groups are effective if all or almost all gains to collective activities can be realized by activities within groups of players bounded in absolute size. A large game is an abstract model of a large economy. The results reported demonstrate that large games with effective small groups share important properties of markets, defined as exchange economies with money where all agents have continuous, concave utility functions. The properties include that approximate cores of large finite games are nonempty ([78,103,113], and others); approximate cores converge to the core of a continuum limit game ([49,114]); values of large games are in approximate cores ([9,80,114]); and large games are approximately market games – ones derived from markets ([105,111]). Since large games are approximately market games, approximate cores of large games converge to competitive payoffs of representing markets ([107]). Moreover, games with a continuum of players and small (finite) effective groups have nonempty cores and the core coincides with the competitive payoffs of representing markets.

Games with the property that all gains to group formation can be realized by groups smaller than the total player set satisfy a monotonicity property – any vector of population changes and any corresponding vector of changes in core payoffs point in opposite directions ([75] and this paper). The monotonicity results give conditions ensuring that player types who become less abundant in a society receive higher (or at least no lower) core payoffs.

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Large games with small effective groups exhibit an approximate monotonicity (this paper). We call the monotonicity property a "weak law of scarcity" since it is a game-theoretic counterpart of the "Weak Law of Demand", which dictates that changes in prices of commodities and amounts demanded point in opposite directions. (See Hildenbrand [44] for a recent treatment.)

If we regard the core of a large game as a game-theoretic counterpart of a competitive equilibrium, then the results reported suggest the conclusion that a game-theoretic counterpart of a competitive economy is a large game with effective small groups. We return to this in the final section.

The framework used to discuss large games is a pregame (in characteristic form) with a finite set of player types. While many of the results have been obtained for games without side payments,[1] the game-theoretic model presented is restricted to the side payments case. The modeling assumptions underlying the framework are:

(1) **Small group effectiveness:** All or almost all gains to collective activities can be realized by groups bounded in absolute size:

(2) **Substitution:** In games with many players most players have many substitutes. Players are described by their types, and players of the same type are substitutes for one another. The extension of the case of a finite set of player types to the case with a compact metric space of player types is indicated in Appendix 2. In either case, large numbers of players ensure the substitution property. The consequences of small group effectiveness are examined within this framework.[2]

The research reported begins with some preliminary results on cores and balancedness. A modification of the Bondareva-Shapley conditions for the nonemptiness of the core is presented. This modification treats cores and approximate cores of games with player types. The Kaneko and Wooders [52] condition of strong balancedness of collections of coalitions is discussed. Strong balancedness of a collection of coalitions ensures nonemptiness of the core of *every* coalition structure game whose admissible coalitions coincide with those in the collection. Shapley and Shubik [81] assignment games are an important class of games satisfying the strong balancedness condition.

Before treating the central case of small group effectiveness, results for an important special case are reported. For this special case all gains to group formation, either for the achievement of feasible outcomes or for improvement upon outcomes not in the core, can be realized by groups bounded in absolute size. Such games have interesting properties: all convergence of the core takes place in finite games and the core correspondence is monotonic. With the restriction to effective groups bounded in size, the results are quite straightforward, and more general results appearing later in the paper become transparent. Games with the property that all gains to scale are exhausted by groups bounded in absolute size approximate games with effective small groups, where all or *almost all* gains to group formation are exhausted by bounded-sized groups.

Next, the central property of small group effectiveness is introduced and related to other properties. The small group effectiveness condition, dictating that small groups are

[1] c.f., [51,86,87,103,115].

[2] The pregame framework is convenient and used throughout this paper. In ongoing research, small group effectiveness and substitution are imposed directly on individual games, without any underlying topology on the space of player types. The techniques and intuition developed in this paper apply, and essentially the same results hold.

able to achieve almost all feasible gains to group formation, is equivalent to the condition that almost all improvement can be done by small groups. When there are sufficiently many players of each type appearing in the games, then small group effectiveness is equivalent to boundedness of per capita payoffs. Results showing nonemptiness of approximate cores of large games with effective small groups are discussed, and conditions ensuring the nonemptiness are compared.

To indicate the amazing power of replication of games we discuss the following result: given a player set and an admissible collection of profiles – descriptions of groups of players in terms of the numbers of players of each type in the group– if the player set is replicated an appropriate number of times, then for *any* structure of payoffs to the admissible groups the core of the replicated game is nonempty ([52]).[3]

We then turn to relationships between market properties and properties of large games. The Market-Game Equivalence Theorem of Shapley and Shubik [79] is sketched. The market-game equivalence allows an introduction to Wooders' asymptotic market-game equivalence ([105,111]). An extension of the Wooders [99] and Wooders and Zame [114] convergence results for approximate cores is stated. The convergence is used to show that for large games, since approximate cores are nonempty and the games asymptotically exhaust all gains to group formation, the approximate core correspondence is asymptotically monotonic. To provide intuition for the results, some special properties of large games with a minimum efficient scale to group size (equivalently, ones where, without affecting the core, all improvement can be restricted to groups bounded in absolute size) are reported. A consequence of the results is that approximate (equal-treatment) cores of large games are typically small. The approximate cores are small in the sense that they are contained in a ball of small radius.

As the discussion of monotonicity may indicate, we consider only core and approximate core payoffs with the equal treatment property — players who are substitutes receive the same payoff. The restriction is justified by the result that when small groups are effective, approximate core payoffs and core payoffs are distributed nearly evenly among most players of the same type ([100]). While the equal-treatment result is subsidiary, it requires the careful setting of some parameters, so a proof is provided in an Appendix.

Wooders' [110] application of the results reported above to surplus-sharing problems, as introduced in Zajac [118] and Faulhaber [31], is discussed. Units of the economic variables generating payoffs, called "attributes", are taken as the "players". The resultant game is called an "attribute game" and its core is called the "attribute core". All our game-theoretic results immediately apply. We also briefly discuss the effects of assigning property rights to bundles of attributes to individual players. The relationship of the attribute core to subsidy-free prices is indicated.

The concluding game-theoretic section introduces a continuum limit model of large games with effective small groups. As in the elegant model of Aumann [6] the player set is an atomless measure space so that individual players and finite groups of players are negligible relative to the total player set. The model discussed here, due to Kaneko and Wooders [50], has a continuum of players and finite coalitions. The total player set is an

[3] Here we mean replicating both the player set and the structure of the characteristic function, so that all possibilities for any group in any replicated game are determined by the payoff structure of the original game. Most of the results reported hold uniformly, for all sufficiently large games; there is no restriction to replication sequences.

atomless measure space but players are atomistic relative to each other; while no individual player or finite group of players can influence aggregate outcomes, individual players and finite groups can influence each other. To motivate the continuum with finite coalitions, recall that small group effectiveness dictates that almost all gains to group formation can be realized by groups bounded in absolute size. This bound does not depend on the size of the total player set so almost all gains realizable by *very large* player sets can be realized by collective activities within groups that are relatively *very small.* The continuum model with finite effective groups, and thus, effective groups of measure zero, is intended to provide a limit model of large finite games with small effective groups where the nature of coalitions and the role of the player in a coalition is the same as in finite games. For the continuum with finite effective groups, the effectiveness of small (finite) groups ensures the nonemptiness of the core ([50]).

The motivation for the game-theoretic results comes from economic models. Small group effectiveness undergirds the equivalence of the core and the competitive outcomes and plays a significant role in economies with shared goods – collectively consumed and/or produced goods, including certain sorts of information. In the penultimate section of the paper, relationships of small group effectiveness to the competitive properties of economic models are reported. In continuum exchange economies with finite dimensional commodity spaces, with or without widespread externalities, the core with small (finite) coalitions coincides with the Walrasian outcomes ([42,50]). In economies with shared goods (including public goods) and with effective small groups, cores converge to outcomes that are Lindahl equilibrium outcomes within groups sharing the public goods ([19,101,106]). When all gains to collective activities and trade can be realized by groups bounded in absolute size, then the convergence can be completed at finite sizes of the economy ([74,98,108]). When crowding depends only on the numbers of players in a group collectively consuming and/or producing the public goods, then core/equilibrium groups consist only of consumers with the same demands ([74,98]). Other related literature is discussed.

While it is apparent that strategic game-theoretic approaches are important for the study of competitive economies and small group effectiveness, any discussion of such approaches is beyond the scope of this paper. Also, we only briefly discuss the Shapley value of large games with effective small groups, since an illustrative example appears in [116]. We omit discussion of a number of very recent results continuing the study of large games (both with and without side payments) reported in this paper and further relating large games to markets. We also omit any discussion of very recent research on small group effectiveness in the context of Arrow-Debreu exchange economies with general preferences.

1.1 Examples

Before introducing the model, we discuss three simple examples. The first illustrates our framework and results. The next two examples illustrate situations not satisfying our assumptions.

Example 1.1. Production with Two Types of Players. There are two types of players – cooks and helpers — and four sorts of cooking teams

(a) 1 cook and 2 helpers can make a cake;

(b) 4 cooks alone can make a cake (too many cooks have difficulty reaching an agreement);

(c) a helper alone can make a cookie; and

(d) a cook alone can do nothing.

A cake is worth \$10.00 and a cookie is worth \$1.00. A group (x, y) consisting of x cooks and y helpers can realize the maximal total payoff possible from splitting into teams of the sorts described above. Let $\Psi(x, y)$ denote this maximal total payoff. The information consisting of a specification of a set of player types and the associated payoffs to groups constitutes a "pregame". As soon as a population $n = (n_1, n_2)$ is specified, a game in characteristic form is determined.

We consider five cases: Case 1: The population $n = (n_1, n_2)$ has $n_1 > 0$ and $n_2 = 0$. In this case, the entire population consists of cooks. If $n_1 \leq 4$ the core is nonempty. If $n_1 > 4$, the core is nonempty if and only if n_1 is an integer multiple of 4. If $n_1 > 4$ and $n_1 \neq 4k$ for some integer k, then in any partition of the population into teams there will be some leftover cooks. These cooks create instability–for any division of the payoff among the employed cooks, unemployed cooks can profit by offering to work for a lower payment. When $n_1 = 4k$ for some integer $k \geq 2$, the core is nonempty and consists only of the payoff imputing \$5/2 to each cook.

Case 2: The population $n = (n_1, n_2)$ consists entirely of helpers; $n_1 = 0$ and $n_2 > 0$. The core is nonempty and assigns \$1 to each helper.

Case 3: With the population $n = (n_1, n_2)$, where $n = r_1(1, 2) + r_2(0, 1)$ for some positive integers r_1 and r_2, there are "many" helpers relative to the number required for teams with composition $(1, 2)$. The core is nonempty and each helper receives \$1. Intuitively, "competition" between helpers keeps the price of a helper down to his opportunity price in a helper-only group. The core assigns \$8 to each cook.

Case 4: If $n = r_1(1, 2) + r_2(4, 0)$ for some positive integers r_1 and r_2, then there are many cooks relative to the number required for teams with composition $(1, 2)$. Competition between cooks for helpers keeps the price (core payoff) for helpers up to \$15/4, while cooks get no surplus from being in "mixed" groups. The core assigns only \$5/2 to each cook.

Case 5: If $n = r(1, 2)$ for some positive integer r then the core contains a continuum of points and its extreme points are described by the cores in Cases 3 and 4 above. At one extreme point cooks each get \$8 and helpers each get \$1, while at the other extreme point, cooks get \$5/2 and helpers get \$15/4.

The reader can verify that if n is a total population with $n_1 \geq 4$ and $n_2 \geq 2$ the core is nonempty only if the population is described by one of the cases 2 to 5 or by Case 1 with $n_1 \geq 4$ or $n_1 = 4k$ for some integer k. In any other case, there will be "leftover" cooks or helpers who cannot realize the payoffs received by other players of the same type. These players create the instability associated with an empty core.

Now let $n = (n_1, n_2)$ be any large population. There are integers r_1, r_2, r_3, and r_4 so that $n = r_1(1, 2) + r_2(0, 1) + r_3(4, 0) + r_4(1, 0)$. Clearly r_4 can be restricted to be less than or equal to 3, and in a large game we can ignore 3 players. For large $n_1 + n_2$, either r_2 or r_3 (or both) can be set relatively small and the situation can be approximated by one of the three cases above. This means that for sufficiently large numbers of players in total, there are feasible partitions of players into groups and distributions of payoff so that no group of players can significantly improve upon the payoff received by the group members – approximate cores of large games are nonempty.

When there is a continuum of players, we let N_1 denote the measure of cooks and N_2 the measure of helpers. Then there is a partition of the players into finite groups so that one of the following holds, except possibly for a set of measure zero;

(a) all players are in groups $(1,2)$ and $(0,1)$ or
(b) all players are in groups $(1,2)$ and $(4,0)$, or
(c) all players are in groups $(1,2)$.

With a continuum player set there are no "leftover" players, (or, at most, a set of measure zero). Thus the core of the continuum game is nonempty and described by one of the three cases above.

While we do not discuss the Shapley value in detail in this paper, we relate it to this example. The Shapley value assigns to players their expected marginal contributions. When there are large numbers of players in total and many helpers, a helper can expect to be in a helper-only group and receive about \$1. A cook can expect that he will be able to join up with two helpers in helper-only groups and thus he can expect to make a marginal contribution of about \$8.00. Thus, in large games with many helpers, Shapley values are in approximate cores. Analysis of the opposite case, with many cooks, is similar. (The case with cooks and helpers in the ratio 1 to 2 is more difficult.) These sorts of insights motivate the results.

Later we will describe games with "bounded essential group sizes", those where all gains to group activities can be achieved by cooperation within groups bounded in absolute size. This example fits that description. It is not necessary, however, that *all* gains to group formation can be realized by groups bounded in size; small group effectiveness requires only that *almost all* gains are realized by groups bounded in absolute size. For example, we could modify the example by taking the payoff function to be $\hat{\Psi}(x,y)$ where $\hat{\Psi}(x,y) := \Psi(x,y) + 1 - 1/(x+y)$ for $x+y > 0$, and where Ψ is as given above. Note that $r\hat{\Psi}(x,y) < \hat{\Psi}(rx,ry)$ for any positive integer $r > 1$; there are "increasing returns to scale". The intuition developed in the example, however, still applies since we can approximate this new situation by games where groups are bounded in size.

Example 1.2. A Pure Public Goods Game. This example illustrates situations to which our results do not apply. Per capita payoffs can go to infinity for large groups, and small groups are not effective. Suppose that "the more, the merrier". The payoff realizable by a group with n members is n^2. Small groups are not effective. (For this example however, cores are nonempty and the Shapley value is in the core. Later we provide an example where large games have empty cores.) The example can be interpreted as one with a pure public good where the players themselves resemble public goods.

Example 1.3. A Game Without Effective Small Groups. The games constructed in this example have the property that small groups become ineffective as the games grow large. Even for large games, cores and approximate cores are empty.

For x in the unit interval [0,1], let $f(x)$ be the median of the three numbers 0, 1, and $3x-1$. Define a k-person game Ψ_k by $\Psi_k(S) = kf(\frac{|S|}{k})$ where k is the total number of players in the game and $|S|$ is the number of players in the coalition S. Unlike the preceding example, the characteristic function is not specified independently of k, the size of the population. Note that the game has an empty core for any $k > 2$. Even for large games, approximate cores are empty, since if the core is nonempty it contains an equal-treatment payoff and any payoff of approximately 1 for each player can be improved upon by any coalition containing more than half the players.

2 Games and Pregames

There is a given finite number T of player types. A *profile* $f = (f_1, \ldots, f_T) \in Z_+^T$, where Z_+^T is the T-fold Cartesian product of the non-negative integers Z_+, describes a group of players by the numbers of players of each type in the group. The profile describing a group with one player of each type is denoted by 1_T. Given a profile f, define $\|f\| = \Sigma_t\, f_t$, called the *norm* or *size* of f; this is simply the total number of players in the group. The set $\{t \in \{1, \ldots, T\} : f_t \neq 0\}$ is the *support of* f. A *partition* of a profile f is a collection of profiles f^k, not all necessarily distinct, satisfying $\Sigma_k\, f^k = f$. A partition of a profile is analogous to a partition of a set except that all members of a partition of a set are distinct. A *replication* of a profile f is a profile $rf = (rf_1, \ldots, rf_T)$ where r is a positive integer, called a *replication number.*

Let Ψ be a function from the set of profiles Z_+^T to $\mathbb{R}_+$ with $\Psi(0) = 0$. The pair (T, Ψ) is a *pregame* with *characteristic function* Ψ. The value $\Psi(f)$ is the total payoff a group of players f can achieve by collective activities of the group membership.

Let (T, Ψ) be a pregame. Define a characteristic function Ψ^*, the *superadditive cover of* Ψ, by

$$\Psi^*(f) = \max \Sigma_k\, \Psi(f^k)\ , \tag{2.1}$$

where the maximum is taken over the set of all partitions $\{f^k\}$ of f. The pregame (T, Ψ) is *superadditive* if the characteristic functions Ψ and Ψ^* are equal.

A *game determined by the pregame* (T, Ψ), which we will typically call a *game* or a *game in characteristic form*, is a pair $[n; (T, \Psi)]$ where n is a profile. When the meaning is clear, a game is denoted by its profile n.[4] Let f be a subprofile of n, that is, f is a profile and $f \leq n$. Then f is a *subgame* of the game n.

A *payoff vector* is a point x in $\mathbb{R}^T$. A payoff vector states a payoff for each type of player. The t^{th} component of x, x_t, is interpreted as the payoff to each player of type t. A payoff vector x is *feasible* for the game $[n; (T, \Psi)]$ if there is a partition $\{n^k\}$ of n satisfying:

$$\Sigma_k \Psi(n^k) \geq x \cdot n\ . \tag{2.2}$$

2.1 Cores of games, balanced games, and strongly balanced games

Let n be a game determined by a pregame (T, Ψ), let ϵ be a non-negative real number, and let x be a payoff vector. Then x is in the ϵ*-core of* n if x is feasible for n and

$$\Psi(s) \leq x \cdot s + \epsilon\|s\| \text{ for all subprofiles } s \text{ of } n\ . \tag{2.3}$$

[4]Observe that with any game n, we can associate a game according to the standard definition as follows: Let N be a finite set with $|N| = \|n\|$ and let α be a function from N into $\{1, \ldots, T\}$ with the property that $|\alpha^{-1}(t)| = n_t$ for each t. Let ν be a function from subsets S of N to $\mathbb{R}_+$, defined by $\nu(S) = \Psi(s)$ where s is the profile given by $s_t = |\alpha^{-1}(t) \cap S|$ for each t. Then the pair (N, ν) satisfies the usual definition of a game in characteristic (function) form. Since we do not keep track of identities of players we can identify a game with a profile.

When $\epsilon = 0$, we call the ϵ-core simply the *core*. The ϵ-core depends on the game n; thus the ϵ-core determines a correspondence, called the *ϵ-core* (or *core*) correspondence, from games to subsets of $\mathbb{R}^T$. The concept of the core was formally introduced by Gilles [37] and the ϵ-core was introduced by Shapley and Shubik [78]. Note that consistent with the idea that large groups might not form, we do not require that ϵ-core payoffs be Pareto-optimal.

In contrast to the usual formulation of the core, only payoffs which treat identical players identically are considered. This suffices since, for large games with effective small groups, ϵ-core payoffs treat "most" players of the same type "nearly" equally. See Proposition A.1 in Appendix 1.

The proof of the following Proposition is left to the reader.

Proposition 2.1. Let (T, Ψ) be a pregame and let (T, Ψ^*) be its superadditive cover. For every $\epsilon \geq 0$ and every game $[n, (T, \Psi)]$ and its superadditive cover $[n, (T, \Psi^*)]$, a payoff vector x is in the ϵ-core of $[n, (T, \Psi)]$ if and only if it is in the ϵ-core of $[n, (T, \Psi^*)]$.

Let $[n, (T, \Psi)]$ be a game and let β be a collection of subprofiles of n. The collection is a *balanced collection of subprofiles of n* if there are positive real numbers γ_f for $f \in \beta$ such that $\sum_{f \in \beta} \gamma_f f = n$. The numbers γ_f are called *balancing weights.* The game n is *ϵ-balanced* (in *characteristic form*) if for every balanced collection β of subprofiles of n it holds that

$$\Psi^*(n) \geq \sum_{f \in \beta} \gamma_f (\Psi(f) - \epsilon \|f\|) \tag{2.4}$$

where the balancing weights for β are given by γ_f for $f \in \beta$. When $\epsilon = 0$, an ϵ-balanced game is called *balanced.* This definition extends that of Bondareva [18] and Shapley [77] to games with player types and to pregames. Roughly, a game is balanced if allowing "part time" groups does not improve the total payoff. A game n is *totally balanced* if every subgame $f \leq n$ is balanced.

For later convenience the notion of the balanced cover of a pregame is introduced. Let (T, Ψ) be a pregame. For each profile f, define

$$\Psi^b(f) = \max_{\beta} \sum_{g \in \beta} \gamma_g \Psi(g) \tag{2.5}$$

where the maximum is taken over all balanced collections β of subprofiles of f with weights γ_g for $g \in \beta$. The pair (T, Ψ^b) is called the *balanced cover pregame* of (T, Ψ). Since a partition of a profile is a balanced collection it is immediately clear that $\Psi^b(f) \geq \Psi^*(f)$ for every profile f.

A pregame (T, Ψ) has the *approximate core property* if, for each $\epsilon > 0$, there is an integer $\eta_0(\epsilon)$ such that every game n with $\|n\| \geq \eta_0(\epsilon)$ has a nonempty ϵ-core. The pregame is *asymptotically balanced* if, for each $\epsilon > 0$, there is an integer $\eta_1(\epsilon)$ such that every game n with $\|n\| \geq \eta_1(\epsilon)$ is ϵ-balanced.

The following Proposition is an extension of the Bondareva [18] and Shapley [77]) result.

Proposition 2.2. Let $\epsilon \geq 0$ be given. A game in characteristic form $[n; (T, \Psi)]$ has a nonempty ϵ-core if and only if it is ϵ-balanced in characteristic form.

A proof is provided in the Appendix. We conclude this section with a Corollary.

Corollary 2.1. A pregame (T, Ψ) has the approximate core property if and only if it is asymptotically balanced. Moreover, the integers $\eta_0(\epsilon)$ and $\eta_1(\epsilon)$ in the definitions of these concepts can be chosen to be equal.

2.2 Minimal balanced collections of subprofiles

A balanced collection β of subprofiles of a profile n is a *minimal balanced collection* if there is no proper subset of β which is also a balanced collection of subprofiles (Shapley [77]). Since balancing weights can be obtained from solutions to systems of linear equations where the variables and the coefficients are all integers, minimal balancing weights are rational numbers. Moreover, the balancing weights for minimal balanced collections are unique. These are very useful observations for us. We note here one consequence:

Proposition 2.3. (Shapley [77]): Let (T, Ψ) be a pregame and let n be a game. Then $\Psi^b(n) = \max_{\beta} \sum_{f\in\beta} w_f \Psi(f)$, where the maximum is taken over only all minimal balanced collections β of subprofiles of n with weights w_f for f in β; the balanced cover pregame is unchanged when the balanced collections in the definition of the balanced cover are restricted to be minimal.

2.3 Strong balancedness

Consider a game with a collection of "admissible" groups. All collective activities are restricted to occur within these groups. In this Section we describe player profiles and associated admissible collections of subprofiles having the property that **every** characteristic function defined on the collection of subprofiles determines a game with a nonempty core.

Let T be a finite number of player types. A finite collection C of profiles f in Z^T_+ is *an admissible collection of profiles* if it contains the singleton profiles χ^t for each t, where $\chi^t = (\chi^t_j : j = 1, \dots, T)$ is defined as follows:

$$\begin{aligned} \chi^t_j &= 1 \text{ if } j = t \\ \chi^t_j &= 0 \text{ otherwise.} \end{aligned}$$

Let (n, C) be a pair consisting of a profile and a collection of admissible subprofiles. A balanced collection β of subprofiles of n is called *C-balanced* if each profile f in β is also in C.

Let (T, Ψ) be a pregame. Define

$$\hat{\Psi}^b(n) = \max \sum_{f\in\beta} \gamma_f \Psi(f)$$

where the maximum is taken over all C-balanced collections of subprofiles β with weights γ_f for $f \in \beta$. Similarly, define

$$\hat{\Psi}^*(n) = \max \Sigma_k \Psi(n^k) ,$$

where each subprofile n^k is in C and the maximum is taken over all partitions $\{n^k\}$ of n into subprofiles in C.

Let (n, C) be a pair consisting of a profile $n \in Z_+^T$ and an admissible collection of subprofiles of n. The pair (n, C) is *strongly balanced* if for every pregame (T, Ψ) with T types of players, $\hat{\Psi}^*(n) = \hat{\Psi}^b(n)$. The collection C is itself *strongly balanced* if for every pregame (T, Ψ) and every profile n, it holds that $\hat{\Psi}^*(n) = \hat{\Psi}^b(n)$.

It is a remarkable fact that there exist strongly balanced collections of admissible profiles. It was shown by Shapley and Shubik [81] that admissible coalitions of assignment games are strongly balanced.

In Kaneko and Wooders [52] six necessary and sufficient conditions are given for a pair (n, C) to have the strong balancedness property. Perhaps the most interesting of these is that every balanced collection of admissible subprofiles of the player profile n contains a partition of n. (This condition was named the *strong balancedness* property by le Breton, Owen, and Weber [15]). The following Proposition is a variation of part of a Theorem in [52]. We refer the reader to [52] for the proof. (The symbol $\approx$ is to be read "is equivalent to".)

Theorem 2.1 Strong balancedness $\approx$ C- balanced collections containing partitions (Kaneko and Wooders [52, Theorem 2.7 (ii)]): Let (n, C) be a pair consisting of a profile $n \in Z_+^T$ and a finite collection C of admissible subprofiles. The pair is strongly balanced if and only if every C-balanced collection β of subprofiles of n contains a partition of n.

Classes of games whose admissible coalitions have the strong balancedness property include assignment games (introduced by Gale and Shapley [36] and Shapley and Shubik [81]), consecutive games (introduced by Greenberg and Weber [38]), and those communication games (introduced by Myerson [61]) whose graphs are "forests". (See le Breton, Owen, and Weber [15] for a description of these games and proofs showing that their admissible coalition structures all satisfy the strong balancedness property.) Other related papers include [22,23,39].

Since consecutive games will be of interest to us later, we provide a brief description here. Informally, and using the standard notation, a consecutive game is one where there is some indexing on the player set $N = \{1, \ldots, Q\}$ so that if i and j are players in some admissible coalition S and $i < k < j$, then k is in S. Every admissible coalition consists of players that are "consecutive".

3 Games with Effective Small Groups, Cores, and Approximate Cores

We introduce small group effectiveness and establish the relationship of small group effectiveness to other conditions. We also discuss nonemptiness of approximate cores of large games.

In Section 3.1, the central condition of small group effectiveness is introduced. Results obtained using this notion are deferred until Section 3.3.

In Section 3.2, some results are demonstrated for games where *all* gains to group formation can be realized by groups bounded in size. Two other conditions ensuring the results are also discussed: all gains to improvement can be realized by groups bounded in absolute size, and the games have a "minimum efficient scale" of group size. For games with bounded essential group sizes, cores and approximate cores cease to shrink after a finite number of replications and remain unchanged with further replication. With the results for games where *all* gains to group formation can be realized by bounded-sized groups in hand, we report in the next Section on games where *almost all* gains can be realized by groups bounded in size – games with small effective groups.

In Section 3.3, we review some nonemptiness of approximate core results in the literature. These results are obtained by continuations of arguments for the case of bounded group sizes. Some new relationships are established to make connections between various results in the literature. For games with sufficient numbers of players of each type, small group effectiveness is equivalent to boundedness of per capita payoffs. (Recall that our framework has the substitution property). Small group effectiveness is equivalent to the condition that the power of improvement is concentrated in small coalitions, that is, any feasible payoff that can be significantly improved upon can be improved upon by a small group. As the convergence results of Mas-Colell [56] and Kaneko and Wooders [49] suggest even in more general contexts, in our framework small groups can realize almost all gains to group formation if and only if small groups are effective for improvement.[5]

3.1 Small group effectiveness

A pregame (T, Ψ) satisfies *small group effectiveness* if, for each $\epsilon > 0$, there is an integer $\eta_3(\epsilon)$ such that for every profile f there is a partition $\{f^k\}$ of f satisfying:

$$\|f^k\| \leq \eta_3(\epsilon) \text{ for each subprofile } f^k, \text{and} \tag{3.1}$$

$$\Psi^*(f) - \Sigma_k \, \Psi(f^k) \leq \epsilon \|f\|; \tag{3.2}$$

given $\epsilon > 0$ there is a group size $\eta_3(\epsilon)$ such that within ϵ per capita of the gains to group formation can be realized by the collective activities of groups containing no more than $\eta_3(\epsilon)$ players.

The term "inessentiality of large groups" has also been used for this property –it is not necessarily the case that only small groups form or that large groups are ineffective; it is only required for small group effectiveness that large groups cannot significantly improve upon the outcomes realizable by small groups. With small group effectiveness, as we discuss in Section 3.3, all sufficiently large games have nonempty ϵ-cores. The Theorem holds uniformly for all sufficiently large games. In Section 4.2 it is shown that small group effectiveness implies all sufficiently large games are close to limiting market games and asymptotically the core correspondence is monotonic.

Games with effective small groups are ones that can be approximated by games with bounded coalition sizes.

3.2 Games with bounded essential group sizes[6]

Games with bounded essential group sizes appear often in game-theoretic and economic models. Some examples include buyer-seller models and assignment games more generally.

[5] In [56] it is shown that with bounded sizes of improving coalitions, approximate cores of economies are close to the Walrasian allocations. In [49] the same conclusion is reached with bounded sizes of trading coalitions.

[6] With the exception of the strong ϵ-core Theorem, motivated by a result of Elul [29], the results in this subsection are primarily variations of ones in [99,100].

Other examples include coalition structure games, partitioning games, and games derived from economies with coalition production and with public goods subject to congestion.[7]

A simple but important result is that for any game n there is a replication number r with the property that if the game is replicated r times, then the core of the replicated game rn is nonempty. If there are sufficiently many players of each type in a game n, then the ϵ-core doesn't shrink when the game is replicated, i.e., when the total player set is increased to rn. (Of course it is only required that there be enough players of each type appearing in the game.) Cores of games with sufficiently many players of each type do not expand when the game is replicated. These results illuminate the special properties of games that "exhaust gains to scale"–no further per capita gains can be realized by replicating the game. In addition, all sufficiently large games have non-empty approximate cores. (See the condition in Proposition 3.2 below.) Since games with bounded essential group sizes (and games that exhaust gains to scale) approximate games with small effective groups, the results in the following sections of nonemptiness of approximate cores can be viewed as continuations of the results of this section.

A pregame (T, Ψ) has *bounded essential group sizes* if there is a real number B with the property that for every profile f, there is a partition $\{f^k\}$ of f with $\|f^k\| \leq B$ for each k, and

$$\Psi^*(f) - \Sigma_k \Psi(f^k) = 0 . \tag{3.3}$$

While larger groups might form, such groups can do no better than some partition into sub-groups bounded in absolute size; groups larger than this bound are not essential. The property appears in Example 1.1 and in a number of previous papers.

A related notion is that there is a "minimum efficient scale of group size". A pregame (T, Ψ) has a *minimum efficient scale* if there is a bound B with the properties that for every profile f, there is a balanced collection β of subprofiles of f with $\|g\| \leq B$ for each $g \in \beta$, and

$$\Psi^b(f) - \sum_{g\in\beta} w_g \Psi(g) = 0 , \tag{3.4}$$

where $\{w_g\}_{g\in\beta}$ is a set of balancing weights for β. The condition does not rule out the efficiency of large groups; it only requires that that there exist "small" efficient groups, that is, efficient groups bounded in size. It is immediately clear that if a pregame satisfies boundedness of essential group sizes, then it has a minimum efficient scale.[8]

The notion of a minimum efficient scale is equivalent to *exhaustion of improvement possibilities by bounded-sized groups*: there is a number B such that for each profile f and

[7] See for example von Neumann and Morgenstern [62 p. 556-586] for an early game-theoretic discussion of buyer-seller models, Gale and Shapley [36] and Shapley and Shubik [81] for assignment games, Crawford and Knoer [21] and Roth and Sotomayer [69] for a discussion of two-sided matching games (where a match may involve several agents on each side of the market), Aumann and Dreze [8]) for coalition structure games, Kaneko and Wooders [52], le Breton, Owen and Weber [15] and Demange [22] for partitioning games (including coalition structure games as in [8]) and also strongly balanced games, Böhm [17] and Ichiishi [46] for coalition production economies, and Ellickson [28], Scotchmer and Wooders [74] and Wooders [98,108] for games derived from economies with congestable public goods, and so on.

[8] The terms "minimum efficient scale" and "exhaustion of gains to scale" to follow were introduced in [103] to make a connection with the same terms in micro-economic production theory, dictating that there exists a minimum average cost of production (c.f. [63]).

payoff x, if $x \cdot f < \Psi(f)$, then there is a profile h with $\|h\| \leq B$ and $x \cdot h < \Psi(h)$.[9]

Proposition 3.1. Minimum efficient scale ≈ exhaustion of improvement possibilities: A pregame (T, Ψ) has a minimum efficient scale of group size with bound B if and only if improving opportunities are exhausted by groups bounded in size by B.

We provide a proof of this claim in Appendix 1.

The above conditions can be used interchangeably to obtain the results of the remainder of this subsection. Each of the conditions implies that for any game n there is some finite replication of that game that exhausts all gains to scale in the sense that eventually there are no further per capita gains to replication. Note that exhaustion of gains to scale is not equivalent to 1-homogeneity of the per capita payoff function, although it does imply asymptotic 1-homogeneity. The property of exhaustion of gains to scale is important for the equivalence of cores of games and economies to competitive outcomes. We comment on the proof after the proof of Proposition 3.4 below.

Proposition 3.2. Exhaustion of gains to scale: Let (T, Ψ) be a pregame and suppose that any one of the above three conditions holds with bound B. Then:

Given any game n there is an integer $r(n)$ such that for all integers k,

$$\frac{\Psi^*(kr(n)n)}{\|kr(n)n\|} = \frac{\Psi^*(r(n)n)}{\|r(n)n\|} .$$

The following Figure may give insight into the results of this Section. For the Figure and following exposition, we ignore indivisibilities.

Let (T, Ψ) be a pregame with bounded essential group sizes (or a minimum efficient scale of group size) and let f be a profile. Define $\theta^b(r) = \Psi^b(rf)/\|rf\|$ for each r. The function $\theta^b(\cdot)$ is non-decreasing; this follows from the observation that a balanced collection of subprofiles of rf is also a balanced collection of subprofiles of r^*f for any $r^* \geq r$. When $\Psi^*(rf)/\|rf\| = \theta^b(r)$ the game $[rf; (T, \Psi)]$ has a nonempty core.

Suppose that $\Psi^*(rf)/\|rf\|$ achieves its maximum at r_0. We ask how much $\Psi^*(rf)/\|rf\|$ can dip below its maximum value as we increase r beyond r_0. Define $\theta(r) = \max_k \theta(\frac{r}{k})$, where the maximum is over all nonnegative integers k. From superadditivity, the function $\Psi^*(rf)/\|rf\|$ must not cross below the function $\theta(r)$, since a possibility open to a group rf is to divide into k^* subgroups, all with the same profile $\frac{r}{k^*}f$. Since $\Psi^b(rf)/\|rf\| \geq \Psi^*(rf)/\|rf\|$ the function $\theta(r)$ cannot rise above the function $\theta^b(r)$. At r_0, $\theta(r_0) = \theta^b(r_0)$. The function $\theta^b(r)$ cannot take on values higher than $\theta(r_0)$ since at r_0 per capita payoffs are maximized over all replication numbers r. From superadditivity, for all integer multiples ℓ of r_0, it follows that $\theta(\ell r_0) = \theta^b(\ell r_0)$. This implies that the dips in the function $\theta(r)$ vanish in the limit and the function $\theta(r)$ converges to $\theta^b(r)$. Since $\theta(r) \leq \Psi^*(rf)/\|rf\| \leq \theta^b(r)$, we have the conclusion that $\Psi^*(rf)/\|rf\|$ converges to $\theta^b(r)$.

The Figure illustrates Propositions 3.2, and 3.4. It also illustrates Proposition 3.6 (but only for replication sequences). Because of its clear interpretation and relationship to several economic models, the following results are stated with boundedness of essential group sizes.

[9] A version of this condition was introduced in [99], and another in [114], discussed later. Related conditions with other names appear in various papers.

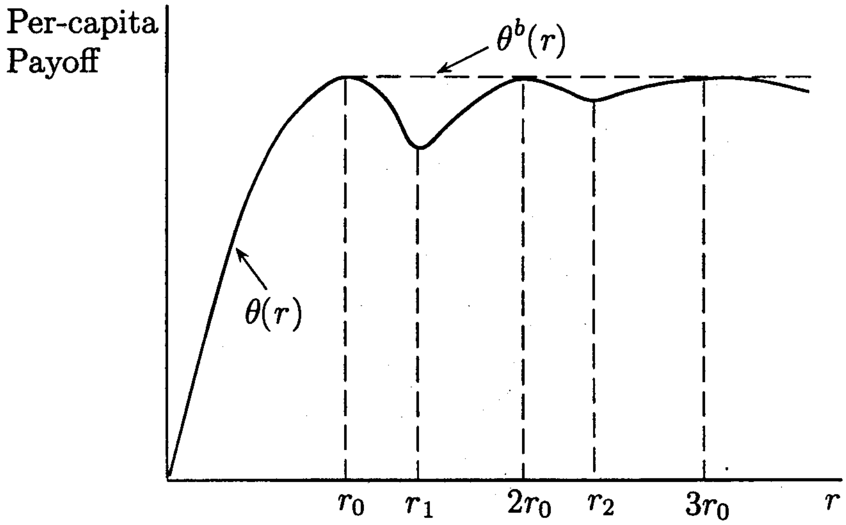

Figure XII.1: Limiting 1-homogeneity with replication.

The next Proposition shows that for any game with bounded essential group sizes and sufficiently many players of each type, the ϵ-core does not shrink when the game is replicated.

Proposition 3.3. No shrinkage of ϵ-the core: Let (T, Ψ) be a pregame with essential group sizes bounded by B. Let n be a game, where, for each t, either $n_t = 0$ or $n_t > B$. Let ϵ be a non-negative real number and suppose that x is a payoff in the ϵ-core of n. Then, for all integers r, x is in the ϵ-core of the game rn.

Proof of Proposition 3.3. Suppose the game n and the payoff x satisfy the conditions of the Proposition. Observe that since x is a feasible payoff for the game n, x is a feasible payoff for all replications r, i.e., $x \cdot rn \leq r\Psi^*(n) \leq \Psi^*(rn)$. Therefore if x is not in the ϵ-core of rn there is a profile $f \leq rn$ such that $\Psi(f) > f \cdot x + \epsilon\|f\|$. From boundedness of essential group sizes, there is a partition $\{f^k\}$ of f such that

$$\Psi(f) - \Sigma_k \Psi(f^k) = 0 \text{ and}$$

$$\|f^k\| \leq B \text{ for each } k\ .$$

Since $\Psi(f) > f \cdot x + \epsilon\|f\|$ it holds that $\Sigma_k \Psi(f^k) > \Sigma_k (f^k \cdot x + \epsilon\|f^k\|)$. It follows that for at least one k, $\Psi(f^k) > f^k \cdot x + \epsilon\|f^k\|$. Since $f^k \leq n$ this contradicts the supposition that x is in the ϵ-core of n.

Q.E.D.

The following Proposition is remarkably simple but the idea is crucial to many of the results to follow. The key observation is that any minimal balanced collection of subprofiles of a game n generates a partition of rn for appropriate choice of replication number r. For the pregame depicted in Figure 3.1, the integer m_0 in the Proposition can be chosen to be the r_0 in the description of the Figure.

Proposition 3.4. Nonemptiness under replication: Let (T, Ψ) be a pregame satisfying boundedness of essential group sizes with bound B. Let n be a game and let ϵ be a non-negative real number. Then there is a positive integer m_0 such that for each positive integer r, the game rm_0n has a nonempty ϵ-core.

Proof of Proposition 3.4. It suffices to prove the Proposition for $\epsilon = 0$. Let n be a profile. Let r_1 be an integer sufficiently large so that for each t with $n_t > 0$, $r_1 n_t > B$. For ease in notation, and without any loss of generality, assume that $r_1 = 1$.

Consider the collection of all minimal balanced collections of subprofiles of n. The key observation that *since minimal balanced collections have rational weights, we can select an integer m_0 with the property that for every balancing weight γ for any member of any minimal balanced collection of subprofiles of n, $m_0\gamma$ is an integer.* It follows that for any minimal balanced collection β of subprofiles of n with weights γ_f for $f \in \beta$, there is a partition of m_0n where $m_0\gamma_f$ members of the partition have profile f.

Let x be in the core of the balanced cover of n. Then, for some minimal balanced collection β with weights γ_f for $f \in \beta$ it holds that $x \cdot n = \sum_{f\in\beta} \gamma_f \Psi(f)$. Furthermore, for any positive integer r, $(m_0\gamma_f)$ is integral for each profile f in β and $\sum_{f\in\beta}(m_0\gamma_f)f = m_0n$. From superadditivity of Ψ^* it follows that $rm_0x \cdot n = r\sum_{f\in\beta}(m_0\gamma_f)\Psi(f) \leq \Psi^*(rm_0n)$. Therefore x is a feasible payoff for the game rm_0n. As in the preceding proof, it follows that x is in the core of rm_0n.

Q.E.D.

Taking $r(n) = m_0$ as defined above, the Proof of Proposition 3.2 follows from the observation that if a payoff x is in the core of $kr(n)n$ for all positive integers k then, from superadditivity and from the fact that x is in the core and therefore cannot be improved upon, $\Psi^*(kr(n)n) = kr(n)n \cdot x = k\Psi^*(r(n)n)$ for all k.

The reader might observe that except for having the property that all gains to group formation can be realized by groups containing no more than B players, the characteristic function Ψ played no essential role in the above proof; the profile n determined m_0. We return to this at the end of the subsection.

The next Proposition demonstrates conditions under which the ϵ-core, if nonempty, does not expand when the game is replicated.

Proposition 3.5. Non-expansion of cores: Let (T, Ψ) be a pregame with essential group sizes bounded by B. Let n be a game with the property that if $n_t \neq 0$ then $n_t > B$ and let ϵ be a non-negative real number. Assume that the core of n is nonempty. For any positive integer r, if x is a payoff in the ϵ-core of rn then x is in the ϵ-core of n.

Proof of Proposition 3.5. Let n, ϵ, and r satisfy the conditions of the Proposition. Let x be in the ϵ-core of rn. There is a balanced collection $\{f^k\}$ of subprofiles of rn with weights γ_k for f^k and with $\|f^k\| \leq B$ for each k, such that $rx \cdot n \leq \Sigma_k \gamma_k \Psi(f^k)$. Dividing by r we obtain $x \cdot n \leq \Sigma_k(\frac{\gamma_k}{r})\Psi(f^k)$. Since $\Sigma_k \gamma_k f^k = rn$, it holds that $\Sigma_k(\frac{\gamma_k}{r})f^k = n$. Since, for each k, $\|f^k\| \leq B$, it follows that $f^k \leq n$ – each f^k is a subprofile of n. We conclude that $\{f^k\}$ is a balanced collection of subprofiles of n. It follows that $x \cdot n \leq \Psi^b(n)$.

Since n has a nonempty core, n is a balanced game and $x \cdot n \leq \Psi^*(n)$. Since x is in the ϵ-core of rn, x cannot be ϵ-improved upon by any subprofile $f \leq n$. It follows that x is in the ϵ-core of n.

Q.E.D.

Combining Proposition 3.3, 3.4, and 3.5 yields the following Corollary, showing conditions under which the core remains unchanged when the game is replicated. The Corollary indicates that in economic models with side payments and bounded essential group sizes, any convergence of the core is completed after a finite number of replications. See, for example, [80], where this property is demonstrated for a "matching model" (a glove market).[10]

Corollary 3.1. Invariance of cores: Let (T, Ψ) be a pregame with essential group sizes bounded by B, let $\epsilon \geq 0$ be a given real number, and let n be a game. Then there is an integer $m_0 > 0$ such that the game $m_0 n$ has a nonempty core and, for all positive integers r, the ϵ-core of the game $r m_0 n$ equals the ϵ-core of the game $m_0 n$.

The above Corollary states nonemptiness of ϵ-cores for a subsequence of multiples of a given game. It also states that the subsequence can be chosen so that the ϵ-core is the same for all games in the subsequence.

The approximate core concept we have been using was called the "weak ϵ-core" by Shapley and Shubik [78]. They also introduced a notion called the strong ϵ-core. Given $\epsilon \geq 0$ a payoff x is in the *strong ϵ-core* of a game n if

$$\Psi^*(n) \geq x \cdot n \text{ and}$$

$$\Psi(s) \leq x \cdot s + \epsilon \text{ for all subprofiles } s \text{ of } n .$$

Shapley and Shubik showed conditions under which exchange economies have nonempty strong and weak ϵ-cores. Both Propositions 3.4 and 3.5 hold for strong ϵ-cores as well as weak ϵ-cores. Since just a bit more work allows us to obtain nonemptiness of the strong ϵ-core for all sufficiently large games, we next demonstrate nonemptiness of both the strong ϵ-core and the (weak) ϵ-core.

The following Proposition gives nonemptiness results holding uniformly for all sufficiently large games; there is no restriction to replication sequences.[11] Roughly, the Proposition follows from the observation that for any large game we can find a core payoff for a large subgame of the game. We can take just a "little bit" (ϵ) of payoff away from players in the subgame, and give the "left-over" players (those not in the subgame) almost the same payoff as those in the subgame. The resulting payoff is in the ϵ-core. The proof involves messy calculations, but the idea is simple in the light of our previous work.

Part (a) of Proposition 3.6 is a special case of [109] (which admits a compact metric space of player types and assumes small group effectiveness). Part (b) is new. The proof of Proposition 3.6 is contained in Appendix 1.

Proposition 3.6. Uniform nonemptiness of approximate cores (Wooders [109] and this paper): Let (T, Ψ) be a pregame satisfying boundedness of essential group sizes with bound B.

(a) Then given $\epsilon > 0$ there is an integer $\eta_2(\epsilon)$ with the property that all games n determined by the pregame with $\|n\| \geq \eta_2(\epsilon)$ have nonempty (weak) ϵ-cores.

[10] This can hold even for situations without side payments, c.f. [98] where core-equilibrium equivalence is demonstrated in finite economies with a local public good (and one private good).

[11] Nonemptiness of strong ϵ-core results for all sufficiently large games in a replication sequence have been obtained by Elul [29] and Scotchmer and Wooders [75].

(b) Assume that for all profiles f with $\|f\| > B$, $\Psi(f) = 0$ and, for each t, $\Psi(\chi^t) > 0$. Then given $\epsilon > 0$ there is an integer $\eta_2'(\epsilon)$ such that each game n with $n_t > \eta_2'(\epsilon)$ for each t has a nonempty strong ϵ-core.

The reason the additional conditions are required for part (b) of the Theorem is that even in a large game with bounded essential coalition sizes, large groups may have good payoff possibilities. A large group may be able to ϵ-improve upon a payoff that cannot be ϵ-improved upon by any small group (where, by "ϵ-improve" it is meant here that the group can, in total, be better off by at least ϵ). For the same reason Proposition 3.3 does not hold for the strong ϵ-core. The restrictions in (b) can be somewhat relaxed. For example, instead of assuming each player can realize a positive payoff we could have assumed that, for each t, some positive numbers of players of type t can realize a positive payoff. The statement given has the advantage of being uncomplicated.

3.2.1 Strong Balancedness of Replicated Games.

Let T be a finite set of player types an let C be an admissible collection of profiles, as defined in Section 2.4. The following result illustrates the power of replication.

Theorem 3.1. Strong balancedness of collections of profiles and replicated player profiles (Kaneko and Wooders [52], Theorem 3.2): Given a number of player types T, let C be a collection of admissible profiles. Let n be a profile with the property that $f \leq n$ for each f in C. Then there is an integer m_0 such that for every replication number r the pair (rm_0n, C) is strongly balanced.

We refer the reader to [52] for a detailed proof. To prove the Theorem we choose m_0 to be a multiple that clears all the denominators of all weights on subprofiles in minimal balanced collections of admissible subprofiles of n. Since for any pregame (T, Ψ) and any game $[n; (T, \Psi)]$ satisfying the required conditions, the game $[m_0n; (T, \Psi)]$ has a nonempty core, the pair (m_0n, C) is strongly balanced. Kaneko and Wooders [52] demonstrate the result for games with and without side payments.

3.3 Characterizations of large games with effective small groups

Building on the results of the preceding section, it is now not difficult to show that if groups bounded in absolute size can achieve *almost* all gains to group formation, then approximate cores of large games are nonempty. The nature of the results depends on whether or not "scarce types" are allowed. Informally, scarce types are allowed if we admit sequences of games where the percentages of players of one or more types are positive but become arbitrarily small as the games become large. Sequences of replica games, those of the form $\{rf\}_{r=1}^{\infty}$ for some fixed profile f, do not allow scarce types. When scarce types are ruled out (and with a finite number of player types) simply boundedness of per capita payoffs will ensure that all sufficiently large games have nonempty approximate cores.

A pregame (T, Ψ) satisfies *weak effectiveness of small groups*, or, in other words, *boundedness of per capita payoffs*, if there is some constant c such that, for all profiles f, $\Psi(f)/\|f\| \leq c$.

Theorem 3.2 Per capita boundedness → nonemptiness of approximate cores of

replicated games (Wooders [99,103])[12]: Let (T, Ψ) be a pregame satisfying boundedness of per capita payoffs. Let n be a derived game and let $\epsilon > 0$ be given. Then there is an integer r_0 sufficiently large so that for all integers $r \geq r_0$, the ϵ-core of the game $[rn, (T, \Psi)]$ is nonempty.

Using the intuition and the ideas of the preceding Section, the reader will perhaps easily see how to prove the Theorem. The per capita payoff function $\Psi^*(rf)/\|rf\|$ might not be increasing, but it does have an increasing trend. Let L denote the limit of $\Psi^*(rf)/\|rf\|$ as r becomes large. From the convergence, we can get arbitrarily close to the limiting value L with groups restricted in size. Therefore, we can put a (sufficiently large) bound on group sizes and obtain the result by applying the analysis suggested by Figure 3.1 to the approximating games with bounded group sizes.

The per capita boundedness condition is easy to visualize in the replication case. It "almost" suffices to ensure that *all* sufficiently large games have nonempty approximate cores. The following example illustrates that it does not.

Example 3.1. Per capita boundedness does not imply the approximate core property (Wooders and Zame [113]): Consider a set $T = \{1, 2, 3, 4\}$ of 4 elements and a function $\lambda : Z_+^T \to \mathbb{R}_+$ defined as follows:

$$\lambda(f) = k^2 \text{ if } \quad \begin{aligned} f &= (k, k, 0, k^2), \text{ or} \\ f &= (k, 0, k, k^2), \text{ or} \\ f &= (0, k, k, k^2). \end{aligned}$$

$$\lambda(f) = 0 \qquad \text{otherwise.}$$

This function is obviously not superadditive, but we can define its superadditive cover function λ^* by setting:

$$\lambda^*(f) = \max \Sigma_j \lambda(f^j)$$

where the maximum is taken over all partitions $\{f^j\}$ of f. It is easily checked that λ^* has a per capita bound, in particular $\lambda^*(f) \leq \|f\|$. We can easily produce many large games for which the ϵ-core is empty (for small ϵ). For example, the game $n = (k, k, k, 2k^2)$ has an empty ϵ-core for each $\epsilon < 1/12$.

The difficulty in the above example is that the percentages of players of types 1, 2, and 3 in the game n become arbitrarily relatively small – players of these types become scarce. Moreover, relatively small groups of players of scarce types have significant effects on per capita payoffs. In contrast, this cannot cause a problem for any sequence of replica games or for sequences of games where the percentages of players of each type is bounded away from zero.

The condition of small group effectiveness introduced in Section 3.1 ensures that scarce types cannot have significant effects on per capita payoffs of large groups. In view of our work in Section 3.2, the following result is now easily proven:

Theorem 3.3. Small group effectiveness → uniform nonemptiness of approximate cores of large games (Wooders [109]). Let (T, Ψ) be a pregame satisfying small

[12] [103] treats games without side payments.

group effectiveness. Then (T, Ψ) has the approximate core property.

Proof of Theorem 3.3. Let (T, Ψ) be a pregame. Suppose (T, Ψ) satisfies small group effectiveness. This implies that given $\epsilon_0 > 0$ there is a bound B such that for any profile g, for some partition of g into subprofiles, say $\{g^k\}$, where $\|g^k\| \leq B$ for all k, it holds that:

$$\Psi^*(g) - \Sigma_k \Psi(g^k) < \epsilon_0 \|g\| .$$

Let (T, Λ) be a pregame with Λ defined by $\Lambda(f) = \max \Sigma_k \Psi(f^k)$ where the maximum is taken over all partitions $\{f^k\}$ of f with $\|f^k\| \leq B$ for each k. The pregame (T, Λ) has bounded essential group sizes. From the nonemptiness of the ϵ-core of a large game with bounded essential group sizes, Proposition 3.6, we can select an integer $\eta(\epsilon_0)$ sufficiently large so that all games $[n; (T, \Lambda)]$ with $\|n\| \geq \eta(\epsilon_0)$ have nonempty ϵ_0-cores. We leave it to the reader to verify that if x is in the ϵ_0-core of $[n; (T, \Lambda)]$ then x is in the $2\epsilon_0$-core of $[n; (T, \Psi)]$, which establishes the result.

Q.E.D.

The important difference between Theorems 3.2 and 3.3 is that 3.3 holds uniformly for all sufficiently large games, while 3.2 is for replication sequences. The result in [109] is obtained with a compact metric space of player types. The proof uses approximation by a finite number of player types and argument by contradiction.

If we bound the percentages of players of each type that appear in a game away from zero (i.e., if we rule out scarce types, or, in economic terms, if the player set is "thick") then small group effectiveness is equivalent to per capita boundedness. A proof is provided in Appendix 1.

Proposition 3.7. With "thickness", per capita boundedness ≈ small group effectiveness[13]: Let (T, Ψ) be a pregame satisfying boundedness of per capita payoffs. For each pair of real numbers $\rho > 0$ and $\epsilon > 0$ there is an integer $\eta_4(\rho, \epsilon)$ such that for every game f with $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$ for each t, for some partition $\{f^k\}$ of f with $\|f^k\| \leq \eta_4(\rho, \epsilon)$ for each k, it holds that

$$\Psi^*(f) - \Sigma_k \Psi(f^k) \leq \epsilon \|f\| ;$$

when arbitrarily small percentages of players are ruled out, the pregame satisfies small group effectiveness.

We now have the following Corollary.

Corollary 3.2. Let (T, Ψ) be a pregame satisfying per capita boundedness. Given $\rho > 0$ and $\epsilon > 0$ there is an integer $\eta_5(\rho, \epsilon)$ such that for all games f with $\frac{f_t}{\|f\|} > \rho$ or $f_t = 0$ for each t, if $\|f\| > \eta_5(\rho, \epsilon)$ then the ϵ-core of f is nonempty.

Small group effectiveness can also be related to the condition in Wooders and Zame [114] that "blocking power" is concentrated in small groups; any feasible payoff that can be significantly improved upon can be improved upon by a small group. This is an asymptotic version of exhaustion of improving opportunities by bounded-sized groups. A pregame

[13]The author is indebted to Jean-François Mertens for suggesting this Proposition.

(T,Ψ) satisfies *small group effectiveness for improvement* if, for each $\epsilon > 0$ there is an integer $\eta_6(\epsilon)$ with the following property:

For any game n determined by the pregame (T,Ψ), if $x \in \mathbb{R}_+^T$ is a feasible payoff not in the ϵ-core of n then there is a profile $f \leq n$ such that $\|f\| < \eta_6(\epsilon)$ and

$$\Psi(f) \geq x \cdot f + \frac{\epsilon}{2}\|f\| \,.$$

Proposition 3.8. Small group effectiveness $\approx$ small group effectiveness for improvement: Let (T,Ψ) be a pregame. Then (T,Ψ) has effective small groups if and only if (T,Ψ) satisfies effectiveness of small groups for improvement. $\blacklozenge$

The proof is contained in Appendix 1.

Remark 3.1 Theorem 3.3 above extends a result in Wooders and Zame [113]. There a more restrictive assumption on boundedness of marginal contributions is used. A pregame (T,Ψ) has an *individual marginal bound* if there is a constant M such that for all profiles f and for all types t it holds that

$$\Psi^*(f+\chi^t) - \Psi^*(f) \leq M \,.$$

The following Proposition relates individual marginal boundedness and effectiveness of small groups.

Proposition 3.9 Boundedness of marginal contributions small $\rightarrow$ group effectiveness (Wooders [109]): Let (T,Ψ) be a pregame with an individual marginal bound. Then (T,Ψ) satisfies small group effectiveness.

The assumption of boundedness of marginal contributions is more restrictive than required for the approximate core property. Roughly, what is required is not that marginal contributions are bounded but that "expected" marginal contributions are bounded. The following is an example illustrating that small group effectiveness does not imply an individual marginal bound.

Example 3.2. A pregame satisfying small group effectiveness but not boundedness of marginal contributions Wooders ([109, Example 2]): The idea of the example is simple. All players are identical. For large games, an additional player may make a very large contribution to the total payoff. However, this happens very very seldom, so small groups are effective.

We consider a sequence of games where the k^{th} game has 10^{2^k} players. The marginal contribution of a player to a group containing $10^{2^k}-1$ players for any positive integer k will be at least $10^{2^k}/10^k$, which goes to infinity as k becomes large. Per capita payoffs, however, are bounded.

Precisely, let (T,Ψ) be a pregame with $T=1$. Define Ψ by

$$\begin{aligned}\Psi(0) &= 0\\ \Psi(1) &= 1\\ \Psi(10) &= 10\end{aligned}$$

$$\Psi(10^{2^k}) = (10^{2^k})\left[\sum_{i=0}^{k} 1/10^i\right] \text{ for } k = 1, 2, \ldots$$

For any k it holds that $\Psi^*(10^{2^k}) = \Psi(10^{2^k}) = \max \Sigma \Psi(10^{2^j})$ for all partitions $\{10^{2^j}\}$ of 10^{2^k}; it is optimal, when there are 10^{2^k} players for some k, to have only one group in the player partition. Note that Ψ satisfies small group effectiveness which, in the 1-player-type case, is equivalent to per capita boundedness. This follows from the observation that $\lim_{k\to\infty} \frac{\Psi(10^{2^k})}{10^{2^k}} = 1 + \frac{1}{9}$. As noted above, marginal contributions can become arbitrarily large.

4 Market Games, Monotonicity, Convergence, and Competitive Pricing

We discuss the Shapley and Shubik [79] characterization of totally balanced games as market games and provide an introduction to the characterization of Wooders [105,107] of large games as market games.[14] That large games are market games suggests that for any large economy with effective small groups and substitution there is at least one set of "commodities" such that relative to those commodities an approximate competitive equilibrium exists.[15] The market-like properties of large games with effective small groups include core convergence and the "law of scarcity" (core payoffs to players of a given type do not increase and may decrease when that type becomes more abundant). The concepts of an attribute game, where attributes/commodities are taken as the players, and the ϵ-attribute core are introduced. The attribute core is related to competitive pricing and also subsidy-free pricing.

4.1 Market games and monotonicity

The next Proposition, 4.1, the first of our "law of scarcity" results, extends a result of Scotchmer and Wooders [75].[16] Proposition 4.3 treats the "law of scarcity" in the continuum limit and Proposition 4.4 provides an asymptotic treatment. [17]

Proposition 4.1. Monotonicity of the core correspondence of finite games with exhaustion (This paper): Let (T, Ψ) be a pregame with minimum efficient scale of group size bounded by B. Let f and g be games with, for each t, $f_t > B$ and $g_t > B$. Suppose that x is in the core of the game f and y is in the core of the game g. Then

$$(x - y) \cdot (f - g) \leq 0\ . \tag{4.1}$$

[14]The author is indebted to Robert J. Aumann, who encouraged this characterization.

[15]We refer the reader to Wooders [107,108] for a more complete discussion of the implications of the result that large games are market games.

[16]In [75] the exhaustion condition is more restrictive and is a condition on the entire pregame. Our condition actually applies to a given game; the pregame structure is unnecessary. See also Wooders [110].

[17]Related results appear in the "matching" literature; see Crawford [20] for a recent treatment. We do not discuss these here. Also, we will not discuss "partial" monotonicity results, where the changes in the player population are restricted to changes in the relative scarcity of one type (c.f.,[30]).

Proof of Proposition 4.1. Let $[f;(T,\Psi)]$ be a game satisfying the conditions of the Proposition and let x in the core of the game $[f;(T,\Psi)]$. From the assumption that x is the core of f, $x \cdot h \geq \Psi(h)$ for all profiles h with $\|h\| \leq B$. Let g be a profile satisfying the conditions required by the Proposition and let y be in the core of $[g;(T,\Psi)]$.

We claim that $x \cdot g \geq \Psi^b(g)$ $(= \Psi^*(g)$ since the game $[g;(T,\Psi)]$ has a nonempty core). Suppose not. From the minimum efficient scale assumption there is a balanced collection β of subprofiles of g with weights w_h for $h \in \beta$, such that, for each profile h in β, $\|h\| \leq B$ and $\Sigma_{h\in\beta} w_h \Psi(h) = \Psi^b(g)$. Then $x \cdot g < \Psi^b(g)$ implies that $x \cdot (\Sigma_{h\in\beta} w_h h) = \Sigma_{h\in\beta} w_h (x \cdot h) < \Sigma_{h\in\beta} w_h \Psi(h)$. This implies that is a profile h with $\|h\| \leq B$ such that $x \cdot h < \Psi(h)$. This is a contradiction as, from our assumptions, $h \leq f$ and if $x \cdot h < \Psi(h)$, x cannot be in the core of $[f;(T,\Psi)]$. Similarly, $y \cdot f \geq \Psi^b(f)$. These observations yield the following estimate:

$$\begin{gathered}(x-y)\cdot(f-g) \leq \\ x\cdot f - x\cdot g - y\cdot f + y\cdot g \leq \\ \Psi^b(f) - \Psi^b(f) - \Psi^b(g) + \Psi^b(g) = 0\ .\end{gathered}$$

Q.E.D.

To show approximate monotonicity as a consequence of small group effectiveness and to provide further economic motivation, we introduce some results on the representation of games as markets. Shapley and Shubik [79] define a *market* as an exchange economy with money and with the property that all agents have continuous, concave utility functions. A *market game* is a game derived from a market. A game is derived from a market by assigning to each coalition the maximal total utility the members of that coalition can realize by the consumption of the total endowment of the coalition membership. Conversely, the authors derive a market from a totally balanced game. Here we discuss only one special sort of derived market, called the "direct market".

The *direct market* derived from a balanced game is a market with the properties that the number of commodities equals the number of player types; all agents have the same utility function; each agent is endowed with one unit of one good; and all players of the same type are endowed with the same commodity. The utility function of every agent is the characteristic function appropriately extended from Z^T_+ to $\mathbb{R}^T_+$. Let (T,Ψ) be a pregame and let $[n;(T,\Psi)]$ be a game determined by the pregame. Assume that the game is totally balanced – the game and all its subgames are balanced.[18] We construct a market from the game by first assuming that there are T types of commodities and, in the market, a player of type t is endowed with one unit of the t^{th} commodity. Define a "utility function" u as follows. For each $x \in \mathbb{R}^T_+$ define

$$u(x) = \max_{\{\gamma_f\}} \sum_{\substack{f\leq n\\ f\in Z^T_+}} \gamma_f \Psi(f)\ , \tag{4.2}$$

maximized over all sets of non-negative γ_f satisfying

$$\sum_{\substack{f\leq n\\ f\in Z^T_+}} \gamma_f f = x\ . \tag{4.3}$$

[18]Balancedness would suffice for our purposes here, but would also increase the amount of description required.

Shapley and Shubik show that u is a continuous and concave function. Note that we have not expressly introduced money, the medium of transferring payoff. To do so would require the addition of another variable, say ξ, and defining the utility function of an agent as $u(x)+\xi$. While money is implicitly one of the commodities of exchange, for our purposes we need not keep this commodity explicitly in view. (See [82] for further analysis of the role of money.)

Let n be a profile and let u denote the utility function constructed above. Taking advantage of the concavity, and following Shapley and Shubik, we derive a game from the market. For each subprofile f of n define $X^f = \{x^T \in \mathbb{R}^T_+ : \Sigma x^t f_t = f\}$; X^f is the set of feasible allocations of goods with the equal treatment property for a subset of agents in the market with endowment f (equivalently, a subset of players with profile f in the game). Define the characteristic function v by

$$v(f) = \max_{X^f} \Sigma f_t u(x^t) .$$

Then (n, v) is the *market game* determined by the market, where n is the profile of the agent set and v is the characteristic function. For totally balanced games, Shapley and Shubik show that for all profiles $f \leq n$, $v(f) = \Psi(f)$ – the game generated by the market coincides with the initially given game $[n; (T, \Psi)]$.

Shapley and Shubik show that the competitive payoffs of the direct market described above coincide with the core of the totally balanced game generating the direct market. From the equivalence of the core and the competitive payoffs, our result shows that if the game satisfies the conditions of Proposition 4.1, then (a) the competitive price correspondence is monotonic in the sense that changes in quantities supplied of player types and corresponding changes in competitive prices point in opposite directions and (b) the competitive price vector is typically unique.

Wooders [105,111] introduces the construction of a "limiting direct premarket" derived from a pregame. To construct a direct premarket from a pregame we need to define an appropriate utility function. Let (T, Ψ) be a pregame with effective small groups. For each vector x in $\mathbb{R}^T_+$ define $U(x)$ by

$$U(x) = \|x\| \lim_{\nu \to \infty} \frac{\Psi^*(f^\nu)}{\|f^\nu\|} \tag{4.4}$$

where $\{f^\nu\}$ is any sequence of profiles such that $\|f^\nu\| \to \infty$ and $\|x\| \left(\frac{1}{\|f^\nu\|}\right) f^\nu$ converges to x as $\nu \to \infty$.

The function U is 1-homogeneous, concave, and continuous.[19] (See Wooders [105,111] for proofs.) The concavity is a consequence of 1-homogeneity and superadditivity and the continuity is a consequence of small group effectiveness.

Observe that when U is restricted to profiles (in Z^T_+) then (T, U) is a pregame with the property that every game $[f; (T, U)]$ has a nonempty core. This is a consequence of the Shapley and Shubik result that market games are totally balanced and the observation that each game $[f; (T, U)]$ is a market game.

[19] This concavity, for the side payments form of [103], was initially noted by Aumann [7]. The concavity is shown to hold with a compact metric space of player types in [105].

Note that given a profile n the Shapley-Shubik direct market utility function u determined by the game $[n;(T,\Psi)]$ does not necessarily equal the utility function U; u depends on n. If the pregame has the property that all gains to group formation can be realized by groups bounded in size, (if, for example, there is a minimum efficient scale of group or, equivalently, all improvement possibilities can be realized by bounded-sized groups) and if n contains enough players of each type, then it will be the case that $u(x)$ equals $U(x)$ for all $x \in \mathbb{R}^T_+$. (See also Propositions 3.1 and Proposition 4.5 below.)

The following Proposition shows that when small groups are effective $U(f)$ is uniformly close to $\frac{\Psi^*(f)}{\|f\|}$ for all sufficiently large profiles f. A proof is immediately clear from results in Wooders [111].

Proposition 4.2. Uniform convergence to the limiting utility function (Wooders [105,111]): Let (T,Ψ) be a pregame.

(a) If (T,Ψ) satisfies small group effectiveness then, letting U denote the function defined by (4.4),

$$\text{for each } \epsilon > 0 \text{ there is an integer } \eta_5(\epsilon) \text{ such that for all profiles } f \text{ with } \|f\| > \eta_5(\epsilon) \text{ it holds that:} \tag{4.5}$$

$$U(f) - \Psi^*(f) \leq \epsilon\|f\| \,.$$

(b) If U is a continuous, concave function satisfying (4.5) then (T,Ψ) satisfies small group effectiveness.

It is perhaps clear to the reader that, as already suggested by Aumann [7], the function U, defined by (4.4), is Lipschitz continuous. We leave a proof of this to the reader.

The core correspondence of any market with a continuum of players, in which all agents have the same continuous and concave utility function, satisfies monotonicity, as in Proposition 4.3 below. Large games converge to continuum games representable as markets with these properties, and the approximate cores of the games converge to the core of the limiting market game [107]. It is thus natural to expect that large finite games derived from pregames with effective small groups will satisfy approximate monotonicity. We first state Proposition 4.3.

In Proposition 4.3, for each t we interpret f_t as the percentage of players of type t. The (set valued) function $C(\cdot)$ is the core correspondence. The set $C(f)$ is interpreted as the core of a game with an atomless measure space of players of T different types, each of whom has the utility function u. Alternatively, when each agent is endowed with one unit of a commodity (perhaps his player type) we can also regard the set $C(f)$ as the set of Walrasian prices for the commodities. The Proposition is an application of the monotonicity of the sub-gradients of a proper concave function and, in fact, Propositions 4.2 and 4.3 can be strengthened to cyclic monotonicity. See Wooders [110].

Proposition 4.3. Monotonicity of the core correspondence of the continuum limit (This paper): Let u be a continuous, concave, and 1-homogeneous function with domain $\mathbb{R}^T_+$. For each $h \in \mathbb{R}^T_+$ let $C(h) = \{x \in \mathbb{R}^T_+ : x \cdot h = u(h) \text{ and } x \cdot m \geq u(m) \text{ for all } m \in \mathbb{R}^T_+,\, m \leq h\}$. Then, for each $f \in \mathbb{R}^T_{++}$ the set $C(f)$ is nonempty and for any f and g

in $\mathbb{R}^T_{++}$, and any x in $C(f)$ and y in $C(g)$,

$$(x-y)\cdot(f-g)\leq 0\ .$$

Proof of Proposition 4.3. Since u is concave, the set $C(h)$ is nonempty for any h in the interior of $\mathbb{R}^T_+$. Let f and g be in $\mathbb{R}^T_{++}$. Let x be in $C(f)$ and let y be in $C(g)$. Since the supports of f and g are equal, there is a positive real number λ such that $\lambda g \leq f$. From the definition of $C(f)$ it holds that $x\cdot(\lambda g)\geq u(\lambda g)$. From 1-homogeneity of u it holds that $x\cdot g\geq u(g)$. Similarly, it holds that $y\cdot f\geq u(f)$. As in the argument in the proof of Proposition 4.1:

$$\begin{gathered}(x-y)\cdot(f-g)\leq\\ x\cdot f-y\cdot f-x\cdot g+y\cdot g\leq\\ u(f)-u(f)-u(g)+u(g)=0\ .\end{gathered}$$

Q.E.D

The next Proposition shows asymptotic monotonicity. Note that under any conditions ensuring the nonemptiness of strong ϵ-cores, the monotonicity applies to the strong ϵ-core as well as to the (weak) ϵ-core.

Proposition 4.4. Asymptotic monotonicity (This paper): Let (T,Ψ) be a pregame satisfying small group effectiveness. Let δ_0 and ρ_0 be positive real numbers. Then there is a positive real number ϵ_0 and an integer $\eta(\delta_0,\rho_0,\epsilon_0)$ such that:
(4.1) for all games $[f;(T,\Psi)]$ with $\|f\|>\eta(\delta_0,\rho_0,\epsilon_0)$ the ϵ_0-core is nonempty; and
(4.2) for all pairs of games f and g with, for each $t=1,\ldots,T$, $\frac{f_t}{\|f\|}>\rho_0$ and $\frac{g_t}{\|g\|}>\rho_0$ and with $\|f\|>\eta(\delta_0,\rho_0,\epsilon_0)$ and $\|g\|>\eta(\delta_0,\rho_0,\epsilon_0)$, if x is a payoff in the ϵ_0-core of f and y is a payoff in the ϵ_0-core of g then

$$(x-y)\cdot\left(\left(\frac{1}{\|f\|}\right)f-\left(\frac{1}{\|g\|}\right)g\right)\leq\delta_0\ .$$

Proof of Proposition 4.4. Suppose the conclusion of the Proposition is false. From Theorem 3.3 we cannot contradict the first conclusion of the Theorem. Therefore, since we are supposing that the Proposition is false, there are positive real numbers δ_0 and ρ_0, a sequence of positive real numbers $\{\epsilon^\nu\}$ with $\|\epsilon^\nu\|\to 0$ as $\nu\to\infty$, and a pair of sequences of games $\{f^\nu\}$ and $\{g^\nu\}$, such that

(a) $\|f^\nu\|\to\infty$ and $\|g^\nu\|\to\infty$ as $\nu\to\infty$,
(b) $\frac{f^\nu_t}{\|f^\nu\|}>\rho_0$ and $\frac{g^\nu_t}{\|g^\nu\|}>\rho_0$ for each t and each ν, and
(c) for each integer ν, for some x^ν in the ϵ^ν-core of f^ν and some y^ν in the ϵ^ν-core of g^ν,

$$(x^\nu-y^\nu)\cdot\left(\left(\frac{1}{\|f^\nu\|}\right)f^\nu-\left(\frac{1}{\|g^\nu\|}\right)g^\nu\right)>\delta_0\ .$$

We can suppose, by passing to a subsequence if necessary, that the sequences $\{(\frac{1}{\|f^\nu\|})f^\nu\}$ and $\{(\frac{1}{\|g^\nu\|})g^\nu\}$ converge. Let $f^*=\lim_{\nu\to\infty}(\frac{1}{\|f^\nu\|})f^\nu$ and $g^*=\lim_{\nu\to\infty}(\frac{1}{\|g^\nu\|})g^\nu$. From small group effectiveness and "thickness", bounding the percentages of players of each type away from zero, it follows that the approximate core payoffs are bounded above. (Small group effectiveness ensures per capita boundedness. With thickness, per capita boundedness dictates that approximate core payoffs are bounded.) We can suppose, by passing to a subsequence if necessary, that the sequences $\{x^\nu\}$ and $\{y^\nu\}$ converge, say to the vectors x^* and y^*.

Define the limiting utility function U as above. Given any vector $h \in \mathbb{R}^T_{++}$ the set $C(h)$ (defined as in Proposition 4.3)) coincides with the "core of the limiting game" introduced in Wooders and Zame [114, Section 9], where, for each t, the percentage of players of type t in the limit game is h_t and the characteristic function of the limit game is given by U. From Wooders [105,111] the function $U(\cdot)$ is concave. As stated in Theorem 4.1 below, since the core correspondence $C(\cdot)$ is the limit of approximate cores, x^* is in $C(f)$ and y^* is in $C(g)$. From Proposition 4.3,

$$(x^* - y^*) \cdot (f^* - g^*) \leq 0\ .$$

Let ν_0 be sufficiently large so that for all $\nu \geq \nu_0$ it holds that

$$\sup_t (x_t^* - y_t^*)(\frac{f_t^\nu}{\|f^\nu\|} - f_t^*) \leq \frac{\delta_0}{4T} \text{ and}$$

$$\sup_t (x_t^* - y_t^*)(\frac{g_t^\nu}{\|g^\nu\|} - g_t^*) \leq \frac{\delta_0}{4T}\ ;$$

this is possible since $f^* = \lim_{\nu\to\infty}(\frac{1}{\|f^\nu\|})f^\nu$ and $g^* = \lim_{\nu\to\infty}(\frac{1}{\|g^\nu\|})g^\nu$. From Proposition 4.3 and the above, for all $\nu \geq \nu_0$ it holds that:

$$(x^* - y^*) \cdot ((\frac{1}{\|f^\nu\|})f^\nu - (\frac{1}{\|g^\nu\|})g^\nu) \leq$$

$$|(x^* - y^*) \cdot ((\frac{1}{\|f^\nu\|})f^\nu - f^*)| + |(x^* - y^*) \cdot ((\frac{1}{\|g^\nu\|})g^\nu - g^*)| + |(x^* - y^*) \cdot (f^* - g^*)|$$

$$\leq \delta_0/2\ .$$

Let ν_1 be sufficiently large so that $\nu_1 \geq \nu_0$ and so that for all $\nu \geq \nu_1$, $\sup_t (x_t^\nu - x_t^*)\ (\frac{f_t^\nu}{\|f^\nu\|} - \frac{g_t^\nu}{\|g^\nu\|}) \leq \frac{\delta_0}{4T}$, and $\sup_t (y_t^\nu - y_t^*)\ (\frac{f_t^\nu}{\|f^\nu\|} - \frac{g_t^\nu}{\|g^\nu\|}) \leq \frac{\delta_0}{4T}$; this is possible since $\{x^\nu\}$ converges to x^* and $\{y^\nu\}$ converges to y^*. We now obtain an estimate:

$$(x^\nu - y^\nu) \cdot ((\frac{1}{\|f^\nu\|})f^\nu - (\frac{1}{\|g^\nu\|})g^\nu) \leq$$

$$|(x^\nu - y^*) \cdot ((\frac{1}{\|f^\nu\|})f^\nu - (\frac{1}{\|g^\nu\|})g^\nu)| + |(x^* - y^*) \cdot ((\frac{1}{\|f^\nu\|})f^\nu - (\frac{1}{\|g^\nu\|})g^\nu)| +$$

$$|(y^\nu - y^*) \cdot ((\frac{1}{\|f^\nu\|})f^\nu - (\frac{1}{\|g^\nu\|})g^\nu)| \leq \delta.$$

This is the desired contradiction. Q.E.D

4.2 Convergence and typical smallness of approximate cores

There are many implications of the fact that large games with effective small groups are market games. The fact that the utility function U is differentiable almost everywhere implies that for "most" large games, the equal-treatment ϵ-core is "small", and for almost all distributions of player types the "limit core" is a singleton set. In this Section we state an extension of the Wooders [99] and Wooders and Zame [114] results that approximate cores of large games with a finite number of player types converge. Rather than the boundedness of marginal contributions of [114] we require only small group effectiveness as in [99]. For the statement of the Theorem, we denote the ϵ-core of a game $[f;(T,\Psi)]$ by $C(f;\epsilon)$ and the set of core payoffs for the game $[f;(T,U)]$ by $\Pi(f)$. The function U is as defined in the preceding section.

Theorem 4.1. Convergence of approximate cores (Wooders [107]): Let (T,Ψ) be a pregame satisfying small group effectiveness. Let $\delta_0 > 0$, $\rho_0 > 0$, and $\epsilon_0 > 0$ be positive real numbers. Then there is an ϵ_1 with $0 < \epsilon_1 \leq \epsilon_0$ and an integer $\eta_7(\delta_0,\rho_0,\epsilon_1)$ such that for all games n with $\|n\| > \eta_7(\delta_0,\rho_0,\epsilon_1)$ and $\frac{n_t}{\|n\|} \geq \rho_0$ for each t,

$$C(n;\epsilon_1) \neq \phi \text{ and}$$

$$\text{dist}\ [C(n;\epsilon_1), \Pi(n)] < \delta_0\ .$$[20]

In the Appendix, we indicate the extension of the proof of the convergence Theorem of [114] to prove Theorem 4.1.

The set $\Pi(f)$ is equivalent to the set of competitive payoffs for the direct market where, for each t, there are f_t participants who each own only one unit of the t^{th} commodity and all participants have the utility function U. Thus Theorem 4.1 shows convergence of the core to competitive payoffs of representing markets.

A note worthy aspect of the above Theorem is the equal-treatment property on the ϵ-core. In Appendix 1 we consider ϵ-cores without this restriction. In this case, small group effectiveness ensures that approximate cores of large games treat most players of the same type nearly equally. Thus, an ϵ-core convergence result can be obtained for the unrestricted ϵ-core: for large games and small ϵ, an (unrestricted) ϵ-core payoff assigns most players nearly their competitive payoffs.

It can be shown that if the percentages of players of each type are bounded away from zero then convergence of the core implies small group effectiveness (Wooders [107]). This suggests that the condition of small group effectiveness defines a boundary of perfect competition.

If a pregame (T,Ψ) has bounded essential group sizes (or a minimum efficient scale of group size) then the market game derived from a pregame has particularly nice properties. These properties may help explain the above results. In addition to the 1-homogeneity and concavity of the utility function U defined in the preceding section, with the assumption of a minimum efficient scale, the function U is "piece-wise linear" – the commodity space $\mathbb{R}^T_+$ can be partitioned into a finite number of cones, and on the domain of any one of these cones, the utility function is linear. More precisely, we have the following Proposition.

[20]see Hildenbrand [51] for a discussion of the Hausdorff distance.

Proposition 4.5. Piece wise linearity of the utility function with exhaustion by bounded sized groups (Winter and Wooders [96]): Let (T, Ψ) be a pregame with a minimum efficient scale of group size, and let U denote the derived utility function. Then U is piece-wise linear, that is,

for some collection of cones, say $C_1, \ldots, C_k$, with $C_k \subset \mathbb{R}^T_+$, with the vertex at the origin for each k, and with the property that $\mathbb{R}^T_+ = \bigcup_{k=1}^{K} C_k$, the function U is a linear function on C_k for each $k = 1, \ldots, K$, that is, for any k and any x, $y \in C_k$ it holds that

$$U(x) + U(y) = U(x + y) \text{ and}$$
$$U(x) \geq 0 \text{ for all } x \in C_k \ .$$

For pregames with a minimum efficient scale of group size, some results are quite immediate and very intuitive. Let f be in the simplex in $\mathbb{R}^T_+$, and suppose that f is in the interior of one of the cones C_k. Then the limiting utility function U is differentiable at f. In the continuum economy with measures of agents of each type t given by f_t, competitive prices (and payoffs) in the limiting market are determined by the slope of the indifference curve of U at the point f (Winter and Wooders [96], Proposition 2). Since the utility function U is differentiable at the point f the competitive payoffs are uniquely determined. From Proposition 4.2 and the fact that for each cone C_k for all population distributions x in the interior of C_k, the competitive payoff vector of the market game where all participants have the utility function U is uniquely determined, we have the following conclusion, discussed in [107].

Proposition 4.6. Typical smallness of approximate cores: Let (T, Ψ) be a pregame with minimum efficient scale of group size B. Let $\delta_0 > 0$ and $\lambda_0 > 0$ be given real numbers. Then there is a subset $S(\delta_0, \lambda_0)$ of the simplex in $\mathbb{R}^T_+$ with the Lebesgue measure of $S(\delta_0, \lambda_0)$ greater than $1 - \lambda_0$, a real number ϵ^*, and an integer $\eta(\delta_0, \lambda_0, \epsilon^*)$, such that for all games $[f; (T, \Psi)]$ with $\|f\| \geq \eta(\delta_0, \lambda_0, \epsilon^*)$ and $(\frac{1}{\|f\|})f \in S(\delta_0, \lambda_0)$,

the ϵ^*-core of $[f; (T, \Psi)]$ is nonempty and contained in a ball of radius less than δ_0.

A similar result can be obtained for large games with effective small groups and "thickness", since the limiting utility function U is differentiable almost everywhere. For exchange economies with money where all agents have the same differentiable utility function, the competitive payoff is unique, and competitive prices are given by the derivatives of the utility function at the total endowment point (with the price of money equal to one).[21] For our case, in a direct market when each agent is endowed with 1 unit of a single commodity, the competitive payoff of an agent can be taken as the price of the commodity he owns. For the case of a pregame with small effective groups the limiting utility function U is differentiable except on a set of measure zero. Asymptotic "typical" uniqueness results follows from the concavity of the limiting utility function U and the fact that it is differentiable almost everywhere.[22]

[21] See Shapley and Shubik [82] for related discussion.
[22] See [107] for further discussion.

4.3 Competitive prices, subsidy-free prices, and the attribute core[23]

A line of literature most closely identified with cost allocation and Aumann-Shapley prices defines characteristic functions on amounts of commodities, and/or prices of commodities, and/or other economically relevant variables. This important line of research, initiated in Zajac [118] and Faulhaber [31], treats the assignment of prices and/or costs to economic variables so that certain desiderata are satisfied.[24] In Wooders [110] the research reported in this paper is applied to problems of the endogenous determination of prices for attributes.

Suppose that (T, Λ) is a pregame. Now, however, interpret profiles as bundles of commodities and/or attributes. The word "attributes" is intended to be more general than the usual connotation of "commodities". For example, an attribute may be the ability to distinguish between fine wines or it may be some private information. When "small amounts of attributes are effective", however, attributes are equivalent to the commodities of general equilibrium theory. This is further discussed in Wooders [108,110]. For the remainder of this Section we will use the word attributes, but the reader may wish to keep in mind that the term "commodities" can be taken as a substitute for "attributes".

In principle, the set of attributes could be a compact metric space as in Mas-Colell [55], or Wooders and Zame [113,114] for example. We discuss only the case where there is a finite number T of types of attributes. We will assume that bundles of attributes are points in Z_+^T, so we can exactly apply all the results that we have obtained and the concepts we've introduced. We assume that *small amounts of attributes are effective*, that is, viewing Λ as the characteristic function of a pregame with types, Λ satisfies small group effectiveness. (We have simply replaced the word "types" by "attributes").

Let $\epsilon \geq 0$ be given and let x be a profile of attributes, $x \in \mathbb{R}_+^T$. Then a (price) vector $p \in \mathbb{R}_+^T$ is in the *attribute ϵ-core* (given the total endowment x) if

$$p \cdot z \geq \Lambda(z) - \epsilon\|z\| \text{ for all } z \leq x \quad \text{and} \quad p \cdot x \leq \Lambda(x) \ .$$

Given a (total) endowment z, we can think of z_t as the number of type t players in the game. The endowment $x \leq z$ represents a subgroup of players with x_t players of type t. The attribute ϵ-core is a natural concept, since it describes situations where commodities form "coalitions". For example, units of the attributes, labor and capital, may be placed in firms (i.e. coalitions) containing units of other productive attributes. Another example is the placement of money into coalitions, mutual funds. Note that the attribute ϵ-core is simply the ϵ-core when we revert to the interpretation of Λ as a function with domain the profiles of players.

From our results on the representation of large games as markets with small effective groups we can define a limiting payoff (or utility) function W to attributes just as we defined the utility function U. The function W is superadditive, 1-homogeneous, and concave.

Recall that Theorem 4.1. states the convergence of approximate cores to competitive payoffs/prices in a market where the player types were the goods. In our current interpretation, the approximate attribute cores converge to competitive prices for attributes. (This is precisely as in the approach above; only the names of the components of profiles have changed.) All of our results for large games apply, including nonemptiness of (attribute)

[23] We are grateful to Ed Zajac and Yair Tauman for helpful discussions on this topic.

[24] See Tauman [93] for a recent survey and Schotter and Schwödiauer [73] for a survey placing more emphasis on the core.

ϵ-cores for large games, core convergence, the representation of games as markets, and asymptotic monotonicity of payoffs (now equivalent to prices for attributes/commodities).

To relate the attribute core to competitive pricing and core convergence in economic models where agents may own bundles of commodities, observe that the pair consisting of the T types of attributes and the function Λ can be regarded as the components of a "pre-economy" where each agent has the same utility (or net revenue) function Λ. To derive an economy from the pre-economy (T, Λ), let N be a finite set, interpreted as a set of agents. Let e be a mapping from N into Z_+^T, where $e(i)$ is interpreted as the endowment of agent i. An *economy* (N, e) is given by the agent set N and the assignment of endowments e. Of course, the payoff function Λ may not be concave and a competitive equilibrium may not exist. However, for large economies, in per capita terms the utility function (for commodities) approaches the function W, and an approximate competitive pricing (of attributes/commodities) exists.

Let (N, e) be an economy. We derive a game from the economy in the usual way. Let S be a coalition in N and let z_S denote the sum of the attributes owned by the members of S. Define $V(S) = \Lambda(z_S)$. Then the pair (N, V) is the game derived from the economy. It is interesting to note that players in the derived game (N, V) are "syndicates" (of attributes) in the attributes game with total endowment z_N. (A syndicate is defined as a group of players which has coalesced into 1 player. In the game on attributes a syndicate is a commodity bundle. See Wooders and Zame [114] for a formal definition of syndicates.)

In exchange economies with money competitive prices are independent of ownership of commodities. This indicates that our convergence and monotonicity results for games apply *immediately* to competitive prices for attributes in economies. Approximate cores of derived games, however, depend on the assignment of control of bundles of attributes to individual players. If assignments of attributes are bounded, the effectiveness of small amounts of attributes of the pre-economy (T, Λ) ensures that approximate cores of economies converge to competitive payoffs and the limiting core payoffs are the sums of the worths of the individual endowments of attributes — the total payoff to each player is (approximately) the value of his endowment at the competitive prices. Equivalently, if the sizes of syndicates are bounded in the attributes game then the ϵ-core payoff to a syndicate in a large game with possibly "many" syndicates is approximately the same as the sum of the payoffs to the syndicate members in a game prior to syndication.[25] The convergence is obtained in [110] for the model discussed here by application of Theorem 4.1 and Aumann's Core-Equilibrium Equivalence Theorem ([6]).[26]

In Wooders [110] an example is provided illustrating that when property rights assignments are unbounded, approximate cores of economies converge to price- equilibrium payoffs but the prices are subsidy-free prices and distinct from competitive prices. Subsidy-free prices are prices for attributes that are feasible and have the property that there exists no alternative price system that is feasible for some group of participants and preferred by all members of the group (c.f. Sharkey and Telser [83], Moulin [59,60], or Wooders [110]).

[25] A related, and more subtle, result is obtained in [114]: if marginal contributions to coalitions are bounded, then the Shapley value of a small syndicate in a large (finite) game is the sum of the Shapley values of the members of the syndicate in the game prior to syndication.

[26] A related proposition was shown in Engl and Scotchmer [30] for sequences of economies with converging distributions of attributes but with assumptions of differentiability of the limiting production function W and of uniform convergence to W.

Formally, let (N, e) be an economy and, for each $S \subset N$, let $e(S) = \sum_{i \in S} e(i)$. A vector $p \in \mathbb{R}^T$ is a *subsidy-free equilibrium price* if

$$\begin{array}{ll} p \cdot e(N) & \leq \Lambda(e(N)) \text{ and} \\ p \cdot e(S) & \geq \Lambda(e(S)) \text{ for all } S \subset N \ . \end{array}$$

For an economy (N, e) where each participant is endowed with one and only one unit of one attribute, a subsidy-free price is an attribute core payoff. It appears that if a pregame on attributes satisfies small scale effectiveness, then approximate cores of large derived economies converge to approximate subsidy-free price payoffs, whether or not property rights assignments are bounded.

5 Continuum Games with Effective Small Groups

Small group effectiveness expresses the idea that all or almost all gains to group formation can be realized by small groups of participants. In a "limit version" of small group effectiveness, all gains to group formation can be realized by groups of measure zero in a game with a continuum of players. A model with effective groups of measure zero, specifically finite groups, has been developed by Mamoru Kaneko and this author [49,50,51].[27] In this section, we describe the model of the continuum with finite groups, state a theorem postulating nonemptiness of the core, and provide some examples. We focus on the model with a finite number of types of players and indicate the extension to a compact metric space of player types.

The purpose of the continuum with finite coalitions is to provide an idealized model of a large game or economy where, just as in finite games or economies, individual participants can interact one with another and within small groups without affecting aggregate outcomes. The negligibility of individual participants and finite groups relative to the total player set suggests an atomless measure space of participants as introduced by Aumann [6] to model situations where individual participants are negligible relative to economic aggregates. That individual participants attempt to pursue their own self-interest, and in doing so, interact with and influence each other, suggests that individuals are atomistic. These sorts of ideas, of a large total player set not subject to the influence of small numbers of players, and atomistic self-interested participants actively engaged in the pursuit of personal gains, appear in early descriptions of competitive economies. These ideas are suggested, perhaps, by Adam Smith [90]. The difficulty is the reconciliation of the apparent paradox of an atomless measure space of players with atomistic individual players and effective small groups. A reconciliation of the apparent paradox is achieved by the adding-up of finite coalitions in a manner consistent with the measure on the total player set. Kaneko and Wooders [51] introduces the concept of measurement-consistent partitions for this purpose.

Most of this Section discusses the model and result of Kaneko and Wooders for continuum games with finite coalitions. We remark, however, that an axiomatization of the core of games with finite coalitions is provided in Winter and Wooders [97]. The games include both finite games and games with a continuum of players and finite coalitions.

[27](and also Hammond – see [41,42] and Winter and Wooders [97].

5.1 Continuum games with a finite set of player types

The pregame construct used in the preceding sections will also be used here. In games with a continuum of players the possibilities open to any finite group are exactly the same as those open to that group in a game with a finite total player set. To describe a game with a continuum of players we must describe the total player set and the permissible partitions of the total player set into finite groups (of measure zero, of course).

Let T be a given number of player types. Let $N = (N_1, \ldots, N_T)$ denote the distribution player types in the total player set, with $\|N\| = 1$. For each t, N_t is the *proportion* of players of type t. Our interpretation is that there is a continuum of players of each type, and N_t is the *measure* of players of type t in the player set. Each player in a game is viewed as an individual. The player is "small" relative to the total player set and also relative to the set of players of the same type, but two players (or the members of any finite group) are the same size and able to meet, face to face, and engage in collective activities. Groups are also small relative to the total player set. Payoffs to (finite) groups are small relative to aggregate payoffs to large masses of players. But the payoff to an individual player is the same size relative to the player as in a finite game.[28]

Just as in the preceding sections, we will denote a group by a profile $f \in Z^T_+$; f_t is again interpreted as the absolute number of players of type t in the group. Clearly, the total player set N, which contains a continuum of players, can form into a continuum of groups. The problem is to partition players into groups in such a way that the "relative scarcities" given by the measures of players of each type in the game N are preserved. To address this problem we first index the collection of all profiles. Since there is a finite number of types and since profiles are vectors of integers, the collection of all profiles is countable. We next assign weights to profiles. These weights determine the proportion of players of each type in each kind of group. The weights are consistent with the proportions, given by the measure, of players of each type.

Throughout the remainder of this Section let (T, Ψ) be a pregame and let N be a continuum player set. Let $\{f^k\}_{k=1}^{\infty}$ be the collection of all profiles. Let $\{\lambda^k\}_{k=1}^{\infty}$ be a countable collection of non-negative real numbers, called *weights*. A collection $\{\lambda^k\}$ of weights is *measurement-consistent* if

$$\Sigma_k \lambda^k f^k = N = (N_1, N_2, \ldots, N_T) \ .$$

A measurement-consistent collection of weights describes a measurement-consistent partition of the players in N into finite groups with profiles in the set $\{f^k\}$. For each t the number $\lambda^k f^k_t / N_t$ is interpreted as the proportion of players of type t in members of the partition with profile f^k. We stress that a profile f^k describes a finite group by the number of players of each type in the group, exactly as in the preceding sections. A measurement-consistent weighting describes a partition of the total player set into a continuum of groups. For any profile f^k with a positive weight λ^k, in the partition there is a continuum of groups with profile f^k.

Example 5.1. A matching model: Let (T, Ψ) be a pregame where $T = 2$. The pregame is a matching pregame with $\Psi(f) = \min(f_1, f_2)$ for each profile f. Let $N = (N_1, N_2)$ and

[28] This sort of description also applies to recent bargaining models of economies with a continuum of agents, c.f. Gale [34,35].

suppose that $N_1 = N_2$. Let $f^1 = (0,1)$, $f_2 = (1,0)$, and $f_3 = (1,1)$. Let λ be some number less than or equal to N_1. Then $\{\lambda^k\}$ is a measurement-consistent collection of weights, where $\lambda^1 = \lambda$, $\lambda^2 = \lambda$, $\lambda^3 = (N_1 - \lambda)$ and $\lambda^k = 0$ for all profiles $f \notin \{f^1, f^2, f^3\}$.

Think of the measure N_t as identical to the interval $[0, N_t)$. Observe that we can associate a partition of the total player measure N with the weighting. One such partition p is given by:

$$p = \{\{i\} : i \in N_1 \cup N_2, i \le \lambda\} \cup \{\{i,j\} : i \in [\lambda, N_1), j \in [\lambda, N_2) \text{ and } i = j\} .$$

This partition leaves the fractions $\lambda/N_1 = \lambda/N_2$ of participants of each type unmatched and matches the fraction $(N_1 - \lambda)/N_1 = (N_2 - \lambda)/N_2$ of players of each type to players of the other type.

In a continuum game with finite coalitions players cooperate only within finite coalitions. The total payoff is the result of such cooperation. To describe feasible payoffs, as in the preceding sections we consider only equal-treatment payoffs. Let $\{f^k\}$ denote the set of all distinct profiles. A payoff vector $x \in \mathbb{R}^T_+$ is *feasible* and there is a measurement-consistent collection of weights $\{\lambda^k\}$ such that $x \cdot f^k \le \Psi(f^k)$ for each k with $\lambda^k > 0$. This definition ensures that, except possibly for a set of measure zero, there is a partition of the total player set into groups so that each group in the partition can achieve the payoff given by x for its membership. In interpretation, if $\lambda^k = 0$ a partition of the total player set consistent with $\{\lambda^k\}$ will contain at most a negligible portion of groups with profile f^k.

A payoff vector x is in the core, called the *f-core*, of the game N if it is feasible and there does not exist a profile f^k such that $\Psi(f^k) < x \cdot f^k$.

The following Theorem is proved in[50], for games without side payments.

Theorem 5.1 Nonemptiness of the core (Kaneko and Wooders [50]): Let (T, Ψ) be a pregame satisfying per capita boundedness. Let N be a player set as described above. Then the f-core of the game is nonempty.

Sketch of the Proof. Suppose, for simplicity, that for each t the measure of players of type t, N_t, is a rational number. Let r be a positive integer and let $rf = (rN_1, \ldots, rN_T)$ be a profile describing a group of players with the same percentage of players of each type as there is in the total player set. The set of weights where the profile rf has weight 1 and all other profiles have weight 0 is measurement-consistent. [For example, suppose $N_1 = 1/3$ and $N_2 = 2/3$. Then we can partition the players into a continuum of finite groups, each consisting of r players of type 1 and $2r$ players of type 2. While groups are finite, there is a continuum of groups with profile $(r, 2r)$.] From Theorem 3.2, given any integer ν there is an integer $r(\nu)$ such that for all $r \ge r(\nu)$ the $(1/\nu)$-core of the finite game with player profile rf is nonempty. Let x^ν be a payoff in the $(1/\nu)$-core. Observe that x^ν is a feasible payoff for the continuum game. Also, $x^\nu \cdot g \ge \Psi(g) + (1/\nu)\|g\|$ for all profiles g with $g \le rf$. It can be shown that the sequence x^ν is bounded, i.e., there is a constant c such that $x^\nu_t \le c$ for each $t = 1, \ldots, T$ and for all ν; otherwise per capita boundedness would be violated. Let x^* be the limit of a converging subsequence of $\{x^\nu\}$. We leave it to the reader to verify that x^* is in the f-core of the game.

We remark that the proof for continuum games without side payments is quite similar.

In the case of games with side payments additional results can be shown. We state some here. For almost all distributions N of player types in the simplex, the f-core consists of a single element, up to sets of measure zero. That is, if x is in the f-core and y is in the f-core, then typically (for almost all proportions of player types) x and y differ only on a set of measure zero. The typical uniqueness can be demonstrated as a consequence of the feature that the limiting utility function U is concave ([105,107]) and approximate cores converge to the f-core (Kaneko and Wooders [49,51]). It can easily be shown that the f-core has the equal treatment property; except possibly for a set of measure zero, all players of the same type are assigned the same payoff by an f-core payoff. As indicated by Proposition 4.3, the f-core satisfies monotonicity.

5.2 Measurement-consistent partitions

The concept of measurement-consistent partitions is central to the model of the continuum with finite coalitions. In the finite type case discussed above, our treatment with weightings of profiles is equivalent to the treatment of Kaneko and Wooders. For situations where the set of types is not finite, the definition of measurement-consistent partitions is more difficult. We provide a general definition of measurement-consistent partitions, which the reader can compare with the definition above.

For the remainder of this subsection let (N, β, μ) be a measure space, where N is a Borel subset of a complete separable metric space, β is the σ-algebra of all Borel subset of N, and μ is a nonatomic measure, with $0 < \mu < \infty$.

Example 5.2 We begin with a simple example. Let $N = [0, 3)$. A measurement-consistent partition is given by

$$p = \{\{i, 1+i\} : i \in [0,1)\} \cup \{\{i\} : i \in [2,3)\}.$$

This satisfies measurement consistency because the mapping $i \to i+1$ of players in $[0,1)$ to their partners is measure preserving. An example of a partition which is not measurement-consistent is given by

$$q = \{\{i, 1+2i\} : i \in [0,1)\}.$$

This partition fails measurement consistency since one-third of the players in the total player set are "matched", in a one to one matching, to the remaining two-thirds of the players.

Let F be the set of all finite subsets of N. Each element S in F is called a *finite coalition* or simply a *coalition*. Let p be a partition of N into finite coalitions. For each integer k define N_k as the subset of players in k-member coalitions in p; we have $N_k = \bigcup_{S \in p\ ,\ |S|=k} S.$ The partition p is *measurement-consistent* if for each positive integer k,

N_k is a measurable subset of N; and

each N_k has a partition into measurable sets $\{N_{kt}\}_{t=1}^{k}$, such that there are measure-preserving isomorphisms $\emptyset_{k1}, \ldots, \emptyset_{k2}, \ldots, \emptyset_{kk}$ from N_{k1} to $N_{k1}, \ldots, N_{kk}$ respectively and $\{\emptyset_{k1}(i), \ldots, \emptyset_{kk}(i)\} \in p$ for all $i \in N_{k1}$.[29]

For any $S \in p$ with $|S| = k$, we have $S = \{\emptyset_{k1}(i), \ldots, \emptyset_{kk}(i)\}$ for some $i \in N_{k1}$. For each integer k, the set N_k consists of all the members of k-player coalitions and N_{kt} consists of the t^{th} members of these coalitions. The measure-preserving isomorphisms dictate that

[29] Let A and B be sets in β. A function Ψ from A and B is a *measure-preserving isomorphism* from A to B if Ψ is 1 to 1, onto, and measurable in both directions, and $\mu(C) = \mu(\Psi(C))$ for all $C \subset A$ with $C \in \beta$.

coalitions of size k should have as "many" (the same measure) first members as second members, as many second members as third members, etc.

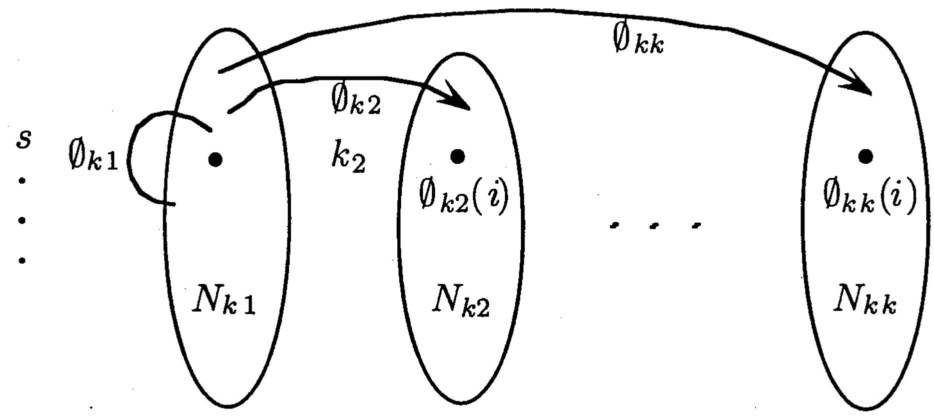

Figure XII.2: Measurement-consistency for k-member coalitions.

Figure XII.2 illustrates measurement consistency. The set of players in k-player coalitions is the union of the sets $N_{k1}, \dots, , N_{kk}$, all of equal cardinality. The isomorphism $\emptyset_{k1}(i)$ maps i to himself; he is the "first member" of the coalition $S = \{\emptyset_{k1}(i), \dots, \emptyset_{kk}(i)\}$. The second member is given by $\emptyset_{k1}(i)$, etc. Kaneko and Wooders [50, Lemma 1] shows that the set of measurement-consistent partitions of a Borel measurable subset is nonempty. In addition to the nonemptiness of the core of a game with finite coalitions, it is also shown that the set of allocations which can be achieved in an exchange economy by trade within finite coalitions coincides with the set of allocations that can be achieved by aggregate trading (the integral of the allocation equals the integral of the endowment, as in Aumann [6]).

With bounded essential group sizes and an additional assumption ensuring that if one player in a coalition can be made better off, all players in the coalition can be made better off, the nonemptiness of the core result of Theorem 5.1 can be obtained when the space of player types is an arbitrary compact metric space. This holds for both games with and without side payments ([51]).

To conclude this section we present a simple example of the f-core of a game with a continuum of players.

Example 5.3. (Kaneko and Wooders [50]). Let $N = [0,3)$ be the total player set, with Lebesgue measure. Each point in the interval is a player. The players in the interval $(0,1]$ are called "women" and the players in the interval $(1,3]$ are called "men". Suppose the marriage of the i^{th} woman and the j^{th} man yields a payoff of $i+j$ utils, while remaining single has a payoff of zero. All other finite coalitions can realize only the total payoffs that could be obtained by partitioning into man-woman coalitions and singleton coalitions. An outcome in the f-core is given by the function

$$x(i) = \begin{bmatrix} 2+i & \text{if } i \in [0,1) \ , \\ 0 & \text{if } i \in [1,2) \\ i-2 & \text{if } i \in [2,3); \end{bmatrix}$$

only the men with high index numbers are married, and the higher the index number of a married player, the higher the index number of his partner.

Figure XII.3 depicts the f-core payoff x.

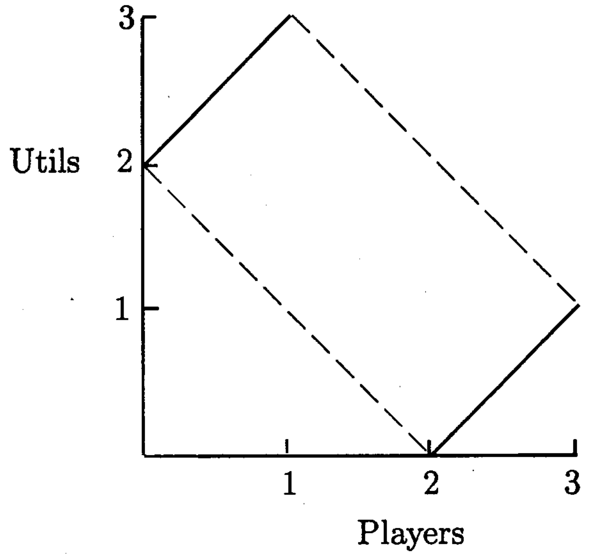

Figure XII.3: A payoff in the f-core of the marriage game.

6 Economies with Effective Small Groups

Economists have applied the concept of the core in several branches of economic theory. For example, the Shapley - Shubik [81] Theorem on the nonemptiness of the core of an assignment game has been used in connection with housing markets (c.f. [48]). The core has also been applied to matching firms and workers (c.f. [21]), and to placement of students and medical interns (c.f., [68]). The equivalence of the core and the set of competitive payoffs is of major importance in the study of competitive markets in general equilibrium theory. In the following, we indicate some relationships of the notion of effective small groups to the asymptotic equivalence of the core and the Walrasian equilibrium, and to economies with shared goods, with asymmetric information, and with coalition production. Special emphasis is given to economics structures with shared goods since they provide a rich framework for the study of small group effectiveness.

6.1 Edgeworth equivalence theorems

Because of the important role of exchange economies in the literature we discuss the relationship of the game-theoretic ideas reported in the preceding section to the equivalence of cooperative outcomes and Walrasian outcomes in exchange economies. The discussion is informal and directed towards establishing connections between core convergence and small group effectiveness. Other literature on core convergence is discussed in Chapter 5.

The model of a continuum game with finite coalitions has been applied to the study of competitive economies. The classic example of a competitive economy is a market with many participants.[30] Since there are many participants, all small relative to the total

[30]See, for example, Stigler [92] for a discussion.

economy, it is argued that each participant will view his own actions as unimportant to others, and therefore each participant will individually (and non-strategically) optimize.

The most widely accepted test of the hypothesis of perfect competition against coalition formation is the convergence of the core to the set of Walrasian payoffs in exchange economies, originating in Edgeworth [26]. The relationship of the core to Edgeworth's contract curve was noted by Shubik [84], and the convergence of the core to the Walrasian allocations was elegantly shown by Debreu and Scarf [24]. Asymptotically, in these models individual participants become negligible relative to the total player set. Since all coalitions are allowed, neither the models nor the results address questions concerning the size of effective coalitions.

One model capturing the idea of a negligible trader is that of Aumann [6]). His model of an exchange economy has a continuum of traders, so that the effect of any one trader is negligible. Coalitions are "large"; all admissible coalitions consist of non-negligible proportions (positive measures) of the total player set. Aumann shows that the core coincides with the set of competitive equilibrium payoffs. In the context of an exchange economy or market with a continuum of participants this result establishes that *even if* large groups of participants act together, the large number of participants will lead to free competition.[31] This is a very important result, and important for the implications of the research we have reported. It will become apparent however that the small group effectiveness property of exchange economies provides an explanation of Aumann's result.

In exchange economies with a continuum of participants all gains to collective activities can be realized by finite groups of participants (Kaneko and Wooders [50]).[32] In economies with many agents, large groups cannot improve upon the set of outcomes attainable by collective activities of small groups — large groups are not essential. Small groups cannot influence broad economic aggregates, including prices — small groups are price takers. Small group effectiveness thus suggests that neither large nor small groups can influence economic aggregates; the core with non-negligible coalitions, the core with finite coalitions (called the f-core), and the Walrasian outcomes are all equivalent (Hammond, Kaneko and Wooders [42]).

The continuum limit results for games and economies with effective small groups have asymptotic analogues. Mas-Colell [56] shows that in exchange economies small groups are effective for improvement — any outcome that can be significantly improved upon can be improved upon by a small coalition.[33] Building on the result that small groups are effective for the achievement of feasible outcomes in exchange economies, Kaneko and Wooders [49] show that small groups are effective for both improvement and feasibility (when there are no widespread externalities) and demonstrate that the f-core is the limit of ϵ-cores when coalitions are constrained to be relatively small.[34] These results together suggest

[31] As pointed out to us by Joseph Ostroy, this depends on our interpretation of the continuum. In Ostroy's interpretation, individuals can influence prices even in a continuum economy unless the economy satisfies a differentiability assumption (see Ostroy [66]).

[32] The exchange economy model in [50] does allow ordinal preferences but does not allow infinite dimensional commodity spaces, the subject of much current interest. It would be of interest to establish conditions on economies with infinite dimensional commodity spaces showing when finite groups of participants are effective. Anderson [4] provides examples indicating this is a subtle problem.

[33] See also Khan [53], where it is shown that in exchange economies relatively small groups are effective for improvement.

[34] The relationship between the continuum with finite coalitions and with coalitions of positive measure is

the equivalence shown in Proposition 3.8 of small group effectiveness for improvement and small group effectiveness for feasibility.

Since finite groups are negligible, it is reasonable to suppose that a finite group views widespread externalities — i.e., externalities determined by the aggregate outcomes of individual behaviours — as independent of its own activities. In the above referenced papers on continuum games with finite groups, the possibility of widespread externalities is allowed. In the presence of widespread externalities, small groups are not effective — there may be significant gains in the coordination of the activities of all members of the economy. Despite that small groups are not effective under these circumstances, it may be that there is no mechanism to coordinate the activities of all participants, and thus only small groups actually form. In this case, even with widespread externalities, the equivalence of the Walrasian equilibrium and the f-core still holds.

In related work, Gretsky, Ostroy, and Zame [40] consider an assignment game with a continuum of players, focusing on the characterization of optimal outcomes as solutions to optimal programming problems. They thus extend the programming characterization for finite economies of Shapley and Shubik [81]. Gretsky, Ostroy, and Zame also relate the "distributional core" (the core with coalitions of positive measure and a statistical or probabilistic interpretation), and "integral" assignments (where each seller sells to at most one buyer and each buyer buys from at most one seller). This is similar to the result discussed above, that the notion of feasibility used in Aumann [6] and the feasibility notion introduced in Kaneko and Wooders [50], where trade can take place only within finite groups, give the same feasible outcomes.

Rather than core-equilibrium equivalence as a criterion for price-taking behaviour, Ostroy [66] discusses the "no-surplus condition", the condition that each agent can be paid his marginal contribution. Roughly, for totally balanced games, no-surplus is assured if the core is a singleton. Since the limiting utility function introduced in Section 4 is concave, and thus differentiable almost everywhere, and approximate cores of large finite games are typically small, a limit model of games or economies with effective small groups as a continuum would "typically" have a core containing only one point. Thus, there is apparently a close relationship between effectiveness of small groups and no-surplus. Since the relationship of small group effectiveness to Ostroy's concept of "no-surplus" has not been formally investigated, we do not discuss this further here.

6.2 Values of large economies and games with effective small groups

In Wooders and Zame [114], using the framework of this paper and assuming boundedness of individual marginal contributions to coalitions, it is shown that for sufficiently large games with sufficiently many players of each type, values are in approximate cores. Recall that boundedness of individual marginal contributions implies small group effectiveness. From results on large games as market games, and the fact that approximate cores converge to limiting Walrasian payoffs, we have the conclusion that the values of large games converge to limiting Walrasian payoffs. As discussed in Section 4, since the Walrasian payoff to the limiting game is typically unique, asymptotically the value payoffs and the Walrasian payoffs usually coincide. It is an open question if small group effectiveness suffices for

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further discussed in Hammond [41].

the result that values are in approximate cores of large games. We refer the reader to a suggestive example in [116].

For a further discussion of values of large economies, see Chapters VII, IX and X in Part B of this volume.

6.3 Economies with public goods and shared goods

The problem of how to determine the optimal level of public good provision is of central interest in public economics. A public good is one with the property that all members of an economy can consume the total output. Examples are radio and television, where an increase in consumption of one person has no effect on the amount supplied to others. As pointed out by Samuelson [70], optimal provision of a public good requires that each participant in the economy reveal his true willingness to pay for the public good. This leads to the "free-rider problem". Each individual has an incentive to under-represent his willingness to pay, and, in consequence, the public good is under-provided.

Tiebout [94] conjectured that when public goods are "local", the free-rider problem disappears, and there is a "market-type", near-optimal outcome. A local public good is a public good subject to exclusion and congestion. As the population jointly consuming a local public good increases in size, eventually the congestion effects outweigh the advantages of sharing costs of providing the good among the membership of a larger population.

In the remainder of this subsection we provide a brief discussion of research on price-taking equilibria and cores in economies with congestible shared goods. We refer the reader to Wooders [112] for a rigorous treatment extending and unifying a number of previous results on large economies with public goods and effective small groups.

There are two problems in applying the test of core convergence to the "Tiebout Hypothesis". The first is that, as has been noted by several authors, the core of an economy with local public goods, or shared goods more generally, may well be empty and a competitive equilibrium may not exist (c.f., [27,67]). It was recognized that this was a serious impediment to the analysis of equilibrium notions for economies with shared goods (c.f. Atkinson and Stiglitz [5]). The other problem is that in economies with shared goods, the "appropriate" notion of a competitive equilibrium may not be immediately apparent. One way to determine whether or not an equilibrium concept is competitive is to ask if it satisfies the core convergence property — the property that the core converges to the equilibrium payoffs. If the core is typically empty, such a procedure cannot be applied.

The problems of the emptiness of the core and an appropriate notion of a competitive equilibrium for economies with public goods have been attacked in a series of papers. Techniques related to those of this paper have been used to study equilibria and cores (both exact and approximate).[35] The public goods are subject to congestion and exclusion.[36] The production possibilities and/or utilities of agents may depend on the numbers of agents of each type with whom the collective goods are jointly produced/consumed, that is, crowding may be *differentiated.*[37] Alternatively, the dependence may be just on the total number of

[35] c.f. [74,98,101,102,104,106].

[36] Models with similarities include McQuire [57], Atkinson and Stiglitz [5], Berglas [16], and Bewley [14].

[37] This has also been called "discriminatory" crowding and "non-anonymous" crowding. The term "discriminatory" may have unintended connotations, and the term "nonanonymous" is inaccurate in this context (only types, not names, of participants are relevant so some anonymity holds). The term "differentiated" is

agents jointly producing and/or consuming the goods, that is, crowding may be *anonymous.* Several notions of equilibrium are considered. Here, we focus on an equilibrium with Lindahl (benefit) prices for the public goods provided within groups and "participation prices" — payments for participation in groups — for agents themselves.[38] These participation prices, which may be positive or negative, are essentially payments from some types of participants in a group to agents of other types to keep those receiving the payments from leaving.[39] Agents who receive positive payments are relatively scarce economically and thus able to command a payment for joining a group. Agents who pay positive prices to belong to groups are relatively abundant economically. In free-entry equilibrium, no entry of new producers of the public goods implies that "profits" must be paid back to participants in equilibrium groups. In the case of constant returns to scale in production (so that there are zero profits) the sum of the participation prices within an equilibrium group is zero, since the benefits produced for group members are consumed by group members. Analogues of the results of Debreu and Scarf [24] and Foley [33] are obtained for a model with possibly several private and public goods and with possible complementarities between agent types (See especially [108,112] and also [106]).

For the special case of anonymous crowding, for all sufficiently large economies all states of the economy with payoffs in the core have groups consisting of agents with the same demands (although all agents in the same group are not necessarily of the same type) (Wooders [98,Theorem 3(iv)]). Again with anonymous crowding, the participation prices are equal shares of the surplus generated by the group, i.e., profit shares; free entry of firms (or of groups) producing the public goods would make any other profit sharing scheme unstable ([98, Definition 10 and Theorem 4, showing equivalence]). In [101] it is shown that no matter how taxes (admission fees to groups) for public goods are determined, if an equilibrium with taxation is stable against entry by profit-maximizing firms and entry is easy, in large economies the taxes must be approximately equal to Lindahl prices times quantities plus group participation prices. (A similar result is obtained in [106] for a broader class of situations.)

The convergence results are of uncertain value without nonemptiness results. For the case of anonymous crowding, conditions showing nonemptiness of the core are given in [98, Theorem 3], and nonemptiness of approximate cores is shown in [101]. For the case of complementarities between agent types, the approximate core theory introduced in [103] is applied in Shubik and Wooders [88] and Wooders [101,104] to show nonemptiness of approximate cores of large economies with shared goods. These nonemptiness results, through the convergence of approximate cores to equilibrium payoffs, enable the test of the Tiebout Hypothesis and of the competitiveness of the equilibrium.

Related core convergence results have been obtained by Conley [19], who shows that when consumers are ultimately satiated with public goods, the core converges to the Lindahl payoffs. (For Conley's model, joint consumption by the group of the whole is optimal —

used since types of participants are analogous to types of differentiated commodities (c.f., Mas-Colell [65]).

[38]The price-taking equilibrium was initially defined in Wooders [98, Definition 10] for situations with anonymous crowding.

[39]An example of such participation prices is the premium paid by university departments to department members who generate externalities for other researchers in the department. All such externalities are both produced and consumed by department members (by definition, for this example), so the net payments must sum to zero.

there is no congestion.) In Conley's model, the Lindahl prices all converge to zero.

Other literature focuses on pricing schemes in economies with small effective groups. Scotchmer and Wooders [74] apply the methods described in this paper and above to an economy with a public good and anonymous crowding. Their equilibrium concept has a single admission price for each type of agent in each group, as is usual in "club theory" (see also Bennett and Wooders [13]). In an equilibrium the admission price for an agent to a group is his Lindahl tax (Lindahl price times quantity). Scotchmer and Wooders obtain similar results for the admission price system to those of Wooders [98] and demonstrate a "Second Welfare Theorem". They also provide an example where in equilibrium agents with different preferences (but the same demands) may share jurisdictions in equilibrium states of the economy.

Barro and Romer [10] emphasize that in a variety of economic models — ski-lifts, amusement parks, labor markets, congested roads — a number of pricing schemes may be consistent with the notion of competitive equilibrium. In economies with shared goods, these pricing schemes may superficially appear quite different from the Walrasian equilibrium. Barro and Romer describe possible co-existence of various sorts of pricing systems, such as lump sum admission prices or admission prices to the facility plus a per-unit-of-use fee (similar to the participation prices and Lindahl pricing for economies with local public goods). As in the coalition production literature (c.f., [46]) or in economies with shared goods Barro and Romer also conclude that when effective groups are small and providers of the goods and services can freely enter the industry, a competitive equilibrium has the property that to the extent wages and prices leave positive surplus, this surplus must be distributed to the workers (more, generally, to the "group members"). Bennett and Wooders [13] stress that even in non-capitalist economies, if firms (or "groups") can freely form, and if participants in the economy are not rewarded according to participation prices[40] (determined by each agent for each group to which he might belong and based on opportunities in other groups), then endogenous divisions of the participants in the economy into groups may not be optimal.

Wooders [112] introduces a model comparable in generality to Debreu and Scarf [63] but with the added features of differentiated crowding and public goods. It is shown that asymptotically the core, the Lindahl equilibrium outcomes, and the admission equilibrium outcomes coincide. Admission pricing has appealing properties and provides an explanation of the asymptotic equivalence of Lindahl outcomes and the core. A group will only accept an additional member if he pays at least the cost he imposes on the group; this places a lower bound on an entrance fee for that potential member. An upper bound on what an individual will pay to join a group is determined by his opportunities elsewhere. With small group effectiveness, in large economies these two bounds come together, thus determining equilibrium admission prices. Asymptotically equilibrium admission prices are Lindahl prices times quantities plus participation prices.

Our conclusion is that economies with public goods and effective small groups are "market-like" and cores and approximate cores converge to competitive equilibrium outcomes with Lindahl pricing within groups that share the public good and with participation prices for agents. While a number of questions are still to be answered, it appears that

[40] These participation prices were called "reservation prices". For situations with nonempty cores, the participation prices are equivalent to core payoffs.

when public goods are local, and economies are large, optimizing individuals reveal their preferences much as they do for private goods. Since an individual cannot affect the prices that prevail, he chooses the community/club/jurisdiction where his wants are best satisfied, subject to his budget constraint.

A number of related models appear in the literature. Ellickson [28] considers local public goods as indivisible private goods and shows the existence of a competitive equilibrium when group composition and size affect only the technology and when there is a small efficient scale of jurisdiction. Silva and Kahn [89] study the effects of deviations from situations where exclusion is costless, in particular the monitoring problem facing a provider of public goods in this situation. Since their model allows the possibility of profit, in an equilibrium with potential entry of competitors the profits must be distributed to the users of the facility. Schweizer [72] studies a model in which communities with immobile populations compete with each other for agents who are mobile; while the model clearly has some relationships to the ones above these have not been fully worked out. A number of other authors have used characterizations of the core to study how various pricing systems lead to outcomes that differ from competitive/core outcomes; consistent with our focus on competitive equilibrium and cooperative solution concepts we do not review these papers here (Indeed, we have not attempted to survey the area of local public goods or shared goods.)

Remark 6.1

Greenberg and Weber [39] consider a different sort of "Tiebout problem". They assume that within each group that forms, all members must pay the same tax. This feature of the model leads to the use of consecutive games. Because of the restrictions on taxation, even if there is no congestion or crowding in public good consumption or production, more than one group may appear in an equilibrium. Unlike much of the "Tiebout literature", the question of optimality of the equilibrium or the convergence to optimal outcomes is not addressed. It seems reasonable to conjecture that if preferences and endowments are constrained to be in some compact metric space and crowding is anonymous, then the core (constructed requiring equal taxation within jurisdictions) converges to an outcome in the core of the economy when taxation is determined endogenously. This conjecture is based on the observation that in the model of Wooders [98], with congestion and/or satiation and anonymous crowding, in a state of the economy in the core (*without* exogenous constraints on taxation), all members of a jurisdiction pay the same tax.

Remark 6.2

There are many other sorts of shared goods. Information is a particularly interesting example. Some recent papers include Allen [2,3], Koutsougeras and Yannelis [54], and Yannelis [117].

6.4 Coalition production

In a coalition production economy, the production possibility set available to a group of participants depends on the membership of the group. Some research studying the convergence of the core to the competitive payoffs includes Böhm [17] and Hildenbrand [43]]. Böhm [17] considered replica economies where the possibilities open to a large group were defined as those available to the group when it divided into groups with profiles no larger than the profile of the original economy. In our language, Böhm made an assumption of "boundedness of essential productive groups". Hildenbrand assumed additivity of the production correspondence. Non-equivalence of the core and the competitive payoffs with "increasing returns" was shown by Sondermann [91] and Oddou [64].

Results on nonemptiness of approximate cores of large games have been used to show nonemptiness of approximate cores of replicated coalition production economies with virtually no assumptions on the production technologies ([88]).

Recently, Florenzano [32], has shown that in a coalition production economy with few restrictions, equivalence of the core and the set of "Edgeworth equilibria" obtains. An "Edgeworth equilibrium" is an attainable allocation whose r-fold replication belongs to the core of the r-fold replica of the original economy, for all integers r. Alternatively, an Edgeworth equilibrium can be defined as an attainable allocation which cannot be blocked by any coalition in which agents participate for possibly only a rational amount of time q for any q in $[0, 1]$. We remark that the approach used ensures that all gains to group formation are realized by the finite groups – no new production opportunities become available as the economy is replicated, so no further gains to scale are possible. With the interpretation of the Edgeworth equilibrium as one in which groups can operate "part-time", all (rational) distributions of group composition are possible in the finite economy and gains to group formation are exhausted. These two properties seem to underlie the results.

A different approach to the study of the concept of equilibrium, initiated by Ichiishi [47], may perhaps be fruitfully extended to large economies. In his model, members cooperate within groups but can also be influenced by members outside the group. Ichiishi combines the concept of the Nash equilibrium and the core to describe social outcomes. A balancedness assumption is made to ensure the existence of an equilibrium. It seems reasonable that in a large society this balancedness assumption could be relaxed, but this has not been demonstrated.

6.5 Demand commitment theory

Another body of related literature is work on "demand commitment" vectors, c.f. Albers [1], Selten [76], Bennett and Wooders [13][41], and Bennett [11][42]. The "demand commitment" theory is an attempt to model observed behaviour suggestive of "price taking" behaviour in coalition formation. The idea is that each player in a game takes as given a set of "payoff demands", one for each of the other players. The demand of a player is then the maximal amount he could realize if he could hire any subset of the remainder of the players at their stated demands. A demand vector (payoff) has the property that the amount stated for each player is his demand, given the demands of the other players. Roughly, a demand vector is "stable" if dependencies are mutual — if player i needs player j to achieve his demand, then j similarly needs i.

Demand commitment theory has the appealing properties of price-taking behaviour — each individual acts independently, and the total demands will be at least as great as the maximal total payoff. Theoretically, there is an obvious unsatisfactory aspect of demand commitment theory. Consider, for example, a 3-person simple majority game. Every pair of players can realize \$1.00 by cooperating. The total payoff to the game is \$1.00. Yet, the set of stable demand vectors includes $(1/2, 1/2, 1/2)$ - stable demand is not necessarily feasible.

[41] Where they are called "equilibrium reservation prices".

[42] Where they are called "aspirations".

The demand commitment approach might be criticized on the basis that in small economies or games, the lack of strategic behaviour is unrealistic. Yet the demand commitment theory was suggested by observations in the laboratory. Moreover, recent research indicates that demand commitment vectors arise as descriptors of noncooperative equilibrium outcomes in theoretical models of coalition formation (c.f., [12,58,76]). These noncooperative models suggest that coalitions which can pay out stable demands to their members will form, for example, *any* 2-person coalition in the simple majority game.

In large games, nonemptiness of approximate cores ensures that some demand commitment vectors (those that are in the core of the balanced cover game) become approximately feasible. Moreover, as we've indicated, games are like markets, where we might expect price-taking behaviour. This may provide some explanation of why the behaviour of subjects in experimental situations resembles price-taking behaviour; it may be a carry over from experiences in markets.

7 Conclusions

The research presented has been motivated by the study of competitive economies. In this concluding section I will attempt to describe the intuition I perceive as underlying the results and the role of small group effectiveness in competitive economies.

In a competitive economy each of the participants is of the opinion that his own transactions do not affect prevailing prices. Each of the participants is aware of his small share of the market and he knows or thinks that no other participant will feel any tangible effects of his actions. In view of his own unimportance to the market outcome, each participant can assume that other participants will not react to his actions. Thus each participant can behave non-strategically in his individual optimizing.

In a private goods economy with a large number of participants and enough substitutability of goods, if one agent refuses to trade with another agent at the competitive price that agent still has a virtually limitless number of alternative trading partners. In an economy with excludable public goods, the total price that a group can successfully demand of a new member is bounded below by the costs the new member would impose on the group, and bounded above by the opportunities of the potential new member to join other groups. Price-taking behaviour arises as a consequence of two features – the individual cannot influence economic aggregates and there is a virtually limitless supply of trading partners/groups. In any economy with effective small groups the same phenomena occur. If an agent cannot get the "right" deal from one group he can go to another. Since effective groups are small, in a large economy there is (at least potentially) a large supply of groups. Thus both individuals and groups are price-takers. Again since effective groups are small, price-taking behaviour, with sufficient opportunities for group formation if it is beneficial, leads to optimal outcomes.[43]

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[43] Novshek and Sonnenschein [63] and other related research reaches similar conclusions using noncooperative game theory in models with production by firms with small efficient scale. Small efficient scale of firms can be related to small group effectiveness.

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App. 1: Approximate Equal Treatment and Other Proofs

We provide the remaining proofs and conclude by showing that small group effectiveness implies an approximate equal-treatment property of the (not-necessarily-equal-treatment) core of a large game.

Proof of Proposition 2.2.[44]

Let $\epsilon > 0$ be given and let n be a game determined by the pregame (T, Ψ). Suppose that the game n is ϵ-balanced. The vector $x = (x_1, \ldots, x_T)$ belongs to the ϵ-core of n if

$$x \cdot n \leq \Psi^*(n) \text{ and} \qquad (a)$$

$$x \cdot f \geq \Psi(f) - \epsilon \|f\| \qquad (b)$$

for every subprofile f of n. We consider the linear programming problem of minimizing $x \cdot n$ subject to the system of linear inequalities given by (b). A solution to this linear programming problem will belong to the ϵ-core of n if the minimum of the objective function is less than or equal to $\Psi^*(n)$. In order to determine if this is so, we consider the dual linear programming problem with variables w_f. The dual linear programming problem consists of maximizing the linear function $\Sigma_f\, w_f(\Psi(f) - \epsilon\|f\|)$ subject to the system of linear equations

$$w_f \geq 0 \text{ for all subprofiles } f \text{ of } n, \text{and}$$

$$\Sigma_f\, w_f f = n\ ,$$

where the sum is over all subprofiles f of n. Note that the dual constraints are equations which are not inequalities since the primal variables , i.e., the x_t, are not restricted in sign.

[44]The proof is a modification of a proof of the Bondareva-Shapley result in [45].

Denote a solution to the dual problem by $\{w^*_f\}$. The duality theorem of linear programming states that, when x is a solution to the primal,

$$x \cdot n = \Sigma_f\; w^*_f(\Psi(f) - \epsilon\|f\|).$$

Consider the family β of those subprofiles for which $w^*_f > 0$. The constraints of the dual require that this family is balanced with the weights w^*_f. Therefore we have $x \cdot n = \Sigma_f\; w^*_f(\Psi(f) - \epsilon\|f\|) \leq \Psi^*(n)$ since the game is ϵ-balanced, and x is in the ϵ-core.

Now suppose that the game n has a nonempty ϵ-core. Let x be in the ϵ-core of the game. For all subprofiles f of n we then have

$$\Psi(f) - \epsilon\|f\| \leq x \cdot f\;.$$

Let β be any balanced collection of subprofiles of n with weights w_f for $f \in \beta$. For each subprofile f we have

$$w_f(\Psi(f) - \epsilon\|f\|) \leq w_f x \cdot f \text{ and}$$

$$\Sigma_f\; w_f(\Psi(f) - \epsilon\|f\|) \leq \Sigma_f\; w_f x \cdot f = x \cdot (\Sigma_f\; w_f f) = x \cdot n \leq \Psi^*(n)$$

since x is in the ϵ-core. This implies that the game is ϵ-balanced.

Q.E.D.

Proof of Proposition 3.1

Let (T, Ψ) be a pregame with a minimum efficient scale bound B. Let $[n; (T, \Psi)]$ be a game and let x be a feasible payoff that can be improved upon by some group f with $f \leq n$, that is, $x \cdot f < \Psi(f)$. From our choice of f and x and from the minimum efficient scale assumption there is a balanced collection $\{f^k\}$ of subprofiles of f with $\|f^k\| \leq B$ for each k, and some set of balancing weights w_k for f^k such that

$$\Sigma_k\; w_k \Psi(f^k) \geq \Psi(f) > x \cdot (\Sigma_k\; w_k f^k) = \Sigma_k\; w_k(x \cdot f^k)\;.$$

It follows that there is at least one group f^k for which $\Psi(f^k) > x \cdot f^k$. Since $\|f^k\| \leq B$ it follows that (T, Ψ) satisfies exhaustion of improvement possibilities by groups bounded in size by B.

Let (T, Ψ) be a pregame satisfying exhaustion of improvement possibilities by groups bounded in size by B. Let Λ be the function on profiles defined by

$$\Lambda(f) = \max \Sigma_k\; w_k \Psi(f^k)$$

where $\{f^k\}$ is a balanced collection of subprofiles of f with weights w_k for f^k and with $\|f^k\| \leq B$ for each k and where the maximum is taken over all such balanced collections. Note that (T, Λ) has a minimum efficient scale bound B and any derived game $[n; (T, \Psi)]$ is balanced.

Suppose that B is not a minimum efficient scale bound for (T, Ψ). Then there is a profile f such that $\Psi^b(f) > \Lambda(f)$. Let x be in the core of $[f; (T, \Lambda)]$, so x is feasible for the game $[f; (T, \Lambda)]$ and for the game $[f; (T, \Psi)]$. Since x cannot be improved upon by any group h with $\|h\| \leq B$, from the boundedness of sizes of improving groups x cannot be improved upon in the game $[f; (T, \Psi)]$. Therefore x is in the core of the game $[f; (T, \Psi)]$. But this implies that $\Psi^b(f) = x \cdot f = \Lambda(f)$, a contradiction.

Q.E.D.

Proof of Proposition 3.6: Let (T, Ψ) be a pregame satisfying boundedness of essential group sizes with bound B. Suppose that statement (a) of the Proposition is false. Then, given some $\epsilon_0 > 0$, for each integer ν there is a profile f^ν with that properties that $\|f^\nu\| \geq \nu$ and f^ν has an empty (weak) ϵ_0-core.

We first obtain a bound on $\Psi^b(f^\nu) - \Psi^*(f^\nu)$. Let $\{g^k\}$ denote the collection of all profiles with $\|g^k\| \leq B$. Since (T, Ψ) satisfies boundedness of essential group sizes with bound B, for each ν there is a balanced collection β^ν of subprofiles of f^ν where each $g \in \beta^\nu$ is in $\{g^k\}$ and

$$\Psi^b(f^\nu) = \sum_{k:g^k \in \beta^\nu} w_k^\nu \Psi(g^k)$$

for some collection of balancing weights $\{w_k^\nu\}$. For each k such that $g^k \notin \beta^\nu$ define $w_k^\nu := 0$. We then have $\Sigma_k\, w_k^\nu g^k = f^\nu$ and $\Psi^b(f^\nu) = \Sigma_k\, w_k^\nu \Psi(g^k)$.

Since there is a finite number of distinct profiles in the set $\{g^k\}$, we can write each w_k^ν as an integer plus a fraction, say $w_k^\nu = \ell_k^\nu + q_k^\nu$ where $q_k^\nu \in [0, 1)$. Intuitively, we think of the players $\Sigma_k\, q_k^\nu g^k$ as "leftovers" –it may not be possible to fit these players (or any subprofile of them) into groups that can achieve core payoffs (for the core of the balanced cover game) for their memberships.

Since the pregame satisfies boundedness of essential group sizes, for each f^ν there are non-negative integers r_k^ν satisfying $\Sigma_k\, r_k^\nu g^k = f^\nu$ and $\Sigma_k\, r_k^\nu \Psi(g^k) = \Psi^*(f^\nu)$.

Now consider the difference $\Psi^b(f^\nu) - \Psi^*(f^\nu)$. Let K denote the cardinality of the set $\{g^k\}$ of profiles with norm less than or equal to B and let M be a positive number with $M > \max\limits_k \Psi(g^k)$. Note that $\Psi^*(f^\nu) \geq \Sigma_k\, \ell_k^\nu \Psi(g^k)$ from the fact that $f^\nu \geq \ell_k^\nu g^k$ and from the definition of Ψ^* (in particular, its superadditivity). Since $\Psi(g) \geq 0$ for all profiles g, it follows that:

$$\begin{aligned}\Psi^b(f^\nu) - \Psi^*(f^\nu) &= \Sigma_k\, w_k^\nu \Psi(g^k) - \Sigma_k\, r_k^\nu \Psi(g^k)\\ &\leq \Sigma_k\, w_k^\nu \Psi(g^k) - \Sigma_k\, \ell_k^\nu \Psi(g^k) = \Sigma_k\, q_k^\nu \Psi(g^k)\\ &\leq KM\,.\end{aligned}$$

Let x^ν be in the core of the balanced cover game $[f^\nu, (T, \Psi^b)]$. We construct another payoff which will belong to the ϵ_0-core of $[f^\nu; (T, \Psi)]$ for all sufficiently large ν. Define the payoff y^ν for each ν by $y^\nu = x^\nu - (\epsilon_0/B)1_T$.

We claim y^ν cannot be significantly improved upon by any subprofile of f^ν. Suppose, on the contrary, that there is a profile h with $h \leq f^\nu$ and with $\Psi(h) > y^\nu \cdot h + \epsilon_0\|h\|$. From boundedness of essential group sizes there is a partition $\{h^k\}$ of h satisfying $\Sigma_k\, \Psi(h^k) = \Psi(h)$ and, for each h^k in the partition, $\|h^k\| \leq B$. From the inequality $\Psi(h) > y^\nu \cdot h + \epsilon_0\|h\|$ it follows that $\Sigma_k\, \Psi(h^k) > y^\nu \cdot (\Sigma_k\, (h^k + \epsilon_0\|h^k\|)) = \Sigma_k\, y^\nu \cdot (h^k + \epsilon_0\|h^k\|)$. We then have, for at least one h^k, that

$$x^\nu \cdot h^k \geq \Psi(h^k) \text{ (since } x^\nu \text{ is in the core of the balanced cover of } f^\nu),$$

$$> y^\nu \cdot h^k + \epsilon_0\|h^k\| \text{ (since } h^k \text{ can improve upon } y^\nu \text{ by at least } \epsilon_0 \text{ per capita)},$$

$$= x^\nu \cdot h^k - \frac{\epsilon_0}{B} h^k \cdot 1_T + \epsilon_0\|h^k\| \text{ (by construction of } y^\nu),$$

$$\geq x^\nu \cdot h^k (\text{since } h^k \cdot 1_T = \|h^k\| \leq B)$$

which is a contradiction. We next show that for all sufficiently large games y^ν is feasible.

Since $x^\nu \cdot f^\nu = \Psi^b(f^\nu)$ and from the construction of y^ν, it follows that

$$\Psi^*(f^\nu) - y^\nu \cdot f^\nu$$

$$\geq \Psi^*(f^\nu) - (\Psi^b(f^\nu) - \frac{\epsilon_0}{B} 1_T \cdot f^\nu)$$

$$\geq -KM + \frac{\epsilon_0 \|f^\nu\|}{B}$$

which, for all ν sufficiently large, is positive. Therefore for all sufficiently large y^ν is feasible and cannot be ϵ_0-improved upon (per capita) and for all ν sufficiently large y^ν is in the ϵ_0-core of f^ν. This is the required contradiction.

To prove part (b) of the Proposition, we use the same proof up to (and including) the definition of y^ν. However, we choose $\epsilon^* \leq \epsilon_0$, $\epsilon^* > 0$ and sufficiently small so that $\Psi(\chi^t) > \epsilon^*$ for each t. We then show nonemptiness of the strong ϵ^*-core; this suffices since the ϵ^*-core is contained in the ϵ_0-core for any $\epsilon^* \leq \epsilon_0$.

We claim that under the conditions of (b), y^ν is in the strong ϵ^*-core for all sufficiently large games. As shown above, y^ν is feasible for $[f^\nu; (T, \Psi)]$ for each sufficiently large ν. We need to show that y^ν cannot be significantly improved upon by any subprofile of f^ν. Suppose, on the contrary, that there is a profile h with $h \leq f^\nu$ and with $\Psi(h) > y^\nu \cdot h + \epsilon^*$. We first show that this implies $\|h\| \leq B$.

If $y^\nu \cdot h > 0$ it is immediate that $\|h\| \leq B$. If $y^\nu \cdot h \leq 0$ then for at least one type t with $h_t > 0$ it holds that $y^\nu_t \leq 0$. To obtain a contradiction we suppose that for some type t, $0 \geq y^\nu \cdot (\chi^t) = (x^\nu - \frac{\epsilon^*}{B} 1_T) \cdot (\chi^t) = \frac{\epsilon^*}{B} \geq x^\nu_t - \epsilon^*$. Since $\Psi(\chi^t) - \epsilon^* > 0$, if follows that $\Psi(\chi^t) > x^\nu_t$. This is a contradiction, since χ^t can improve upon x^ν (and x^ν is in the core of $[f^\nu; (T, \Psi^b)]$). From the contradiction, we can suppose that $\|h\| \leq B$.

We now have that

$$x^\nu \cdot h \geq \Psi(h) \text{ (since } x^\nu \text{ is in the core of the balanced cover of } f^\nu),$$

$$> y^\nu \cdot h + \epsilon^* \text{ (since } h \text{ can improve by at least } \epsilon^*)$$

$$= x^\nu \cdot h - \frac{\epsilon^*}{B} h \cdot 1_T + \epsilon^* \text{ (by construction of } y^\nu),$$

$$\geq x^\nu \cdot h \text{ (since } h \cdot 1_T = \|h\| \leq B)$$

which is a contradiction. Therefore, for all sufficiently large ν, y^ν is in the strong ϵ^*-core, and therefore in the strong ϵ_0-core.

Q.E.D.

Proof of Proposition 3.7: Let (T, Ψ) be a pregame with bounded per-capita payoffs. Suppose that the Proposition is false. Then there are real numbers ρ_0 and ϵ_0 and a sequence of profiles $\{f^\nu\}$ such that:

$\|f\| \to \infty$ as $\nu \to \infty$;

for each t, either $\frac{f^\nu_t}{\|f^\nu\|} \geq \rho_0$ or $f^\nu_t = 0$; and

for each f^ν, for every partition $\{f^{\nu k}\}$ of f^ν with $\|f^{\nu k}\| < \nu$ for each k,

$\Psi^*(f^\nu) - \Sigma_k\ \Psi(f^{\nu k}) > 3\epsilon_0\|f^\nu\|$.

Without loss of generality we can suppose that $\frac{f_t^\nu}{\|f^\nu\|} \geq \rho_0$ for all $\nu = 1, \ldots$ and for all $t = 1, \ldots, T$.

By passing to a subsequence if necessary, we can suppose that the sequence $\{(\frac{1}{\|f^\nu\|})f^\nu\}$ converges, say to $f \in \mathbb{R}^T_{++}$. Again passing to a subsequence if necessary, from per-capita boundedness we can suppose that the sequence $\{\frac{\Psi^*(f^\nu)}{\|f^\nu\|}\}$ converges, say to the real number L. Since the sequence $\{\frac{\Psi^*(f^\nu)}{\|f^\nu\|}\}$ converges, there is an integer ν_0 sufficiently large so that for all $\nu \geq \nu_0$ it holds that

$$\left|\frac{\Psi^*(f^\nu)}{\|f^\nu\|} - L\right| < \epsilon_0 \text{ and}$$

$$\left|\frac{\Psi^*(f^\nu)}{\|f^\nu\|} - \frac{\Psi^*(f^{\nu_0})}{\|f^{\nu_0}\|}\right| < 2\epsilon_0 \ .$$

Let ν_1 be sufficiently large so that for each $\nu \geq v_1$, for some integer r_ν and profile m^ν it holds that

$$f^\nu = r_\nu f^{\nu_0} + m^\nu \text{ , and}$$

$$\frac{\|m^\nu\|}{\|f^\nu\|}(L + \epsilon_0) \leq \epsilon_0;$$

as is shown by Wooders and Zame [114, Lemma 2] this is possible when the percentage of players of each type is bounded away from zero. From the above inequalities, for all ν sufficiently large we obtain the estimate:

$$\left|\frac{\Psi^*(f^\nu)}{\|f^\nu\|} - r_\nu\frac{\Psi^*(f^{\nu_0})}{\|f^\nu\|}\right|$$

$$\leq \left|\frac{\Psi^*(f^\nu)}{\|f^\nu\|} - \frac{\Psi^*(f^{\nu_0})}{\|f^{\nu_0}\|}\right| + \left|\frac{\Psi^*(f^{\nu_0})}{\|f^{\nu_0}\|} - r_\nu\frac{\Psi^*(f^{\nu_0})}{\|f^\nu\|}\right|$$

$$\leq 2\epsilon_0 + \frac{\Psi^*(f^{\nu_0})}{\|f^{\nu_0}\|}\frac{\|m^\nu\|}{\|f^\nu\|}$$

$$\leq 2\epsilon_0 + (L + \epsilon_0)\frac{\|m^\nu\|}{\|f^\nu\|} \leq 3\epsilon_0 \ .$$

This yields a contradiction since for all ν sufficiently large it follows that

$$\Psi^*(f^\nu) - r_\nu\Sigma_k\ \Psi(f^{\nu_0}) - \Sigma_t\ m_t^\nu\Psi(\chi^t) \leq 3\epsilon_0\|f^\nu\| \ .$$

We leave the other direction to the reader.

Q.E.D.

Proof of Proposition 3.8: Let (T, Ψ) be a pregame satisfying small group effectiveness for improvement. Suppose that the pregame does not satisfy effectiveness for feasibility (small group effectiveness). Then there is a sequence of profiles $\{f^\nu\}$ with $\|f^\nu\| \to \infty$, and a real number $\epsilon_0 > 0$ such that, for each ν, for any partition $\{f^{\nu k}\}$ of f with $\|f^{\nu k}\| \leq \nu$ for each k it holds that:

$$\Psi^*(f^\nu) - \Sigma_k \, \Psi(f^{\nu k}) > 2\epsilon_0 \|f^\nu\| \, .$$

Let $\eta_6(\epsilon_0)$ be the integer given in the definition of effectiveness for improvement. Define another pregame (T, Λ) be setting $\Lambda(f) = \max_P \sum_{g \in P} \Psi(g)$ where the maximum is taken over all partitions P with $\|g\| \leq \eta_6(\epsilon_0)$ for all g in P. Note that (T, Λ) has a minimum efficient scale with bound B. From Proposition 2.2 and 3.6 it holds that for all ν sufficiently large $\left|\frac{\Lambda^b(f^\nu)}{\|f^\nu\|} - \frac{\Lambda^*(f^\nu)}{\|f^\nu\|}\right| < \epsilon_0$.

Suppose that for arbitrary large terms ν it holds that $\left|\frac{\Psi^b(f^\nu)}{\|f^\nu\|} - \frac{\Lambda^*(f^\nu)}{\|f^\nu\|}\right| < \epsilon_0$. From this relationship, since $\Psi^b(f^\nu) \geq \Psi^*(f^\nu) \geq \Lambda^*(f^\nu)$, it follows that for arbitrarily large terms ν, $\left|\frac{\Psi^*(f^\nu)}{\|f^\nu\|} - \frac{\Lambda^*(f^\nu)}{\|f^\nu\|}\right| < \epsilon_0$ and, for some partition $\{f^{\nu k}\}$ of f^ν into subprofiles where $\|f^{\nu k}\| \leq \eta_6(\epsilon_0)$ for each $f^{\nu k}$ in the partition, $\Psi^*(f^\nu) - \Sigma_k \, \Psi(f^{\nu k}) < \epsilon_0 \|f^\nu\|$; a contradiction. Therefore, passing to a subsequence if necessary, we suppose that, for all ν, $\left|\frac{\Psi^b(f^\nu)}{\|f^\nu\|} - \frac{\Lambda^*(f^\nu)}{\|f^\nu\|}\right| > \epsilon_0$.

For each ν let x^ν denote a payoff in the core of $[f^\nu; (T, \Lambda^b)]$. From the assumption that $\left|\frac{\Psi^b(f^\nu)}{\|f^\nu\|} - \frac{\Lambda^*(f^\nu)}{\|f^\nu\|}\right| > \epsilon_0$ we claim it follows that for all ν sufficiently large x^ν is feasible for $[f^\nu; (T, \Psi)]$. If x^ν is not feasible for $[f^\nu; (T, \Psi)]$, then $x^\nu \cdot f^\nu = \frac{\Lambda^b(f^\nu)}{\|f^\nu\|} > \frac{\Psi^*(f^\nu)}{\|f^\nu\|}$. But since $\left|\frac{\Lambda^b(f^\nu)}{\|f^\nu\|} - \frac{\Lambda^*(f^\nu)}{\|f^\nu\|}\right| < \epsilon_0$ and $\Psi^*(f) \geq \Lambda^*(f)$, it follows that $\left|\frac{\Psi^*(f^\nu)}{\|f^\nu\|} - \frac{\Lambda^*(f^\nu)}{\|f^\nu\|}\right| > \epsilon_0$. If this holds for arbitrarily large ν we again have a contradiction to the supposed properties of $\{f^\nu\}$. Therefore for all ν sufficiently large $\Psi^*(f^\nu) \geq \Lambda^b(f^\nu)$ and x^ν is feasible for $[f^\nu; (T, \Psi)]$.

We next show that x^ν is in the ϵ_0-core of $[f^\nu; (T, \Psi)]$ for all ν sufficiently large. If x^ν is not in the ϵ_0-core of $[f^\nu; (T, \Psi)]$ then, from effectiveness for improvement, there is a profile g with $\|g\| \leq \eta_6(\epsilon_0)$ and $\Psi(g) > x^\nu \cdot g + \frac{\epsilon_0}{2}$. This, however, contradicts the assumption that x^ν is in the core of $[f^\nu; (T, \Lambda^b)]$. Therefore x^ν is in the ϵ_0-core of $[f^\nu; (T, \Psi)]$.

We now have that $\Psi^b(f^\nu) - x^\nu \cdot f^\nu \leq \epsilon_0 \|f^\nu\|$ from Proposition 2.2. Since $x^\nu \cdot f^\nu = \Lambda^b(f^\nu)$, $\Psi^b(f^\nu) \geq \Psi^*(f^\nu)$, and $|\Lambda^b(f^\nu) - \Lambda^*(f^\nu)| < \epsilon_0 \|f^\nu\|$ it follows that $|\Psi^*(f^\nu) - \Lambda^*(f^\nu)| \leq 2\epsilon_0 \|f^\nu\|$. This contradicts our suppositions on the sequence $\{f^\nu\}$ and completes the first part of the proof.

To prove the other direction, we again will suppose the implication does not hold and obtain a contradiction. Let (T, Ψ) be a pregame satisfying effectiveness of small groups (for feasibility). Suppose that the pregame does not satisfy effectiveness for improvement. Then there is a real number $\epsilon_0 > 0$, a sequence of games $\{f^\nu\}$ and a corresponding sequence of payoffs $\{x^\nu\}$ such that, for each ν, x^ν is a feasible payoff for f^ν, x^ν is not in the ϵ_0-core of f^ν, and for all profiles $g \leq f^\nu$ with $\|g\| \leq \nu$, $\Psi(g) \leq x^\nu \cdot g + \frac{\epsilon_0}{2}\|g\|$.

Let $\eta_3(\frac{\epsilon_0}{2})$ be the integer given in the definition of small group effectiveness of the pregame (T, Ψ) and choose $\nu > \eta_3(\frac{\eta_0}{2})$. Since x^ν is not in the ϵ_0-core of f^ν it holds that there is some profile h such that $\Psi(h) > x^\nu \cdot h + \epsilon_0\|h\|$. From effectiveness for feasibility and the choice of $\eta_3(\frac{\epsilon_0}{2})$ there is a partition $\{h^k\}$ of h such that $\|h^k\| \leq \eta_3(\frac{\epsilon_0}{2})$ for each k and $\Sigma_k(\Psi(h^k) + \frac{\epsilon_0}{2}\|h^k\|) \geq \Psi(h)$. But $\Psi(h) > x^\nu \cdot h + \epsilon_0\|h\| = \Sigma_k(x^\nu \cdot h^k + \epsilon_0\|h^k\|)$. This implies that for at least one k, $\Psi(h^k) > x^\nu \cdot h^k + \frac{\epsilon_0}{2}\|h^k\|$, the required contradiction.

Q.E.D.

Proof of Theorem 4.1. We indicate how the result follows from Wooders and Zame [114, Theorem 4]. Suppose the claim of the Theorem is false. Then there are real numbers $\delta_0 > 0$ and $\rho_0 > 0$, a sequence of profiles $\{f^\nu\}$ satisfying the condition that $\|f^\nu\| \to \infty$ and $f^\nu_t/\|f^\nu\| > \rho_0$ for each t, and a sequence of real numbers $\{\epsilon^\nu\}$ such that $\lim_{\nu\to\infty} \epsilon^\nu = 0$ and, for each ν, either:

for each ν' there is a $\nu^* \geq \nu'$ such that

(a) $C(f^{\nu^*}, \epsilon^{\nu'}) = \phi$ or

(b) $\text{dist}[C(f^{\nu^*}; \epsilon^{\nu'}), \Pi(f^{\nu^*})] > \delta_0$.

From Theorem 3.3 we can exclude the first possibility. Without loss of generality we may assume that $\{\frac{1}{\|f^\nu\|}f^\nu\}$ converges. For any vector h in the simplex in $\mathbb{R}^T_+$ the set $\Pi(h)$ equals the set $C(h)$ defined in Wooders and Zame [114, Theorem 4]. To prove the Theorem we now need only obtain a contradiction to the Wooders and Zame result. Looking at the proof in their paper, we see that we need only to show that, under small group effectiveness, there is a real number A such that if $x \in C(f^\nu, \epsilon)$, then for each t,

$$x_t \leq \Psi(f^\nu)/f^\nu_t \leq A\|f^\nu\|/f^\nu_t$$

Since small group effectiveness implies per capita boundedness we can take A to be the bound on per capita payoffs. Following the proof in [114], we reach a contradiction.

Q.E.D.

We next redefine the ϵ-core so that players who are of the same type may be treated unequally. Let (T, Ψ) be a pregame. Given any profile n define the player set $N = \{(t,q) : t = 1, \ldots, T \text{ and } q = 1, \ldots, n_t \text{ for each } t\}$. Let x be a function from N to $\mathbb{R}$, called a *payoff function*. For each nonempty subset S of N define $x(S) = \Sigma_{(t,q)\in S} x^{tq}$. For any $\epsilon \geq 0$ a payoff function x is in the *ϵ-core* of the derived game with player set N if $x(N) \leq \Psi^*(n)$ and if, for all nonempty subsets $S \subset N$ (*coalitions*), $x(S) \geq \Psi(s) - \epsilon\|s\|$, where s is the profile given by:

$$s_t = |\{(t,q) \in S : q = 1, \ldots, s_t\}|$$

for each t.

For replications rn of the game n, define the set N_r by $N_r = \{(t,q) : t = 1, \ldots, T \text{ and } q = 1, \ldots, rn_t \text{ for each } t\}$. Clearly, the above definitions of a payoff function and the ϵ-core can be applied to replications of a game.

Proposition A.1.1.[45] Let (T, Ψ) be a pregame satisfying per capita boundedness and let n be a game. Given any real numbers $\delta > 0$ and $\lambda > 0$ there is a real number ϵ^* and an

[45]This Proposition originally appeared in [99].

integer r^* such that for each $\epsilon \in [0, \epsilon^*]$ and for any $r \geq r^*$, if $x \in \mathbb{R}^N$ is in the ϵ-core of rn then

$$|\{(t,q) : |x^{tq} - z_t| > \delta\}| < \lambda r ,$$

for each t, where $z_t = \frac{1}{rn_t}\sum_{q=1}^{rn_t} x^{tq}$, the average payoff received by players of type t.

Proof of Proposition A.1.1: Given real numbers λ and δ greater than zero, select ϵ^*, r^*, and $r_0 \leq r^*$ so that:

(a) $\frac{\epsilon^* r_0}{r^*} < \frac{\lambda}{2\|n\|}$;

(b) $\epsilon^* > 0$ and $\epsilon^* < \min_t\{\frac{\lambda\delta}{4\|n\|}, \frac{\delta n_t}{2\|n\|}\}$ where the minimum is over all t with $n_t \neq 0$; and

(c) for all $r \geq r_0$, $\left|\frac{\Psi^*(rn)}{r\|n\|} - \frac{\Psi^b(r_0 n)}{r_0\|n\|}\right| \leq \epsilon^*$.

Since $\lambda > 0$ and $\delta > 0$, and $\left|\frac{\Psi^*(rn)}{r\|n\|} - \frac{\Psi^b(r_0 n)}{r_0\|n\|}\right| \to 0$ as $r \to \infty$ and $r_0 \to \infty$, such a selection is possible.

Select $r \geq r^*$ and let x be in the ϵ^*-core of rn. For each t, define z_t as in the statement of the Proposition. It can be verified that z is in the equal-treatment ϵ^*-core of the game rn (the ϵ-core is convex). It follows then that for all profiles $s \leq rn$, $z \cdot s \geq \Psi(s) - \epsilon^*\|s\|$ and $z \cdot rn \leq \Psi^*(rn)$.

It is convenient to establish the convention that for each coalition $S \subset N_r$, S_t denotes the subset of players in S of type t, i.e., $S_t = \{(\hat{t}, q) : (\hat{t}, q) \in S \text{ and } \hat{t} = t\}$ for each $t = 1, \ldots, T$. We define the *profile of a coalition* S by $s \in Z_+^T$ with i^{th} component given by $|S_t|$ for each t (where $|S_t|$ denotes the cardinal number of the set).

Select a subset W of N_r so that the profile of W is $r_0 n$ and W contains the "worst-off" players of each type; i.e., if $(t,q) \notin W$ then $x^{tq} \geq x^{tq'}$ for all q' with $(t,q') \in W$. Suppose that, for some type t^*,

$$|W \cap \{(t^*, q) \in N_r : x^{t^*q} < z_{t^*} - \delta\}| = r_0 n_{t^*};$$

i.e. all players of type t^* in W receive less than the average payoff for players of that type minus δ. We then have

$$\Psi^b(r_0 n) - \epsilon^* r_0\|n\| \leq x(W) < r_0(z \cdot n) - \delta r_0 n_{t^*} \leq r_0\frac{\Psi^*(rn)}{r} - \delta r_0 n_{t^*} .$$

The first inequality follows from the fact that x in in the ϵ-core of N_r. The second follows from the facts that $z_{t^*} \geq x^{t^*q} - \delta$ for each q with (t^*, q) in W_t and $x(W_t) \leq r_0 z_t n_t$ for each t. The final inequality is from the feasibility of z; $z \cdot rn \leq \Psi^*(rn)$. It now is apparent that the following relationships hold:

$$\Psi^b(r_0 n) - \epsilon^* r_0\|n\| < r_0\frac{\Psi^*(rn)}{r} - \delta r_0 n_{t^*} .$$

Subtracting $\Psi^b(r_0 n)$ from both sides of the expression, adding $\delta r_0 n_{t^*}$ to both sides, and dividing by $r_0\|n\|$ we obtain

$$\frac{\delta n_{t^*}}{\|n\|} - \epsilon^* < \frac{\Psi^*(rn)}{\|rn\|} - \frac{\Psi^b(r_0 n)}{\|r_0 n\|} .$$

From (b) above, $\frac{\delta n_{t^*}}{\|n\|} - \epsilon^* > \epsilon^*$ which, along with the preceding expression, implies that $\epsilon^* < \frac{\Psi^*(rn)}{\|rn\|} - \frac{\Psi^b(r_0 n)}{\|r_0 n\|}$, a contradiction to (c). Therefore, for each $t^* = 1, \ldots, T$ it holds that

$$|W \cap \{(t^*, q) \in N_r : x^{t^*q} < z_{t^*} - \delta\}| < r_0 n_{t^*};$$

of the worst off players of type t^*, fewer than $r_0 n_{t^*}$ can be treated worse than the average payoff for that type minus δ. This means that $\{(t,q) : x^{tq} - z^t < -\delta\} \subset W$.

From the facts that:

$$r_0 \frac{\Psi^*(rn)}{r} - 2\epsilon^* r_0\|n\| \leq \Psi^b(r_0 n) - \epsilon^* r_0\|n\| \text{ (from (c))},$$

$$\leq x(W) \quad \text{(since } x \text{ is in the } \epsilon^*\text{−core)},$$

$$\leq r_0 \cdot n \quad \text{(from the definition of } W\text{)},$$

$$\leq r_0 \frac{\Psi^*(rn)}{r} \quad \text{(from feasibility of } x\text{)},$$

it follows that

$$0 \leq r_0 z \cdot n - x(W) \leq 2\epsilon^* r_0\|n\| .$$

Informally, the above expression says that, for each t, on average players of type t in W are receiving payoffs within $2\epsilon^*$ of z_t.

We now turn to those players who are receiving payoffs significantly more (more than δ) than the average for their types and put an upper bound on the number of such players. Define the set of "best off" players B by

$$B = \{(t,q) \in N_r : x^{tq} > z_t + \delta\} .$$

Define the set of "middle class" players M by

$$M = N_r/(B \cup W) .$$

Observe that, since $\sum_{(t,q)\in N_r} (x^{tq} - z_t) = 0$, it follows that

$$\delta|B| \leq \sum_{(t,q)\in B} (x^{tq} - z_t) = \sum_{(t,q)\in W\cup M} (z_t - x^{tq}) .$$

From the preceding paragraph and the above expression,

$$\delta|B| < \epsilon^* r_0\|n\| + \sum_{(t,q)\in M} (z_t - x^{tq}) .$$

Obviously, the larger the value of $\sum_{(t,q)\in M}(z_t - x^{tq})$, the larger it is possible for $|B|$ to be. We claim that $\sum_{(t,q)\in M}(z_t - x^{tq}) \le 2\epsilon^*|M|$. This follows from the fact that the players in W are the worst off, and they are, on average, each within $2\epsilon^*$ of the average payoff for their types. Since those players in M are better off, they must receive on average no less than the average for their types minus $2\epsilon^*$. Therefore, $\sum_{(t,q)\in M}(z_t - x^{tq}) \le 2\epsilon^*|M|$. It now follows that

$$\delta|B| \le 2\epsilon^* r_0\|n\| + 2\epsilon^*|M| .$$

From $|M| + |B| = r\|n\| - r_0\|n\|$, $|M| \le r\|n\| - r_0\|n\|$, and

$$\delta|B| \le 2\epsilon^* r_0\|n\| + 2\epsilon^*(r\|n\| - r_0\|n\|) \le 2\epsilon^* r\|n\| ,$$

it follows that $\frac{|B|}{r\|n\|} \le 2\frac{\epsilon^*}{\delta}$.

Counting the number of players who may be treated significantly differently than the average we see that:

$$\frac{|W|}{\|rn\|} + \frac{|B|}{\|rn\|} \le \frac{\epsilon^* r_0}{r} + \frac{2\epsilon^*}{\delta} < \frac{\lambda}{\|n\|} \text{ from (a) above.}$$

The conclusion of the Proposition is immediate from the observation that if x is in the ϵ-core of rn for $r \ge r^*$ and $0 \le \epsilon \le \epsilon^*$, then x is in the ϵ^*-core of rn.

Q.E.D.

We can remove the restriction to replication sequences when we assume small group effectiveness.

Proposition A.1.2. Let (T, Ψ) be a pregame satisfying small groups effectiveness. Given any real numbers $\delta > 0$ and any $\lambda > 0$ there is a real number ϵ^* and an integer $\rho(\delta, \lambda, \epsilon^*)$ such that for each $\epsilon \in [0, \epsilon^*]$ and for every game n with $\|n\| \ge \rho(\delta, \lambda, \epsilon^*)$, if x is in the ϵ-core of the derived game with player set N then

$$|\{(t,q) : |x^{tq} - z_t| > \delta\}| < \lambda\|n\| ,$$

where $z_t = \frac{1}{rn_t}\sum_{q=1}^{rn_t} x^{tq}$, the average payoff received by players of type t.

Proof of Proposition A.2: The proof of the Proposition can be obtained by contradiction. The critical feature is that a large game is approximately a replica game and with small group effectiveness, we can ignore the part of the total player set that does not fit into the replicated profile. In other words, if a profile f equals $rh + m$ for some large multiple r of a profile h, and $\|m\|/\|f\| < \lambda$ for some small λ, when we calculate the maximal number of players who can be treated significantly differently than average, we can just assume that the "leftovers" are also treated significantly differently than the average. We omit the details.

Propositions A.1 and A.2 indicate that only a relatively small set of players can be treated significantly different than the average for their types. Theorem 4.1 says that the average payoffs must be close to Walrasian payoffs. Thus, for sufficiently large groups in a player population, the per capita payoff to a group in an approximate core payoff must be approximately the per capita payoff imputed to the group by some Walrasian payoff and conversely.

App. 2: Pregames with Compact Metric Space of Types

We indicate the extension of the framework and results to a compact metric space of types.[46]

Let Ω be a compact metric space, and let f be a function from Ω to the non-negative integers with finite support. As in the previous section, the function f is called a profile, and describes a group of players by the numbers of players of each type in the group. Let Ψ be a function from profiles into $\mathbb{R}_+$. Then the pair (Ω, Ψ) is a *pregame with a compact metric space of types.*

The reader can verify that the definitions introduced so far for pregames with a finite set of types and for games determined by such pregames can be applied to a pregame with a compact metric space of types. For example, the definitions of the balanced cover, replica games, bounded essential group sizes, and small group effectiveness extend immediately. Payoff functions can replace payoff vectors, where a payoff function is a mapping from Ω to $\mathbb{R}$.

The topology used is the weak $*$ topology on the space of profiles.[47] With appropriate continuity conditions, ensuring that players who are similar types are approximately substitutes, Theorem 3.3 continues to hold. The proof of non-emptiness is be obtained by approximation by finite types and contradiction. Most other Propositions also extend, including 4.1 and 4.3.

The continuity condition required to enable the approximation by a finite number of types is given by: Let f be a profile on Ω. Then for any $\epsilon > 0$ there is a $\delta > 0$ such that for all ω_1, ω_2 in Ω with $\text{dist}(\omega_1, \omega_2) < \delta$, it holds that $|\Psi(f + \chi^{\omega_1}) - \Psi(f + \chi^{\omega_2})| < \epsilon$. Since small group effectiveness implies that we can approximate games by ones with bounded norms, and since the space of norm-bounded probability measures is compact, this mild continuity condition suffices.

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[46] The compact metric space framework was introduced in Wooders and Zame [113] for games with side payments, and in Kaneko and Wooders [51] for games without side payments.

[47] See [43] or [55] for discussions of the weak-star topology. For a further discussion in our context, see [105].