Ciphers and Cryptography

Vanderbilt Math Club

April 1, 2019

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Group Question

If you had to send a secret message to a friend, how would you make sure no one else can read it?

History

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- Here is a short video explaining one method the US used in the war: https://www.youtube.com/watch?v=5rSvm3m8ZUA
- The Germans had their famous Enigma machine, with daily changes in the code. Possibilities:

$$\approx 3000000000000000000000000000000000000000$$

$$000000000000000000000000000000000000000$$

$$000000000000000000000000000000000000(!)$$

(Photo credit: Alessandro Nassiri (CC BY-SA 4.0))

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- Do you see anything unusual about this image?

- Pixels were changed to hide the picture of a cat, but its unnoticeable! (Photo Credit: Wiki user Cyp (CC BY-SA 3.0))

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- Decryption: The process of turning ciphertext back into readable plaintext.

Substitution ciphers

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plaintext	A	B	C	D	E	F	G	H	I	J	K	L	M
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Split into pairs. Pick a message, make a shift cipher by shifting by some number of letters, and encrypt your message. Now swap messages with your partner. Can you break their code?

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Make a new code and exchange with a friend again. Can you break their code this time if you couldn't last time?

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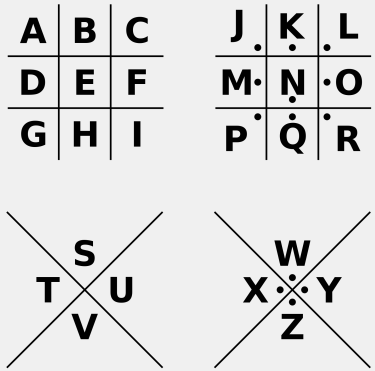

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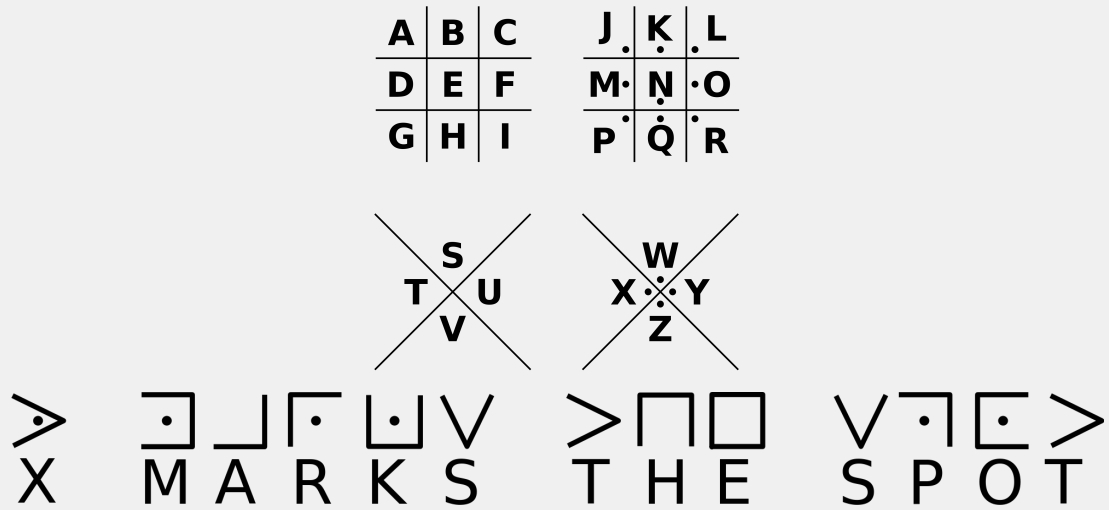

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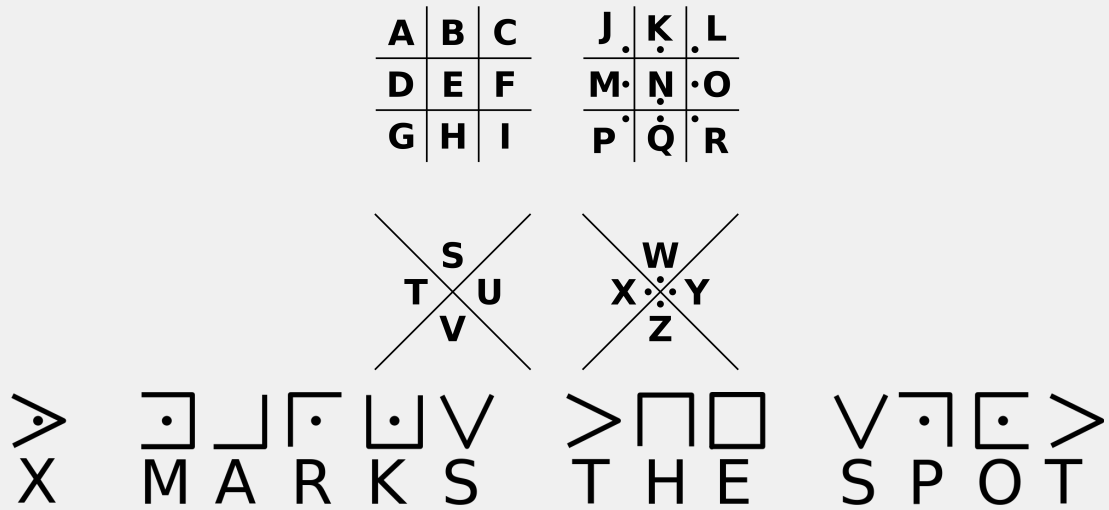

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Encrypt the message "Superspy" in pigpen.

Probabilistic attacks

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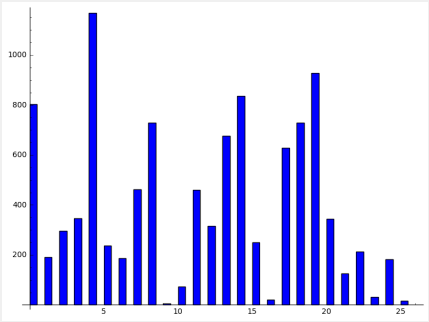

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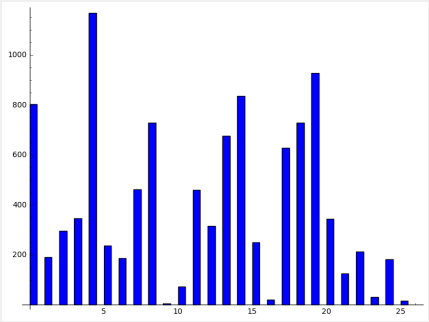
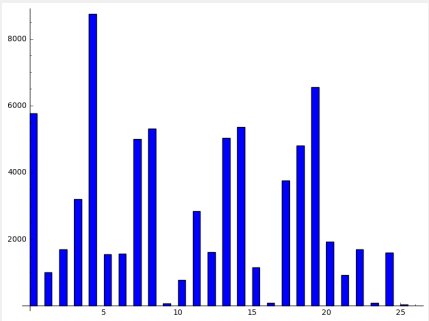


- Bar chart of how often the letters ‘a’ through ‘z’ appear in the first 10 chapters of Anna Karenina:



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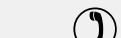
Group Question

You had a large set of text and drew the bar charts. You are pretty confident that the chart has identified the 7 most common letters as: E=  , T=  , A=, O=  , I=  , N=  , S=  .
The first sentence is given below. What does it say?
      , 
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- Then we can do mathematical operations to the numbers.

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- Clock arithmetic rules: to add or multiply numbers "modulo N", add or multiply them as usual. If the number gets outside of the set of numbers $1, 2, \ldots, N$, then shift by N as many times as you need to get back in this range. Instead of writing $=$, we right $\equiv$.

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- Example: Modulo 5, $3 + 4 = 7 \equiv 2$, and $3 \cdot 4 = 12 \equiv 12 - 5 \equiv 7 \equiv 7 - 5 \equiv 2$. On a 5-hour clock, if you add 3 hours 4 times, you've moved 2 hours ahead.

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Find the following numbers in clock arithmetic:

- *Modulo* 2*:*

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- *What does it mean for number to be* 0 *in clock arithmetic?*

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- Decryption: Convert to numbers, add 3, convert to letters: "JXQE"$\to 10, 24, 17, 5 \to 13, 1, 20, 7 \to$ "MATH".

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- Bad choice: $C = 2,\ D = 1$. Then $A \to 1 \mapsto 2 \cdot 1 + 1 = 3 \to C$ but $N \to 14 \mapsto 14 \cdot 2 + 1 = 29 \equiv 3 \to C$. Can't be undone!

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Fact (From number theory)

If C is even or a multiple of 13 (note: $26 = 2 \cdot 23$), then there is no way to undo the affine cipher. If C is odd and not a multiple of 13, then there is a magic choice of C_2, D_2 which undoes the cipher.

Affine ciphers can also be broken!

- Take a few minutes and try to crack the following code:

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"Gwn qzadg unadro zo gwn arrl gr gnii ln gwn hrira rq gwn dfv tngd b hbokv eba".

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- More statistics: what double letters, other patterns are most common in English?

Group analysis (Frequencies: E,T,A,O,I,N,S)

- IWO WBCVO VIBBT BD Q VULPWI ELVO ZCVI BD IWO OTPO BS IWO XLUUQPO. LI VIBBT BD LIV BAD QDT UBBMOT BXOE Q REBQT VYEOQT BS AOVI KBCDIEN SQEHUQDT. DBI Q EOHQEMQRUO WBCVO RN QDN HOQDV LI AQV QRBCI IWLEIN NOQEV BUT, VJCQIILVW, VJCQELVW, HQTO BS RELKM, QDT WQT SBCE ALDTBAV VOI LD IWO SEBDI BS Q VLFO QDT YEBYBEILBD AWLKW HBEO BE UOVV OGQKIUN SQLUOT IB YUOQVO IWO ONO. IWO BDUN YOEVBD SBE AWBH IWO WBCVO AQV LD QDN AQN VYOKLQU AQV QEIWCE TODI, QDT IWQI AQV BDUN ROKQCVO LI WQYYODOT IB RO IWO BDO WO ULXOT LD. WO WQT ULXOT LD LI SBE QRBCI IWEOO NOQEV, OXOE VLDKO WO WQT HBXOT BCI BS UBDTBD ROKQCVO LI HQTO WLH DOEXBCV QDT LEELIQRUO. WO AQV QRBCI IWLEIN QV AOUU, TQEM WQLEOT QDT DOXOE JCLIO QI OQVO ALIW WLHVOUS.

At-home resource

- Fun web applet by Darrin Doud at BYU:
  `https://math.byu.edu/~doud/Substitution/`

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Is this secure from spies? Is it time-efficient?

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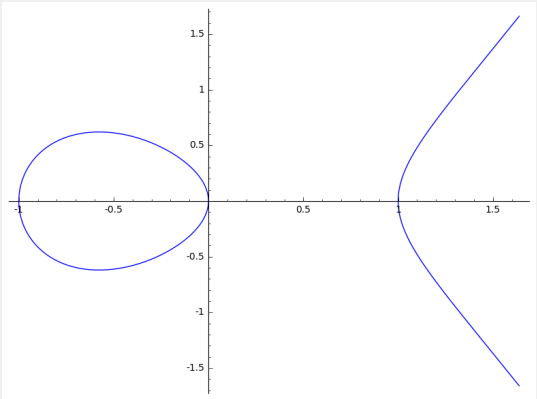
- More recent ideas (more secure than RSA): use geometry!
- Future: quantum computers can break all these codes, we need new ones!



Invent your own cipher!

Group Question

Come up with as many new ciphers as you can! Test these out on the people around you and see if you can invent an unbreakable code.