Take home problems

Nashville Math Club

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1 Take-Home Problems

1. Think back on the key steps that made the proof of the infinitude of primes of the form $4n+3$ work. Would something similar work for primes of the form $3n+1$ or for primes of the form $3n+2$? If so, determine which one and recreate the proof again for these primes.

2. (a) A *Mersenne prime* is a prime of the form $2^n - 1$ for some integer $n \geq 1$. Show that if $2^n - 1$ is a prime, then n must be a prime. (**Hint:** Write $n = ab$, and think of a way to try to factor $2^{ab} - 1$ if $a, b > 1$.)

   (b) Similar to how we asked if we could find a polynomial all of whose values are prime, we can ask if $2^p - 1$ will be prime for all primes p. Try the first 5 primes and see what happens, i.e. plug-in $p = 2, 3, 5, 7, 11$ and see if $M_p := 2^p - 1$ is prime for all of them.

3. Another collection of primes that are of interest are the *Fermat primes*, which are primes of the form $2^n + 1$. Similar to 2(a), we can show that n must be of the form 2^k. (This is much harder to show, but if you want to try, try seeing that if $n = ab$ with b odd, then $2^n + 1$ is divisible by $2^a + 1$.)

   (a) Try writing out the first four Fermat primes, i.e. write out $F_k := 2^{2^k} + 1$ for $k = 0, 1, 2, 3$ and check if they are prime. Although it is harder to compute in this case, we also get numbers that are not prime when we plug in larger values of k. In fact, the only known Fermat primes occur when $k \leq 4$.

   (b) Prove the recurrence relation $F_k = (F_{k-1} - 1)^2 + 1$.