OPERATOR THEORETIC ASPECTS OF CONFORMAL FIELD THEORY

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Conformal field theory concerns the study of vertex operator algebras and their representations. Three approaches include the operator algebraic approach, using loop groups and Connes fusion of bimodules over von Neumann algebras; the holomorphic approach of Segal and Neretin, associating Hilbert–Schmidt operators to Riemann surfaces with boundaries; and the algebraic geometric approach via complex algebraic curves with singularities that are at worst double points. This minicourse will explain how to develop the holomorphic approach for free complex fermions. The main idea in the course will be that of complex semigroups and their representations in the semigroup of non–degenerate Hilbert–Schmidt operators. In addition a major technique will be that of singular integral operators related to the Hilbert transform in one and two dimensions; already these provide operator–theoretic methods for uniformising and gluing in geometric function theory.

The simplest example is the semigroup of Möbius transformations of the Riemann sphere leaving the unit disc or upper half–plane invariant. It is a maximal semigroup in $SL(2,\mathbb{C})$ and is represented by Gaussian kernels on $L^2(\mathbb{R})$. In fact the Stone–von Neumann theorem for the Heisenberg group leads to the "metaplectic" projective representation of $SL(2,\mathbb{R})$ (or equivalently $SU(1,1)$) which extends to the Möbius semigroup. The scaling transformations on the disc correspond to the heat kernel of the harmonic oscillator. Theta functions appear naturally in this context and their modular transformation properties can be deduced using the subgroup $SL(2,\mathbb{Z})$ of $SL(2,\mathbb{R})$. This one–dimensional quantum mechanical example can be generalised to finite dimensions, replacing $SL(2,\mathbb{R})$ by the symplectic group. It also has an infinite–dimensional version. By incorporating the discrete Heisenberg system on $L^2(\mathbb{T})$, this gives positive energy representations of the loop group $LU(1)$ and the diffeomorphism group of the circle. The latter is an infinite–dimensional generalisation of $SU(1,1)$. There is a corresponding semigroup, due to Neretin and Segal, containing the composition semigroup of univalent holomorphic self–mappings of the unit disc. From the point of view of conformal field theory, this is the theory of a charged free boson. Semigroup aspects, however, are often easier to understand in the equivalent description using free fermions.

The fermionic semigroup is formed from projections on Hilbert space that arise as graphs of Hilbert–Schmidt operators. It has a representation on fermionic Fock space. Hardy spaces of half–order differentials on multiply connected regions in $\mathbb{C}$ yield elements of the semigroup. The doubly connected case of an annulus gives the Neretin–Segal semigroup. The triply connected case of a two–holed disc or "trinion" encodes entirely the vertex operator algebra of a single complex fermion. This is a consequence of the fact that the vertex operator algebra can be constructed directly using the action of the Lie superalgebra of fermions on vertex operators. Using the equivalence with a charged free boson, it can also be constructed using the action of the Lie algebra of bosons on standard vertex operators corresponding to the lattice $\mathbb{Z}$. There is also a larger system of vertex operators for $\frac{1}{2}\mathbb{Z}$ that yields twisted modules. Twisting arises because on an annulus there are two choices of half–order differential $dz^{1/2}$ and $(z^{-1}dz)^{1/2}$ corresponding to aperiodic (Neveu–Schwarz) or periodic (Ramond) boundary conditions: only the former extends to the disc. The twisted modules are encoded in the Hardy space of the two–holed disc with appropriate boundary conditions.

The holomorphic picture of conformal field theory predicts that the Hilbert–Schmidt operators for bordered Riemann surfaces should be constructed by composing the operators corresponding to a trinion decomposition of the surface. Hardy spaces on Riemann surfaces require a theory of half–order differentials. These corresponds to spin structures on the surface Σ. Classically they were studied by Riemann as "theta–characteristics" in terms of theta functions with half–periods. For a surface of genus g there are 2^{2g} possible choices. Equivalently they correspond to taking a square root of the cotangent line bundle; or to lifting the fundamental group from $PSL(2,\mathbb{R})$ to $SL(2,\mathbb{R})$. Any spin structure defines a quadratic form on $H_1(\Sigma,\mathbb{Z}_2)$, with value 0 or 1 according to whether the restriction to a closed curve is aperiodic or periodic. The Arf invariant distinguishes two types of spin structures, even or odd. The parity is the same as that of the dimension of the space of half–order differentials, which in the generic "regular" case is either 0 or 1. Trinion decompositions correspond precisely to regular even spin structures. Marked points can be incorporated by using Hardy spaces with poles or zeros of a given order at the marked points; and more general trinion decompositions can be included by allowing twisted boundary conditions.