Asymptotic Shape of Equivariant Random Metrics on Nilpotent Groups

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First Passage Percolation

- Cayley Graph $Cay(\Gamma, S)$ with Edges $= E$
- $X = [a, b]^E, \qquad 0 < a < b < \infty$
  the space of assignments of a length to each edge
- $\mu \in Prob([a, b]) \qquad m = \mu^E \in Prob(X)$
- For each $x \in X$ define

$$d_x(\gamma_1, \gamma_2) = \inf \sum x(e_i)$$

  where the *inf* is taken over paths $(e_1, \ldots, e_k)$ from γ_1 to γ_2.
- What is the asymptotic shape of the ball of radius R in the metric d_x?

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- What is the asymptotic shape of the ball of radius R in the metric d_x?
- $\Gamma = \mathbb{Z}$ — Law of Large Numbers (1713)
- $\Gamma = \mathbb{Z}^d$ — Boivin (1990)
- $\Gamma =$Nilpotent — Benjamini-Tessera (2014)
  C-Furman (2015)
- e.g.

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

- Benjamini-Tessera require weaker integrability, use statistical methods, use that FPP is i.i.d.s
- We consider more general random metrics

Ergodic Equivariant Random Metrics

Definition

An *ergodic equivariant random metric* on a finitely generated group Γ is a probability measure space (X, m) and a measurable family of metrics d_x on Γ satisfying

$$d_{\gamma\cdot x}(\gamma_1, \gamma_2) = d_x(\gamma\gamma_1, \gamma\gamma_2) \qquad \forall \gamma, \gamma_1, \gamma_2 \in \Gamma \textit{ a.e. } x$$

such that $\Gamma \curvearrowright (X, m)$ is measure-preserving and ergodic.

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- First Passage Percolation is a special case of EERM with i.i.d.s.

The Asymptotic Cone Of A Nilpotent Group

Theorem (Pansu's Theorem)

If Γ is a finitely generated nilpotent group equipped with a left-invariant inner metric d (e.g. word metric) then there exists a unique connected, simply-connected, graded nilpotent Lie group G_∞ and a homogeneous Carnot-Caratheodory metric d_∞ such that in Gromov-Hausdorff convergence

$$(\Gamma, \tfrac{1}{n}d, id) \longrightarrow (G_\infty, d_\infty, id).$$

Example

$$\Gamma = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}$$

$$G_\infty = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}$$

The Asymptotic Cone of an Ergodic Equivariant Random Metric on a Nilpotent Group

Theorem (C-Furman 2015)

Suppose (X, d_x) is an ergodic equivariant random metric on a finitely generated nilpotent group Γ such that

1. *d_x is inner a.e. $x \in X$*
2. *$\exists\ 0 < a < b < \infty$ such that $a < d_x/d < b$ for a.e. $x \in X$ for some (any) word metric d*

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Then there exists a homogeneous Carnot-Caratheodory metric d_∞ on G_∞ such that in Gromov-Hausdorff convergence for a.e. x

$$(\Gamma, \frac{1}{n} d_x, e) \;\longrightarrow\; (G_\infty, d_\infty, e).$$

From ERM to Subadditive Cocycles

Definition

Given $\Gamma \curvearrowright (X, m)$ a probability measure-preserving action a *subadditive cocycle* is $c : \Gamma \times X \to \mathbb{R}_+$ such that

$$c(\gamma_1\gamma_2, x) \leq c(\gamma_1, \gamma_2 \cdot x) + c(\gamma_2, x).$$

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- $\{$*subadditive cocycles*$\} \leftrightarrow \{$*equivariant random metrics*$\}$
- Given $c : \Gamma \times X \to \mathbb{R}_+$ define $d_x(\gamma_1, \gamma_2) = c(\gamma_1\gamma_2^{-1}, \gamma_2 \cdot x)$.
- Given $\{d_x\}$ define $c(\gamma, x) = d_x(e, \gamma)$.
- Cocycle inequality $\leftrightarrow$ triangle inequality.

Previous Subadditive Ergodic Theorems

- $\Gamma = \mathbb{Z}$ - Kingmann (1973): For a.e. x as $n \to \infty$

$$\frac{1}{n} c(n,x) \longrightarrow \inf_{n \geq 1} \frac{1}{n} \int_X c(n,x) dm(x).$$

  - implies Birkhoff's Ergodic Theorem
  - implies Asymptotic Shape Theorem for FPP on $\mathbb{Z}$
  - many other applications...e.g. LLN for matrices

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- $\Gamma = \mathbb{Z}^d$ - Björklund (2010): There exists a norm L on $\mathbb{R}^d$ s.t. for a.e. x as $\mathbb{Z}^d \ni \bar{n} \to \infty$

$$\frac{c(\bar{n}, x) - L(\bar{n})}{|\bar{n}|} \longrightarrow 0.$$

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- $\Gamma =$ Nilpotent?

Subadditive Ergodic Theorem for Nilpotent Groups

Theorem (C-Furman 2015)

Suppose Γ is a finitely generated nilpotent group, $\Gamma \curvearrowright (X, m)$ is an ergodic probability measure preserving and $c : \Gamma \times X \to \mathbb{R}_+$ is a subadditive cocycle such that

1. $\exists\, 0 < a < b < \infty$ *such that* $a < c(\gamma, x)/|\gamma| < b$ *for a.e.* $x\ \forall \gamma$ *for some (any) word norm* $|\cdot|$
2. d_x *is inner for a.e.* x*; i.e. for a.e.* $x\ \forall \epsilon > 0\ \exists$ *finite set* $F \subset \Gamma$ *s.t. every* $\gamma \in \Gamma$ *may be written* $\gamma = \gamma_n \cdots \gamma_1$, $\gamma_i \in F$ *and*

$$c(\gamma_1, x) + c(\gamma_2, \gamma_1 \cdot x) + \cdots + c(\gamma_n, \gamma_{n-1} \cdots \gamma_1 \cdot x) \leq (1+\epsilon) c(\gamma, x)$$

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Then there exists a unique Carnot-Caratheodory norm ϕ *on* G_∞ *s.t. for a.e.* x *whenever* $\mathbb{N} \times \Gamma \ni (t_i, \gamma_i) \to g \in G_\infty$

$$\frac{1}{t_i} c(\gamma_i, x) \longrightarrow \phi(g).$$

Where Does ϕ Come From?

- Idea: integrate and apply Pansu's construction.
- Carnot-Caratheodory construction

- I. Benjamini, R. Tessera. *First passage percolation on nilpotent cayley graphs and beyond*. http://arxiv.org/pdf/1410.3292.pdf
- M. Björklund. *The asymptotic shape theorem for generalized first passage percolation*. Ann. Probab. 38 (2010), 632-660.
- J. F. C. Kingman. *Subadditive ergodic theory*. Ann. Probab. 1 8830-909
- P. Pansu. *Croissance des boules et des géodésiques fermées dan les nilvariétés*. Ergodic Dyn. Syst. 3(1983), 415-445.