Powers group methods for locally compact groups acting on trees

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Outline

Group C*- and von Neumann algebras

Definition

Let G be a locally compact group.

- $\mathrm{L}(G) := \lambda(G)'' \subset \mathcal{B}(\mathrm{L}^2(G))$ group von Neumann algebra.
- $\mathrm{C}^*_{\mathrm{red}}(G) := \overline{\lambda(\mathrm{C_c}(G))}^{\|\cdot\|}$ where $\lambda(f)$ acts via convolution.

- Note that $u_g = \lambda(g) \notin \mathrm{C}^*_{\mathrm{red}}(G)$ if G is not discrete.
- But always $u_g \in \mathrm{M}(\mathrm{C}^*_{\mathrm{red}}(G)) \subset \mathrm{L}(G)$.

The Plancherel weight

- No canonical trace on $C^*_{\mathrm{red}}(G)$ and $\mathrm{L}(G)$, but Plancherel weights.
- $\varphi(f) = f(e)$ for $f \in \mathrm{C_c}(G)$ extends to proper KMS-weight on $C^*_{\mathrm{red}}(G)$ and faithful normal weight on $\mathrm{L}(G)$ after choice of a Haar measure μ.
- For $K \leq G$ compact open and $g \in G$, we have $\varphi(u_g p_K) = \frac{1}{\mu(K)} \mathbb{1}_K(g)$.
- Modular automorphism group: $\sigma_t^{\varphi}(u_g) = \Delta(g)^t u_g$.

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- Modular automorphism group: $\sigma^\varphi_t(u_g) = \Delta(g)^t u_g$.

Question

What can we tell about $\mathrm{C}^*_{\mathrm{red}}(G)$ and $\mathrm{L}(G)$ if G is not discrete?

Group von Neumann algebras of discrete groups

Proposition

Let G be a discrete group.

- $\mathrm{L}(G)$ is a factor if and only if G is icc.
- $\mathrm{L}(G)$ is amenable if and only if G is amenable.

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Proof of the factoriality criterion

- If G is icc, let $\mathrm{L}(G) \mapsto \ell^2(G) : x \mapsto \hat{x}$ via the canonical trace.
- If $x \in \mathcal{Z}(\mathrm{L}(G))$, then $\hat{x}$ is constant on conjugacy classes.
- So $\mathcal{Z}(\mathrm{L}(G)) = \mathbb{C}1$.

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There is no embedding of $\mathrm{L}(G)$ into $\mathrm{L}^2(G)$ if G is not discrete.

Group C^*-algebras of discrete groups: simplicity

- C^*-simplicity is more difficult to prove.
- Powers: $C^*_{\mathrm{red}}(\mathbb{F}_2)$ is simple. (Question of Kadison).

Group C*-algebras of discrete groups: simplicity

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- De la Harpe: Powers groups extract essence of Powers's proof.
- Other method: Property RD type criterion by Bekka-Cowling-de la Harpe.
- Recent breakthrough via boundary actions: Kalantar-Kennedy and Breuillard-Kalantar-Kennedy-Ozawa.

Definition

A discrete group G is a Powers group if for all finite sets $F \subset G \setminus \{e\}$ and all $n \in \mathbb{N}$ there are elements $g_1, \dots, g_n \in G$ and a partition $G = C \sqcup D$ such that

- $\forall f \in F : fC \cap C = \emptyset$, and
- $g_1 D, \dots, g_n D$ are pairwise disjoint.

Group C*-algebras of locally compact groups

If G is discrete, then $C^*_{\max}(G)$ is amenable if and only if G is amenable. However:

Theorem (Connes)

If G is connected, then $C^*_{\max}(G)$ is amenable.

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Even more, there are many type I groups ($C^*_{\max}(G)$ is type I).

Type I groups

- Connected semisimple Lie groups. (Harish-Chandra)
- Reductive algebraic groups over non-Archimedian fields. (Bernstein)
- The group $\mathrm{Aut}(T)$ of all automorphism of a regular tree (Figà-Talamanca and Nebbia).

Non-discrete C^*-simple groups

- Bekka-Cowling-de la Harpe observed that a C^*-simple group cannot be connected.
- Even more is true: C^*-simple groups must be totally disconnected.

Non-discrete C*-simple groups

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- Even more is true: C*-simple groups must be totally disconnected.

Every C*-simple group is totally disconnected

- $G \geq G^0 \triangleright H$, with H amenable, G^0/H connected semisimple Lie group with trivial centre.
- Take $\pi, \pi' \prec \lambda_{G^0/H}$ irreducible, $\pi \not\sim \pi'$: representations of G^0.
- $\mathrm{Res}^G_{G^0}\mathrm{Ind}^G_{G^0}(\pi) \prec \sum_{\alpha \in \mathrm{Aut}(G^0)} \pi \circ \alpha$
- Apply Langlands's disjointness theorem to parabolic induced representations from (P, triv) and (P, σ) with σ non-trivial.

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Question of de la Harpe (2007 survey)

Are there non-discrete C*-simple groups?

Group von Neumann algebras of non-discrete groups

Theorem (Paterson)

Let G be a locally compact inner amenable ($1 < \mathrm{cong}$) group. Then G is amenable if and only if $\mathrm{L}(G)$ is amenable.

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Group factors of non-discrete groups

- If $H \leq \mathrm{GL}_2(\mathbb{R})$ acts ergodically on $\mathbb{R}^2$, then $\mathrm{L}(\mathbb{R}^2_{>0} \rtimes H) \cong \mathrm{L}^\infty(\mathbb{R}^2) \rtimes H$ is a factor.
- Schlichting completions of Baumslag-Solitar groups give rise to group factors. (Ciobotaru-R unpublished)

Motivation: connect two developing fields

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Motivation I

Connect two developing fields of mathematics.

Motivation: Groups acting on trees

- If T is a tree, then $\mathrm{Aut}(T)$ with the topology of pointwise convergence is a totally disconnected group.
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If $G \leq \mathrm{Aut}(T)$ is a closed subgroup such that some compact open subgroup of G acts transitively on ∂T, then G is a type I group.

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Motivation II

Study this conjecture by operator algebraic means.

C^*-simplicity

Theorem (R)

Let $G \leq \mathrm{Aut}(T)$ be a closed non-amenable subgroup. Assume that G contains a compact open subgroup with non-compact normaliser. Then $C^*_{\mathrm{red}}(G)$ is simple.

C*-simplicity

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Corollary (R)

Let T be thick tree and $\Gamma \leq \mathrm{Aut}(T)$ some not necessarily closed group acting without proper invariant subtrees. Let Λ be some vertex stabiliser of in Γ and assume that there is a finite index subgroup $\Lambda_0 \leq \Lambda$ whose normaliser satisfies $[\mathcal{N}_\Gamma(\Lambda_0) : \Lambda_0] = \infty$. Then $C^*_{\mathrm{red}}(\Gamma, \Lambda)$ is simple.

Factoriality

Theorem (R)

Let $G \leq \mathrm{Aut}(T)$ *be a closed non-amenable subgroup. Assume that* G *contains a compact open subgroup with non-compact normaliser. Further assume that some compact open subgroup of* G *is topologically finitely generated. Then* $\mathrm{L}(G)$ *is a factor and* $\mathrm{S}(\mathrm{L}(G)) = \overline{\Delta(G)}$.

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- *If G is discrete, then $\mathrm{L}(G)$ is a II_1 factor.*
- *If G is unimodular but not discrete, then $\mathrm{L}(G)$ is a II_∞ factor.*
- *If $\Delta(G) = \lambda^{\mathbb{Z}}$ for some $\lambda \in (0,1)$, then $\mathrm{L}(G)$ is a III_λ factor.*
- *If $\Delta(G)$ is not singly generated, then $\mathrm{L}(G)$ is of type III_1.*

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- Topological finite generation plays a completely different role here than in the previous theorem.
- Sufficient: G locally compact group containing a compact open subgroup whose normaliser is non-amenable.

Reformulation of the hypotheses I

Proposition

Every element in $\mathrm{Aut}(T)$ either

- fixes a vertex of T (elliptic elements),
- fixes no vertex, but an edge of T (inversions), or
- fixes points in $x, y \in \partial T$ and translates along the axis (x, y) (hyperbolic elements).

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Proposition

Let $G \leq \mathrm{Aut}(T)$ be a closed non-amenable subgroup. Then there is an invariant subtree $T' \leq T$ such that G acts minimally on $\partial T'$.

Reformulation of the hypotheses II

Proposition

Let $G \leq \mathrm{Aut}(T)$ be a closed subgroup. The following statements are equivalent.

- There is $x \in \partial T$ such that G_x is open and $G_x \cap G_0$ contains a hyperbolic elements.
- There is a compact open subgroup of G whose normaliser is non-amenable.

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Geometric reformulations of our hypothesis

Let T be a thick tree (i.e. valencies ≥ 3) and $G \leq \mathrm{Aut}(T)$ be a closed subgroup acting minimally on ∂T. Assume that there is some $x \in \partial T$ such that G_x is open and $G_x \cap G_0$ contains a hyperbolic element. (Further assume that some compact open subgroup of G is topologically finitely generated). Then ...

Some remarks

- Topological freeness of the boundary action $G \curvearrowright \partial T$ does not play a role for our results.
  A posteriory reason: $\mathrm{C}^*_{\mathrm{red}}(G) \sim_{\mathrm{Morita}} \mathrm{C}^*_{\mathrm{red}}(G, K)$ is a global object. But only elliptic elements possibly act not topolgically free.

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- In contrast to the discrete case we have to make stronger assumptions for von Neumann algebraic results than for C^*-results. Reason: to estimate $\varphi(x)$ we need to estimate $\|x\|$ and the support of x.

Examples

Schlichting completions of Baumslag-Solitar groups

- $\mathrm{BS}(m,n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle$ Baumslag-Solitar group.
- $\mathrm{BS}(m,n) \curvearrowright T$ acts on Bass-Serre tree. Faithful if $|m| \neq |n|$.
- $\mathrm{G}(m,n) := \overline{\mathrm{BS}(m,n)} \leq \mathrm{Aut}(T)$ Schlichting completion.

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- $\mathrm{G}(m, n) := \overline{\mathrm{BS}(m, n)} \leq \mathrm{Aut}(T)$ Schlichting completion.

- Satisfies hypothesis of all our main results.
- We obtain non-discrete C^*-simple groups and non-amenable type $\mathrm{III}_{|m|/|n|}$ group factors.
- Products $\mathrm{G}(2, p) \times \mathrm{G}(2, q)$ give rise to type III_1 factors.

The type problem for group von Neumann algebras

Possible types of group factors

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- What about type III_0?

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- Is type III_λ for $\lambda \in [0,1] \setminus \mathbb{Q}$ possible?
- What about type III_0?
- Can we obtain discrete group factors from totally disconnected groups?

Definition

Definition (R)

Let G be a locally compact group. G has the Powers property with control $r \in \mathbb{N}$ with respect to a compact open subgroup $K \leq G$ and some compact set $F \subset G \setminus K$ if the following statement holds. For all $n \in \mathbb{N}$ there are elements $g_1, \dots, g_n \in G_0$ and a decomposition $G = C \sqcup D$ into K-invariant subsets such that

- $\forall f \in F : fC \cap C = \emptyset$,
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Difference to de la Harpe's Powers property (discrete groups)

- Control introduced.
- Powers property quantified in terms of K and F.

Elements of the proof: Powers averaging

Proposition

Let G be a locally compact group with the Powers property with control $r \in \mathbb{N}$ with respect to $K \leq G$ and $F \subset G \setminus K$. Let φ by the Plancherel weight on $\mathrm{C}^*_{\mathrm{red}}(G)$ normalised to $\varphi(p_K) = 1$. Then for all $\varepsilon > 0$ and all $x \in \mathrm{C_c}(G)$ with $\mathrm{supp}(x) \subset F$ there are elements $g_1, \ldots, g_n \in G_0$ such that

- $\| \frac{1}{n} \sum_{i=1}^n u_{g_i}(x - \varphi(p_K x p_K) p_K) u_{g_i}^* \| < \varepsilon$, and
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- Proof works exactly as in de la Harpe's work.
- K-invariance of the decomposition $G = C \sqcup D$ is crucial.

Elements of the proof: invertibility

Proposition

Let $p_K = \int_K \lambda_k \mathrm{d}k$ be the averaging projection. For all $g \in G$

$$p_K u_g p_K u_g^* p_K \geq [K : K \cap gKg^{-1}]^{-2} p_K \,.$$

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- Proposition ensures invertibility in $p_K \mathrm{C}^*_{\mathrm{red}}(G) p_K$ of averages

$$\frac{1}{n} \sum_{i=1}^{n} p_K u_{g_i} x u_{g_i}^* p_K$$

for small

$$\| \frac{1}{n} \sum_{i=1}^{n} u_{g_i} (x - c p_K) u_{g_i}^* \| \, , \quad c \in \mathbb{C} \setminus \{0\} \, .$$

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- Control in Powers property is necessary here.

Elements of the proof: averaging projections

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Let $K \leq L \leq G$ be compact open subgroups. Then $p_L \leq p_K$ and

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- Powers averaging combined with the previous proposition shows that p_K are full projections in $\mathrm{C}^*_{\mathrm{red}}(G)$.
- We only need to show $p_K \in I$ for every non-trivial ideal $I \lhd \mathrm{C}^*_{\mathrm{red}}(G)$.
- Is there a criterion for projections p_K to be full in $\mathrm{C}^*_{\mathrm{red}}(G)$?

Elements of the proof: the boundary action

Proposition

$L \leq G$ comptact open. Then G has Powers property with respect to $K := L \cap G_x$ and $F := L \setminus K$ for all $x \in \partial T$ such that G_x is open and $G_x \cap G_0$ contains a hyperbolic element.

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Proof.

- Lx is finite and $\forall f \in F : fx \neq x$.
- $\exists$ K-invariant open $x \in \mathcal{O}$ such that $\forall f \in F : f\mathcal{O} \cap \mathcal{O} = \varnothing$.
- Put $C = \{g \in G \mid gx \in \mathcal{O}\}$, $D = \{g \in G \mid gx \notin \mathcal{O}\}$.
- There are pairwise transverse hyperbolic elements $(g_i)_i$ such that $[K : K \cap g_i^l K g_i^{-l}]$ is uniformly bounded in $i \geq 1$ and $l \in \mathbb{Z}$.
- After conjugation we can move fixed points of all g_i into $\mathcal{O}$ and assume $g_i(\partial T \setminus \mathcal{O})$ pairwise disjoint.
- This proves that $(g_i D)_i$ are pairwise disjoint. □

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- Classical estimate:

$$|\tau(u_g)| = |\tau(\frac{1}{n}\sum_{i=1}^{n} u_{g_i g g^{-1}})| \leq \|\frac{1}{n}\sum_{i=1}^{n} u_{g_i g g^{-1}}\| < \varepsilon\,.$$

- For weights we need to estimate supports in addition.

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- Use $g_i \in G_0$, i.e $\Delta(g_i^{-1}) = 1$.
- Find $L \leq G$ compact open such that $g_i L g_i^{-1} \leq K$ for all i.
- A topologically finitely generated group has only finitely many closed subgroups of fixed finite index.

Elements of the proof: a non-amenability criterion

Proposition

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There is a faithful normal conditional expectation $\mathrm{L}(G) \to \mathrm{L}(H)$. So it suffices note that $p_K \mathrm{L}(H) p_K = \mathrm{L}(H/K)$ is not-amenable, since H/K is discrete and non-amenable. □

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- In our work, take pairwise transverse hyperbolic elements $g_i \in G_0$, $i \geq 1$ whose fixed points have open stabilisers.
- For any $K \leq G$ compact open, $[K : K \cap g_i^l K g_i^{-l}]$ is uniformly bounded in $i \geq 1$, $l \in \mathbb{Z}$.
- By topological finite generation $\bigcap_l g_i^l K g_i^{-l} = \bigcap_l g_j^l K g_j^{-l} \leq K$ is open and normalised by $\langle g_i, g_j \rangle$ ($\cong \mathbb{F}_2$ wlog).

Future work

Adapt Kennedy-Kalantar's approach to non-discrete groups

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Find truly non-discrete non-amenability criteria

- Our non-amenability result follows is based on discrete groups.
- No satisfying non-amenability criterion known.

References

P. de la Harpe
On simplicity of reduced C*-algebras of groups
Bull. London Math. Soc. 39 (2007) 1–26

P. de la Harpe, J.-P. Préaux
C*-simple groups: amaglamated free products, HNN extensions and fundamental groups of 3-manifolds.
Journal of Topology and Analysis 3, No. 4 (2011) 451–489.

S. Raum
Powers group methods for locally compact groups acting on trees.
To be posted on the ArXiv soon.