Weak Amenability and Haagerup Property for Generalized Baumslag Solitar Groups

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1 Haagerup property and weak amenability

G: a locally compact group, 2nd countable.

Definition 1.1. *G has the Haagerup property (or: is a-(T)-menable) if there exists a sequence of continuous positive definite functions on G, vanishing at infinity, converging to 1 uniformly on compact sets.*

Definition 1.2. *G is weakly amenable if there exists a sequence $(\phi_n)_{n>0}$ of continuous, compactly supported functions on G, converging to 1 uniformly on compact sets, with*

$$\sup_n \|\phi_n\|_{cb} < +\infty.$$

Here $\|.\|_{cb}$ is the completely bounded norm on multipliers of Fourier algebra $A(G)$. The *Cowling-Haagerup constant* is the best possible Λ with $\sup_n \|\phi_n\|_{cb} \leq \Lambda$. A group with $\Lambda = 1$ has the *complete metric approximation property* (CMAP).

2 Cowling's question

Around 1998, M. Cowling conjectured that: *the class of Haagerup groups coincides with the class of CMAP groups.*
Disproved in 2007: wreath product $C_2 \wr \mathbf{F}_2$ is Haagerup (Cornulier-Stalder-V) but does not have CMAP (Ozawa-Popa).
However: Cowling's conjecture holds for interesting subclasses:

- closed subgroups of $SO(n,1)$ and $SU(n,1)$;
- groups acting properly on finite-dimensional $CAT(0)$ cubical complexes (Guentner-Higson).

Observations:

- There is no known direct connection between the two properties. In all cases, it is an *a posteriori* observation that a given class of groups satisfy both properties.
- It seems that the discrepancy is related to some lack of finiteness condition. Could it be that Cowling's conjecture holds for groups admitting a finite-dimensional $\underline{EG}$?

3 Generalized Baumslag-Solitar groups (joint with Y. Cornulier)

3.1 A class of groups

Take $n \geq 1$.

Definition 3.1. *The class $\mathcal{GBS}(n)$ is the class of finitely generated groups acting co-compactly on some tree T, such that all edge- and vertex-stabilizers are isomorphic to $\mathbf{Z}^n$*

Example 1. *The standard Baumslag-Solitar groups $BS(p,q) =< a, b | ab^p a^{-1} = b^q >$ are exactly groups acting on a tree, transitively on vertices and edges, with all stabilizers isomorphic to $\mathbf{Z}$. They form a subclass of $\mathcal{GBS}(1)$.*

3.2 Main result

For $G \in \mathcal{GBS}(n)$, there is a *holonomy representation* $hol : G \to GL_n(\mathbf{R})$ whose image controls both the Haagerup property and $CMAP$.

More precisely: let $N \triangleleft G$ be the (normal) subgroup generated by vertex groups. Let $X = G \backslash T$ be the quotient graph, let d be the rank of $\pi_1(X)$. By Bass-Serre, there exists a subgroup H_d of G, free of rank d, such that $G = N \rtimes H_d$ (H_d is generated by the *stable letters*). Then hol factors through H_d.

Theorem 3.2. *For $G \in \mathcal{GBS}(n)$, TFAE:*

a) $\overline{hol(H_d)}$ is amenable;

b) G has that Haagerup property;

c) G is weakly amenable;

d) G has CMAP.

Sometimes $\overline{hol(H_d)}$ is clearly amenable:

Corollary 3.3. *$G \in \mathcal{GBS}(n)$ is Haagerup + CMAP in the following cases:*

- *$d = 0$, i.e. X is a tree;*
- *$d = 1$, i.e. X has the homotopy type of a circle;*
- *$n = 1$.*

For $n = 1$, we recover the result that $BS(p, q)$ is Haagerup + CMAP (Gal-Januszkiewicz 2003).

3.3 Complaint from the group theorists

C'mon guys! For $n = 1$, we knew that already!

Indeed:

Theorem 3.4. *(Kropholler 1990) Any group in* $\mathcal{GBS}(1)$ *is free-by-solvable.*

Aargh! But fortunately:

Proposition 3.5. *(C+V)* $G \in \mathcal{GBS}(n)$ *is not free-by-amenable, as soon as* $hol(H_d)$ *is not virtually solvable.*

Happy, group theorists?

4 The holonomy representation

4.1 Definition

H: a group

Definition 4.1. *The abstract commensurator of H is the set $Comm(H)$ of isomorphisms between finite index subgroups of G (with obvious composition).*

Example 2. $Comm(\mathbf{Z}^n) = GL_n(\mathbf{Q})$.

Definition 4.2. *If $H \subset G$, say that G commensurates H if $[H : H \cap gHg^{-1}] < +\infty$ for every $g \in G$.*

If G commensurates H, there is a *holonomy map* $hol : G \to Comm(H)$.

4.2 When X is a bouquet of circles

Assume for simplicity that the quotient graph X is a bouquet of d circles (i.e. G acts transitively on vertices of X). Then inclusions of edge groups in the unique vertex group $G_v = \mathbf{Z}^n$ are given by $A_i^{\pm} : \mathbf{Z}^n \to Z^n$, i.e. $A_i^{\pm} \in M_n(\mathbf{Z})$ and $\det(A_i^{\pm}) \neq 0$. Then:

$$G = (\mathbf{Z}^n * < t_1, ..., t_d >)/ << t_i A_i^{-}(v) t_i^{-1} : v \in \mathbf{Z}^n, i = 1, ..., d >> .$$

It his clear from this that G commensurates $G_v = \mathbf{Z}^n$ and hol factors through $H_d =< t_1, ..., t_d >$. Actually

$$hol(G) =< (A_i^{+})^{-1} A_i^{-} : i = 1, ..., d > \subset GL_n(\mathbf{Q}).$$

Extend hol to an affine action of G on $\mathbf{R}^n$, with $G_v = \mathbf{Z}^n$ acting by translations.

Example 3. $hol(BS(p, q)) = (p/q)^{\mathbf{Z}}$

4.3 Ideas for the proof of the main result

Lemma 4.3. *Let* Γ *be a group acting by homeomorphisms on locally compact spaces* X, Y*, where* X *is a connected graph. The action of* Γ *on* $X \times Y$ *is proper if and only if, for every vertex* $v \in X$*, the action of the stabilizer* Γ_v *on* Y *is proper.*
$\square$

We apply this to $G \in \mathcal{GBS}(n)$ acting on $T \times \mathbf{R}^n$. Since $G_v = \mathbf{Z}^n$ acts properly on $\mathbf{R}^n$, the group G acts properly on $\mathbf{R}^n$, i.e. its image in $Aut(T) \times (\mathbf{R}^n \rtimes_{hol} H_d)$ is discrete.
Moreover the image of $\mathbf{R}^n \rtimes_{hol} H_d$ in $(\mathbf{R}^n \rtimes \overline{hol(G)}) \times H_d$ is closed. So the main result follows from:

Proposition 4.4. *Let* $\rho : \mathbf{F}_d \to GL_n(\mathbf{R})$ *be a representation. TFAE:*

i) $\mathbf{R}^n \rtimes_\rho \mathbf{F}_d$ *is Haagerup.*

ii) $\overline{\rho(\mathbf{F}_d)}$ *is amenable.*

(and similarly for CMAP).

Idea of proof: $(ii) \Rightarrow (i)$ If $\overline{\rho(\mathbf{F}_d)}$ is amenable, then $\mathbf{R}^n \rtimes_\rho \mathbf{F}_d$ is a closed subgroup of the Haagerup group $(\mathbf{R}^n \rtimes \overline{\rho(\mathbf{F}_d)}) \times \mathbf{F}_d$.

$(i) \Rightarrow (ii)$ **Claim:** If $\overline{\rho(\mathbf{F}_d)}$ is non-amenable, there exists a non-zero ρ-invariant subspace $V \subset \mathbf{R}^n$ such that $(V \rtimes_\rho \mathbf{F}_d, V)$ has the relative property (T). Since any affine isometric action of $\mathbf{R}^n \rtimes_\rho \mathbf{F}_d$ has a V-fixed point, such an action can't be proper. $\square$

For the claim: by a result of Cornulier (2006), the Claim is equivalent to the existence of a non-zero ρ-invariant subspace $V \subset \mathbf{R}^n$ such that there is no ρ^*-invariant probability measure on the projective space $\mathbb{P}(V^*)$. Taking for V a minimal ρ-invariant subspace with $\overline{\rho|_V(\mathbf{F}_d)}$ non-amenable, the claim follows from:

Lemma 4.5. *Let $\rho : \Gamma \to GL_k(\mathbf{R})$ be a representation of some group Γ, with $\overline{\rho(\Gamma)}$ non-amenable and $\overline{\rho|_W(\Gamma)}$ amenable for every proper ρ-invariant subspace $W \subset \mathbf{R}^k$; then there is no invariant probability measure on $\mathbb{P}(\mathbf{R}^{n*})$.*

Sketch of proof: Assume $\mathbb{P}(\mathbf{R}^{n*})$ carries an invariant probability measure ν. By a classical result of Furstemberg: ν is supported in a canonical finite collection of proper projective subspaces $W_1, ...W_\ell$; in particular Γ permutes the W_i's. Assume W_1 of minimal dimension, let $\Lambda \subset \Gamma$ be a finite index subgroup with $\Lambda(W_1) = W_1$. Then $\mu = (\nu(W_1))^{-1}\nu|_W$ is a Λ-invariant probability measure on W_1. Appealing to Furstemberg again (and using minimality of $\dim W_1$): $\Lambda|_{W_1}$ is relatively compact.
Let $V_1 \subset \mathbf{R}^k$ be the annihilator of W_1, a $\rho(\Lambda)$-invariant subspace. Then, in a basis of $\mathbf{R}^k$ extending one of V_1:

$$\overline{\rho(\Lambda)} = \{ \begin{pmatrix} H_1 & * \\ 0 & H_2 \end{pmatrix}$$

with H_1 amenable and H_2 abelian-by-compact. So $\overline{\rho(\Lambda)}$ is amenable, contradiction. □

5 Haagerup property and CMAP are not quasi-isometric invariants

5.1 A result by K. Whyte

Theorem 5.1. *Let* $G_1, G_2 \in \mathcal{GBS}(n)$*; assume that the quotient graphs* X_1, X_2 *are bouquets of circles. If* G_1 *is quasi-isometric to* G_2*, then there exists* $g \in GL_n(\mathbf{R})$ *and a compact subset* $K \subset GL_n(\mathbf{R})$*, such that* $hol(G_1) \subset (g\, hol(G_2)\, g^{-1}).K$*. If* G_1, G_2 *are non-amenable, the converse holds.* □

5.2 A result by M. Carette (2014)

Take $n = 2$.

Take for X_1 a bouquet of 2 circles, with $A_1^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1^- = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $A_2^+ = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2^- = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. For the corresponding group G_1, we have $hol(G_1) =< \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} >$ so $\overline{hol(G_1)} \simeq \mathbf{R} \rtimes_{4^k} \mathbf{Z}$ (amenable).

Take for X_2 a bouquet of 3 circles, with $A_1^{\pm}, A_2^{\pm}$ same as before, and $A_3^+ = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_3^- = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. For the corresponding group G_2, we have $hol(G_2) =< \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} >$, so $\overline{hol(G_2)} = SL_2(\mathbf{R})$.

Observe that $\overline{hol(G_1)}$ is co-compact in $\overline{hol(G_2)}$. So:

Theorem 5.2. *(Carette) The groups G_1, G_2 are quasi-isometric; G_1 has the Haagerup property and CMAP; G_2 has neither properties.* $\square$

Experimental observation Disproving quasi-isometry invariance, either for property (T) of for the Haagerup property, involves non residually finite groups!

5.3 A remark by Arnt-Pillon-V.

If Γ is a finitely generated group.

Definition 5.3. *The equivariant L^p-compression of Γ is $\alpha_p^*(\Gamma) \in [0,1]$, the supremum of those a's for which there exists an affine isometric action α of Γ on L^p such that $|g|_S^a \prec \|\alpha(g)(0)\|_{L^p}$.*

Theorem 5.4. *(Naor-Peres 2011) For amenable groups, α_p^* is a quasi-isometry invariant.* □

By a result of Chatterji-Drutu-Haglund: for $p \in [1,2]$: if $\alpha_p^*(\Gamma) > 0$, then Γ has the Haagerup property.

Proposition 5.5. *(A.-P.-V.) For Carette's examples: $\alpha_p^*(G_1) = \max\{1/p, 1/2\}$ for every $p \geq 1$; and $\alpha_p^*(G_2) = 0$ for $1 \leq p \leq 2$.*
□