Maximal injectivity and disjointness for the radial masa

Chenxu Wen
chenxu.wen@vanderbilt.edu

NCGOA Spring Institute, May 1-7, 2015

Some history on maximal injective subalgebras

- Fuglede & Kadison('51): existence of maximal hyperfinite subfactor
- Popa('83): generator masa in $L(\mathbb{F}_N)$
- Ge('96): types of maximal amenable subalgebras in $L(\mathbb{F}_N)$
- Shen('06): tensor products of generator masa's

Some history on maximal injective subalgebras

- Cameron et al('10): radial masa in $L(\mathbb{F}_N)$
- Brothier('14): cup subalgebra from a subfactor planar algebra
- Houdayer('13,'14): Bojoljubov actions, Gamma-stability for free products
- Boutonnet & Caderi('13,'14): subalgebras from maximal amenable subgroups
- Popa('14): ultraproduct of singular masa

Peterson's Conjecture

Peterson's Conjecture (J. Peterson)

Any diffuse amenable subalgebra of $L(\mathbb{F}_N)$ *has a unique maximal injective extension.*

The generator masa

Theorem (Houdayer'14)

Any diffuse subalgebra of the generator masa has a unique maximal injective extension in $L(\mathbb{F}_N)$.

The radial masa

- $2 \leq N$ finite
- $\omega_n := \sum_{g \in \mathbb{F}_N, |g|=n} u_g$.
- $C = vN\{\omega_1\}$ — the radial masa.

Some facts

- $\{\omega_n\}_{n\geq 0}$ forms an orthogonal basis of C;
- Pytlik('81): C is a masa in $L(\mathbb{F}_N)$.
- Rădulescu('91): C is singular;
- Cameron et al.('10): C is maximal injective.

The radial masa

Theorem (Wen'15)

Let C be the radial masa. C is the unique maximal injective extension of any diffuse subalgebra of itself.

Strong-AOP

Definition (Popa'83, Houdayer'14, W'15)

An inclusion of finite von Neumann algebras $A \subset M$ satisfies the strong asymptotic orthogonality property (s-AOP), if for any diffuse subalgebra B of A and any $(x_n)_n \in B' \cap M^\omega \ominus A^\omega$ and $y_1, y_2 \in M \ominus A$, we have that $y_1(x_n)_n \perp (x_n)_n y_2$.

s-AOP + singularity + s-solidity $\Rightarrow$ Peterson's conjecture

Key result

Theorem (W'15)

$C \subset L(\mathbb{F}_N)$ *satisfies the s-AOP.*

Revisiting Popa's proof

Data:

- $A = vN\{u_a\}$ generator masa
- $u_i \in B \subset A$ unitaries, supported on S_i
- large gaps between different S_i
- L_{k_0} span of all words that starts with some power of a^m with $|m| \leq k_0$.
- L_{k_0} the left projection corresponds to L_{k_0} above.

Revisiting Popa's proof

- $(x_n)_n \in B' \cap M^\omega \ominus A^\omega$, $\|x_n\| \leq 1$.
- On one hand,

$$\lim_{n\to\omega} \sum_{i=N_1}^{N_2} \langle L_{k_0}(u_i x_n), L_{k_0}(u_i x_n)\rangle \approx (N_2 - N_1) \lim_{n\to\omega} \|L_{k_0}(x_n)\|_2^2$$

- On the other hand:

$$\sum_{i=N_1}^{N_2} \langle L_{k_0}(u_i x_n), L_{k_0}(u_i x_n)\rangle \leq \|x_n\|_2^2 \leq 1.$$

Locating the essential support of (x_n)

- Conclusion:

$$\lim_{n\to\omega} \|L_{k_0}(x_n)\|_2^2 = 0.$$

- Then easy to see the s-AOP.

Back to the radial case

Need:

- Find a proper basis for $L^2(M) \ominus L^2(C)$;
- Locate the essential support of $(x_n)_n \in B' \cap M^\omega \ominus C^\omega$;
- Conclude s-AOP.

The Rădulescu basis

Theorem (Rădulescu, '91)

- $L^2(M) \ominus L^2(C) = \bigoplus_{i \geq 1} \mathcal{H}_i$
- *each* $\mathcal{H}_i$ *has a distinguished unit vector* $\xi^i \in \mathcal{K}_{l(i)}$*, for some* $l(i) \in \mathbb{N}$*, s.t.* $\mathcal{H}_i = \overline{C \xi^i C}$.
- $\{\xi^i_{n,m}\}_{i \geq 1, m, n \geq 0}$ *forms a Riesz basis for* $L^2(M) \ominus L^2(C)$.

Locating the essential support

- Suppose $(x_n)_n \in B \cap M^\omega \ominus C^\omega$.
- Define the "left projection" L_{k_0} from $L^2(M) \ominus L^2(C)$ to $\text{span}\{\xi^i_{n,m}\}_{n \le k_0}$.
- Thanks to the work of Cameron et al, only need

Proposition

$$lim_{n \to \omega} \| L_{k_0}(x_n) \|_2^2 = 0, \forall k_0 \in \mathbb{N}.$$

A technical issue

L_{k_0} is not right-C-modular!

An auxiliary basis

- A more natural basis:

$$\{\eta^i_{n,m} := \frac{\omega_n \xi^i \omega_m}{\|\omega_n \xi^i \omega_m\|_2}\}_{i \geq 1, n,m \geq 0}.$$

- Define similarly the left projections L'_{k_0} w.r.t. this basis.
- From this basis to the Rădulescu basis–not too bad.

Thank you!