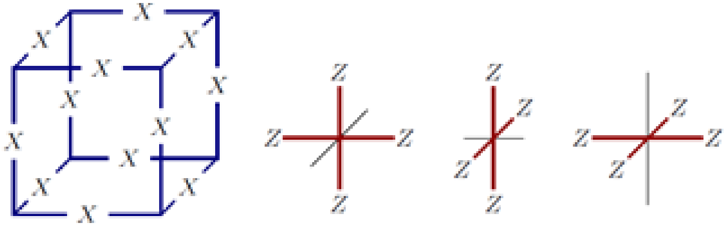
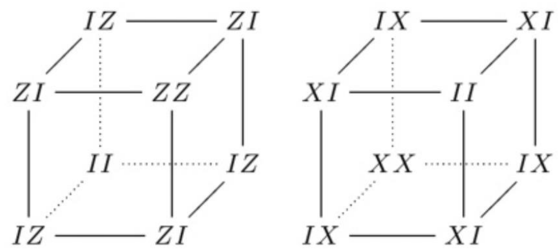
Beyond Anyons: From Fractons to Black Holes





Zhenghan Wang
Microsoft Station Q & UC Santa Barbara

Vanderbilt, May 8, 2019

“Periodic Table” of Topological Phases of Matter

Classification of enriched topological order (TQFT) in all dimensions

Too hard!!!

Special cases:

Symmetry	$d=0$	$d=1$	$d=2$	$d=3$
$U(1) \times Z_2^T$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
Z_2^T	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_1$	$\mathbb{Z}_2$
$U(1)$	$\mathbb{Z}$	$\mathbb{Z}_1$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3)$	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}_1$
$SO(3) \times Z_2^T$	$\mathbb{Z}_1$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$
Z_n	$\mathbb{Z}_n$	$\mathbb{Z}_1$	$\mathbb{Z}_n$	$\mathbb{Z}_1$
$Z_2^T \times D_2 = D_{2h}$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$

Symmetry classes	Physical realizations	$d=1$	$d=2$	$d=3$
D	SC	p-wave SC	$(p+ip)$-SC	0
DIII	TRI SC	Z_2	$(p+ip)(p-ip)$-SC	He^3-B
AII	TRI ins.	0	HgTe Quantum well	$Bi_{1-x}Sb_x$, Bi_2Se_3, etc.
CII	Bipartite TRI ins.	Carbon nanotube	0	Z_2
C	Singlet SC	0	$(d+id)$-SC	0
CI	Singlet TRI SC	0	0	Z
AI	TRI ins. w/o SOC	0	0	0
BDI	Bipartite TRI ins. w/o SOC	Carbon nanotube	0	0

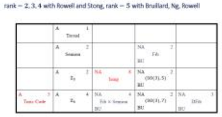
1): short-range entangled (or symmetry protected--SPT) including topological insulators and topological superconductors: X.-G. Wen et al (group cohomology), …, and A. Ludwig et al (random matrix theory) and A. Kitaev (K-theory)---generalized cohomologies,…

2): **Low dimensional: spatial dimensions D=1, 2, 3, n=d=D+1**

2a: **classify 2D topological orders without symmetry**

2b: enrich them with symmetry

2c: 3D much more interesting and harder


Classification of Unitary Modular Categories

rank = 2, 3, 4 with Rowell and Stong, rank = 5 with Bruillard, Ng, Rowell

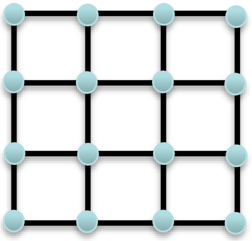
The ith-row lists all rank = i unitary modular tensor categories.

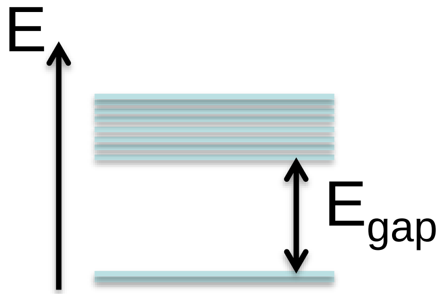
Middle symbol: the fusion rule.
Upper left corner: A = abelian theory, NA = non-abelian.
Upper right corner number = the number of distinct theories.
Lower left corner BU = there is a universal braiding anyon.

Model Topological Phases of Quantum Matter

Local Hilbert Space $\mathcal{H} = \bigotimes_{i=1}^{N} \mathcal{H}_i$

Local, Gapped Hamiltonian $H : \mathcal{H} \to \mathcal{H}$



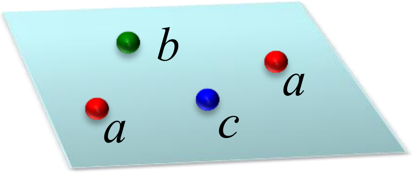


Two gapped Hamiltonians H_1, H_2 realize the same topological phase of matter if there exists a continuous path connecting them without closing the gap/a phase transition.

A topological phase, to first approximation, is a class of gapped Hamiltonians that realize the same phase. Topological order in a 2D topological phase is encoded by a TQFT or anyon model.

Anyons in Topological Phases of Matter

Finite-energy topological quasiparticle excitations=anyons



Quasiparticles a, b, c

Two quasiparticles have the same topological charge
or anyon type if they differ by local operators

Anyons in 2+1 dimensions described mathematically by a
Unitary Modular Tensor Category $\mathcal{C}$

Anyonic Objects

- **Anyons:** simple objects in unitary modular categories
- **Symmetry defects:** simple objects in G-crossed braided fusion categories
- **Gapped boundaries in doubled theories:** Lagrangian algebras
- …

New Directions

- **Fracton models:** (3+1)-gapped lattice models
- **Quantum cellular automatas**
- **BTZ (**Banados-Teitboim-Zanelli**) black holes in (2+1)-gravity with negative cosmological constant as anyon-like objects**
- …

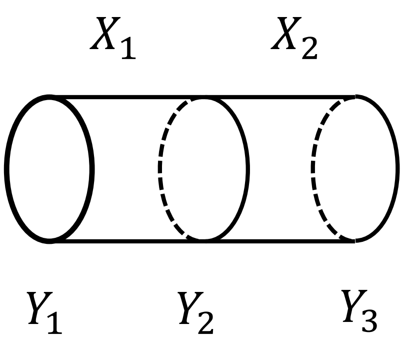
Fracton Physics

- Challenging conventional notion of topological phases of matter
- Sub-dimensional symmetry interpolating between gauge symmetry (0-dim) and global symmetry (n-dim)
- Self-correcting quantum memory
- Interesting physics of loop excitations
- …

Atiyah-Segal Type (3+1)-TQFTs

A symmetric monoidal "functor" (V, Z):
category of 3-4-mfds → Vec
3-mfd Y → vector space $V(Y)$
4-bord X from Y_1 to Y_2 → Z(X): $V(Y_1) \to V(Y_2)$

- $V(S^3) \cong \mathbb{C}$ ⟶ $V(\emptyset) = \mathbb{C}$
- $V(Y_1 \sqcup Y_2) \cong V(Y_1) \otimes V(Y_2)$
- $V(-Y) \cong V^*(Y)$
- $Z(Y \times I) = \mathrm{Id}_{V(Y)}$
- $Z(X_1 \cup X_2) = Z(X_1) \cdot Z(X_2)$



Examples:
1. DW theories
2. State-sum TQFTs: Crane-Yetter premodular, Cui G-crossed

Key feature:
V(Y) is finitely dimensional and depends on only topology

Beyond Atiyah-Segal

- Donaldson-Witten/Seiberg-Witten/Heegaard-Floer, …,
- Fractons:

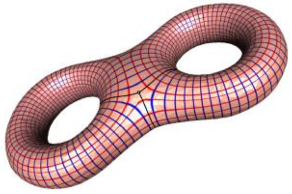
Type I: X-cube,…

Type II: Haah code

V(Y)→V(Y, **t**)

for some topological structure t of the space Y,

e.g. (TV TQFTs with t=triangulation)



$\times S^1$

for fermions, **t**=spin structure

for CS theory, **t**=framing

for X-cube, **t**=a singular compact total foliation

for Haah code, **t**=???

X-cube Model

defined on a cubic lattice with qubit degrees of freedom on the edges. The Hamiltonian

$$H = -\sum_v (A_v^x + A_v^y + A_v^z) - \sum_c B_c \qquad (1)$$

contains two types of terms: cube terms B_c which are products of the twelve Pauli X operators around a cube c, and cross terms A_v^μ which are products of the four Pauli Z operators at a vertex v in the plane normal to the μ-direction where $\mu = x, y,$ or z (Fig. 1).

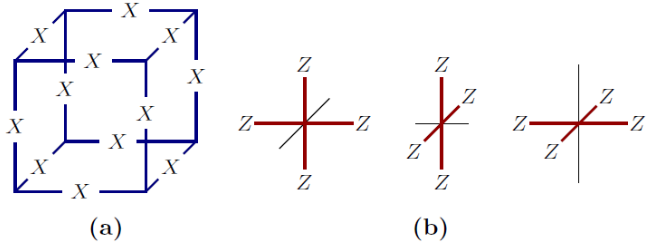


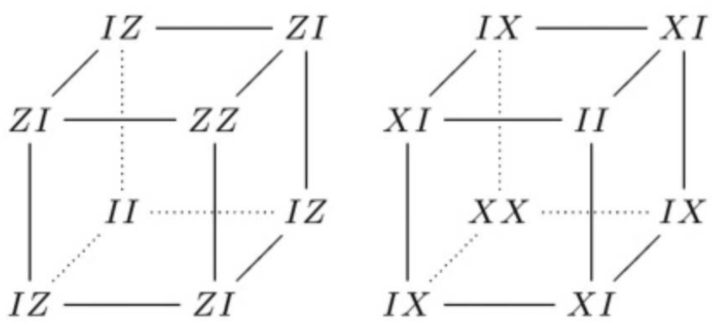
FIG. 1. (a) Cube and (b) cross operators of the X-cube model Hamiltonian on a cubic lattice.

arXiv 1712.05892
Phys. Rev. X 8, 031051

W. Shirley, K. Slagle, Z.W, X. Chen

Haah Code

- On 3-torus T^3:



$$\frac{k+2}{4} = \deg_x \gcd\left((1+x)^L + 1,\ (1+\omega x)^L + 1,\ (1+\omega^2 x)^L + 1\right)_{\mathbb{F}_4}$$
$$= \begin{cases} 1 & \textit{if } L = 2^p + 1, \\ L & \textit{if } L = 2^p, \\ L-2 & \textit{if } L = 4^p - 1, \\ 1 & \textit{if } L = 2^{2p+1} - 1. \end{cases}$$

- Intricate ground state degeneracy 2^k
- Particles are not mobile and fractons
- Self-correcting feature

Groupons or Fractons?

- Particle excitations in Haah code live on vertices of tetrahedra, so come in groups---groupon
- A groupon

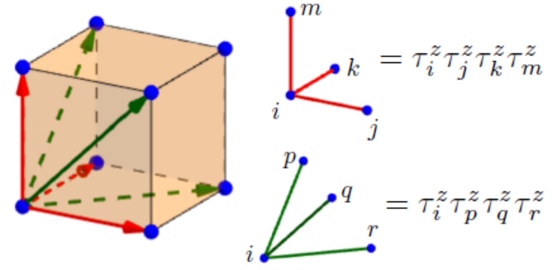
  can be inflated

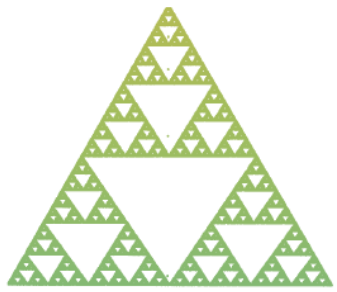
  to be arbitrarily large,

  so each parton of the

  groupon is a fracton

  ---cannot be created with a string operator





- Loop excitations in Haah code are mobile

LR (left/right) or Hastings codes

Let (Γ, S_1, S_2) be a triple such that Γ is a finite group and $S_i, i = 1, 2$ be any two subsets of Γ. Let $L(\Gamma) = \bigotimes_{g\in\Gamma}(\mathbb{C}^2 \otimes \mathbb{C}^2)_g$ be the Hilbert space that assigns a bi-qubit $\mathbb{C}^2 \otimes \mathbb{C}^2$ to each group element g. On each bi-qubit, we define two stabilizers

$$A_g = (\prod_{v\in S_1} \sigma^{z,1}_{gv})(\prod_{w\in S_2} \sigma^{z,2}_{w^{-1}g}),$$

and

$$B_g = (\prod_{v\in S_1} \sigma^{x,2}_{gv^{-1}})(\prod_{w\in S_2} \sigma^{x,1}_{wg}).$$

LR Hamiltonian

- LR Hamiltonian on finite groups: attach a bi-qubit $C^2 \otimes C^2$ (regarded a bi-layer) to each group element g,

$$H = -\sum_g A_g - \sum_g B_g$$

- All local terms commute with each other:

A_g, B_g do no commute on qubits in the top layer

iff $gv = wg, v \in S_1, w \in S_2^{-1}, ie\ \ w^{-1}g = gv^{-1}$,

so A_g, B_g have a corresponding term in the bottom

that do not commute either.

There is no sense of locality.

Generalized Haah codes on general 3-mfds

(with K. Tian and E. Samperton, arXiv:1812.02101, arXiv:1902.04543)

Given a 3-mfd M, fix a pair of finite subsets $\boldsymbol{S_1}, \boldsymbol{S_2}$ of G=$\pi_1(M)$, and **a FIN N (finite index normal subgroup of G)**, then S_1, S_2 are sets of each quotient G_N=G/N.

The LR code of M with structure **t=($\boldsymbol{S_1}, \boldsymbol{S_2}$, N)** is the LR codes on G_N. The code space depends on t.

Bi-qubit can be generalized to 2q qubits with 2q subsets.

For q=2, this generalizes the Haah-B code.

Fundamental groups as lattices

- $S_1 \sqcup S_2$ is a unit cell
- N is a periodic boundary condition
- Fundamental groups of geometric 3-manifolds are “lattices” in the universal covers.
- Loop lattices in 3-manifolds.

Recover Haah code

Fundamental group of T^3 is Z^3, so subgroups are N=L_xZ x L_yZ x L_zZ.

Let $S_1 = \{1, x, y, z\}, S_2 = \{1, xy, yz, xz\}$. Then the LR code on $Z_{L_x} \times Z_{L_y} \times Z_{L_z}$recovers the Haah code.

LR Codes on Thurston Geometries

- Thurston 8 geometries: E^3,H^3,S^3,$H^2 \times R$,$S^2 \times R$, Sol, Nil, SL(2,R)
- Find a closed 3-mfd in each geometry and study the LR codes for various FINs.
- Haah code on T^3 for E^3 geometry. Can do $H^2 \times R$, $S^2 \times R$.
- For $S^2 \times R$, consider $RP^3 \# RP^3$.

Fracton Phases

- When some scaling limit exists? LERF relevant?

- V(Y,t) will be infinitely dimensional in the scaling limit. What are the background physics or free resources for RG?

- Lagrangian formulation? Probably no.

- Categorical structure of elementary excitations, especially loops:
  Representation of motion groups,
  e.g. 3-loop statistics?

- A step to Seiberg-Witten theory?

Edge Physics of Topological Phases

Gapless boundary is described by a chiral conformal field theory (χCFT)

B$ulk - Edge$: **CFTs → TQFTs**

Ads/CFT: **CFTs → (2+1)-gravity**

Primary fields --- Anyons
Primary fields --- BTZ black holes

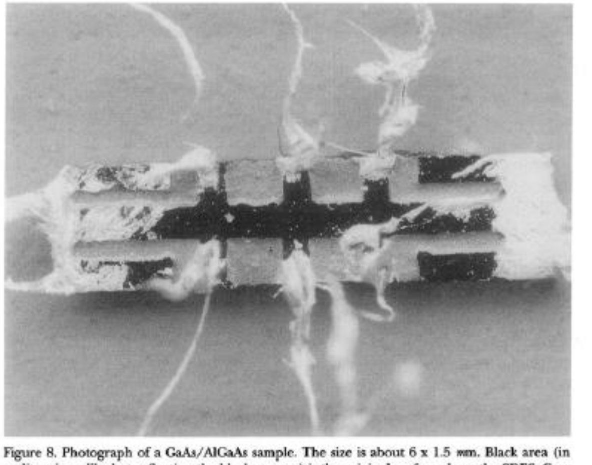


Figure 8. Photograph of a GaAs/AlGaAs sample. The size is about 6 x 1.5 mm. Black area (in reality mirror-like but reflecting the black camera) is the original surface above the 2DES. Gray areas have been scratched away to confine the current path to the center of the sample. White areas are indium blotches used to make contact to the 2DES. Gold wires are attached. Specimens like this one, prepared with little attention to exact dimension nor to tidiness, show quantization of the Hall resistance to an accuracy of a few 10 parts in a billion. The specimen shown is the sample in which the fractional quantum Hall effect (FQHE) was discovered in 1981.

Ising CFT as Gravity Dual

(with C. Jian, H. Sun, A. Ludwig, Z. Luo)

- 3d gravity:

Einstein–Hilbert action is

$$I(g) = \int_{X^3} d^3x \sqrt{g}(R - 2\Lambda)\,,$$

where R is the scalar curvature and Λ the cosmological constant. The equation of motion gives rise to the Einstein equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0\,,$$

where $R_{\mu\nu}$ is the Ricci curvature.

The $3d$ anti-de Sitter space is the subspace of $\mathbb{R}^4$ defined as

$$\{-x_1^2 - x_2^2 + x_3^2 + x_4^2 = -l^2\}\,,$$

for some constant $l > 0$ with the metric $ds^2 = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$. Direct computation gives

$$R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu}\,, \qquad R = -\frac{6}{l^2}\,,$$

therefore the gravitational field $ds^2 = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2$ is a solution of the Einstein equation if we choose the negative cosmological constant $\Lambda = -\frac{1}{l^2}$. Topologically the anti-de Sitter space is simply $S^1 \times \mathbb{R}^2$.

- Ising CFT is a dual of (2+1)-gravity with negative cosmological constant $\Lambda = -\frac{1}{l^2}$ and $c = \frac{1}{2} = \frac{3G}{2l}$.

Three Basic Facts

- (2+1)-gravity is doubled CS theory with G=SL(2,R) classically, but there are subtleties in quantization.
- Brown-Henneaux theorem:

  Virasoro algebra is asymptotic

  symmetry
- For negative Λ, there are BTZ black holes

Equality of Partition functions

- Partition functions of 3d gravity:

$$Z_{grav}(\tau,\bar{\tau}) = \int_{\partial X = T^2} \mathcal{D}g \; e^{-cS_E[g]},$$

$$Z_{grav}(\tau,\bar{\tau}) = \sum_{\gamma\in\Gamma} Z_{vac}(\gamma\tau,\gamma\bar{\tau}) \qquad Z_{\text{grav}} = \sum_{\gamma\in\Gamma} |\chi_{1,1}(\gamma\tau)|^2,$$

- Equal to from Ising CFT:

$$Z_{\text{grav}} = 8Z_{\text{Ising}}. \qquad Z_{\text{grav}} = 384Z_{Ising}. \qquad Z_{grav} = c_g Z_{Ising}$$

genus=1 genus=2 some c_g for each genus g.

Castro et al.
Physical Review D 85.2 (2012)

Why Ising?

- Ising TQFT representations of mapping class groups have finite images (G. Wright). We have a new proof.
- They are all irreducible.
- $h_i \geq \frac{c}{24}$ for non-vacuum.

BTZ Black Holes?

a new type of AdS-Schwarzschild black hole with Lorentzian metric

$$ds_L^2 = -N_L^2 dt_L^2 + N_L^{-2} dr^2 + r^2(d\phi + N_L^\phi dt)^2,$$

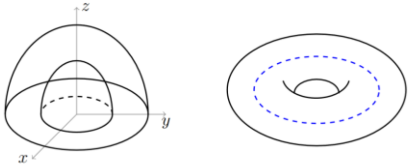
Let $t_L = it$ and $J_L = iJ$, and we do the Wick rotation to get

$$ds^2 = N^2 dt^2 + N^{-2} dr^2 + r^2(d\phi + N^\phi dt)^2,$$

with $N^2 = -8GM + \frac{r^2}{l^2} - \frac{16G^2J^2}{r^2}$, $N^\phi(r) = -\frac{4GJ}{r^2}$. The horizons are ı

$$r_\pm^2 = 4GMl^2 \left(1 \pm \sqrt{1 + \frac{J^2}{M^2 l^2}}\right).$$

The Euclidean BTZ black hole is locally isometric to the hyperbolic tl
globally described by $\mathbb{H}^3/\Gamma$ with $\Gamma \cong \mathbb{Z}$. The topology is a solid torus



Anyons or gapped boundaries?

Driving Force of Convergence of Math & Sciences

Physics	Mathematics
Newtonian Mechanics	Calculus (arranged marriage)
General Relativity and Gauge theory	Differential Geometry
Quantum Mechanics	Linear Algebra and Functional Analysis
Many-body Entanglement Physics	??? (Quantum topology/algebra)

Tip of iceberg: 2D Topological Phases of Matter

(Microscopic physics?) **Topological Quantum Field Theory (TQFT)**
Conformal field theory (CFT)

Name: Nathan Seiberg
Event: Workshop: Future Prospects for Fundamental Particle Physics and Cosmology
Title: What is Quantum Field Theory?
Date: 2015-05-05 @2:45 PM

What is quantum field theory mathematically?
New calculus for the second quantum revolution and future.