Von Neumann equivalence and M_d type approximation properties

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May 8 - 11, 2023

Research supported by JSPS

The main result

Theorem 1 (B., 2023)

M_d type approximation properties are stable under von Neumann equivalence.

Corollary 2

M_d type approximation properties are stable under Measure equivalence and W-equivalence.*

Outline

1. M_d type approximation properties
2. Von Neumann equivalence
3. Proof of the main theorem

Recall

Let G, Γ, and Λ be discrete groups.

- **Regular representation**:

$$\lambda : G \to \mathcal{U}(\ell^2(G)), \quad \lambda(x)f(y) = f(x^{-1}y)$$

- **Group von Neumann algebra**:

$$L(G) := \lambda(G)'' \subseteq \mathcal{B}(\ell^2(G))$$

- **Fourier algebra**:

$$A(G) = \{\varphi = \langle \lambda(\cdot)f, g\rangle \mid f, g \in \ell^2(G)\}$$
$$\|\varphi\|_A = \inf \|f\|_2\|g\|_2$$

- **Fourier-Stieltjes algebra**:

$$B(G) = \{\varphi = \langle \pi(\cdot)v, w\rangle \mid (\pi, \mathcal{H}) \text{ unitary rep, } v, w \in \mathcal{H}\}$$
$$\|\varphi\|_B = \inf \|v\|\|w\|$$

Recall

- **Completely bounded Fourier multipliers**

$$
\begin{aligned}
M_{cb}A(G) &= \{\varphi \in \ell^\infty(G) \mid \|\varphi\|_{cb} < \infty\} \\
\|\varphi\|_{cb} &= \|M_\varphi : f \in A(G) \mapsto \varphi f \in A(G)\|_{cb} \\
&= \|M_\varphi^* : \lambda(x) \in L(G) \mapsto \varphi(x)\lambda(x) \in L(G)\|_{cb} \\
&= \sup_n \|M_\varphi^* \otimes \mathrm{id} : L(G) \otimes M_n \to L(G) \otimes M_n\|
\end{aligned}
$$

Theorem 3 (Bożejko-Fendler, 1984; Jolissaint, 1992)

A function $\varphi \in \ell^\infty(G)$ is in $M_{cb}A(G)$ iff there exist a Hilbert space $\mathcal{H}$ and bounded maps $\xi, \eta : G \to \mathcal{H}$ such that

$$\varphi(y^{-1}x) = \langle \xi(x), \eta(y) \rangle, \quad (\forall x, y \in G).$$

Moreover, we have

$$\|\varphi\|_{cb} = \inf \|\xi\|_\infty \|\eta\|_\infty.$$

The space $M_d(G)$ of M_d-multipliers

Definition 4 (Pisier, 2005)

Let $d \geq 2$ be an integer. A function $\varphi \in \ell^\infty(G)$ is called *M_d-multiplier* if there exist Hilbert spaces $\mathcal{H}_i$ and bounded maps $\xi_i : G \to \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i-1})$ such that $\mathcal{H}_0 = \mathcal{H}_d = \mathbb{C}$

$$\varphi(x_1 \cdots x_d) = \xi_1(x_1) \cdots \xi_d(x_d)(1), \quad (\forall x_1, \dots x_d \in G).$$

We write

$$\|\varphi\|_{M_d} = \inf \|\xi_1\|_\infty \cdots \|\xi_d\|_\infty$$
$$M_d(G) = \{\varphi \in \ell^\infty(G) \mid \|\varphi\|_{M_d} < \infty\}.$$

We can embed $\ell^1(G)$ in $M_d(G)^*$ using (ℓ^1, ℓ^∞)-duality.
We denote by $X_d(G)$ the completion of $\ell^1(G)$ in $M_d(G)^*$.

Theorem 5 (Pisier, 2005)

We have $X_d(G)^* \cong M_d(G)$ isometrically.

$$A(G) \subseteq B(G) \subseteq M_{d+1}(G) \subseteq M_d(G) \subseteq M_2(G) = M_{cb}A(G) \subseteq \ell^\infty(G)$$

Theorem 6 (Bożejko, 1985)

$B(G) = M_{cb}A(G)$ *iff* G *is amenable.*

Theorem 7 (Pisier, 2005)

$B(F_\infty) \neq M_{d+1}(F_\infty) \neq M_d(F_\infty)$ *for any* $d \geq 2$.

$M_d(G)$ in Similarity Problem

Definition 8 (Unitarizable group)

A group is *unitarizable* if any bounded representation $\pi : G \to GL(\mathcal{H})$ is similar to a unitary representation, i.e. there is $S \in GL(\mathcal{H})$ such that $(S^{-1}\pi(\cdot)S, \mathcal{H})$ is a unitary representation.

Conjecture 9 (Dixmier's Similarity Problem, 1950)

Unitarizability = Amenability.

Theorem 10 (Pisier, 2005)

If G is unitarizable, then $B(G) = M_d(G)$ for all large enough $d \in \mathbb{N}$.

M_d type approximation properties

Definition 11 (Vergara, 2023)

We say that G has *M_d-approximation-property* (M_d-AP) if there is a net (φ_i) in $A(G)$ such that $\varphi_i \to 1$ in $\sigma(M_d(G), X_d(G))$-topology.

Proposition 12

Amenability $\Rightarrow M_{d+1}$-AP $\Rightarrow M_d$-AP $\Rightarrow$ AP.

Theorem 13 (Vergara, 2023)

A discrete countable group acting properly on a finite dimensional CAT(0) cube complex has M_d-AP (actually M_d-WA) for all $d \geq 2$.

Theorem 14 (Vergara, 2023)

If G is unitarizable and has M_d-AP for any $d \geq 2$, then G is amenable.

AP is ME and W*E invariant

Theorem 15 (Haagerup-Kraus, 1994)

*AP is stable under ME and W*E.*

Theorem 16 (Ishan, 2021)

AP is stable under vNE.

Von Neumann equivalence

(M, Tr): vN-algebra with a faithful normal semifinite trace
$\mathrm{Aut}(M, \mathrm{Tr})$: the trace preserving *-automorphisms

Definition 17 (Action and fundamental domain)

A group homomorphism $\sigma : G \to \mathrm{Aut}(M, \mathrm{Tr})$ is called a **trace preserving action**. A **fundamental domain** for σ is a projection $p \in M$ such that $\sum_{x \in G} \sigma_x(p) = 1$ in SOT.

Definition 18 (Von Neumann equivalence)

We say that Γ and Λ are **von Neumann equivalent** (vNE in short) and write $\Gamma \sim_{vNE} \Lambda$ if there is a trace preserving action

$$\sigma : \Gamma \times \Lambda \to \mathrm{Aut}(M, \mathrm{Tr})$$

such that each of Γ and Λ actions admits a finite trace fundamental domain.

Measure equivalence

Definition 19 (Measure equivalence)

We say that Γ and Λ are **Measure equivalent** (ME in short) and write $\Gamma \sim_{ME} \Lambda$ if there is a standard measure space (X, μ) and a measure preserving action $\Gamma \times \Lambda \curvearrowright (X, \mu)$ such that each of Γ and Λ actions admits a finite measure fundamental domain, say $A \subseteq X$ and $B \subseteq X$, respectively.

Example 20

1. Lattices in a second countable locally compact group are ME.
2. (Ornstein-Weiss, 1980) Infinitely countable amenable groups form a single ME-class.
3. (Furman, 1999) $SL_n(\mathbb{Z})$, $n \geq 3$ are pairwise not ME.

$\Gamma \sim_{ME} \Lambda$ implies $\Gamma \sim_{vNE} \Lambda$. Indeed, consider

$$\sigma : \Gamma \times \Lambda \to \mathrm{Aut}\left(L^\infty(X, \mu), \int d\mu\right), \quad \sigma_{\gamma,s}(f)(x) = f((\gamma, s)^{-1}.x).$$

W*-equivalence

Definition 21 (W*-equivalence)

We say that Γ and Λ are **W*-equivalent** (W*E in short) and write $\Gamma \sim_{W^*E} \Lambda$ if $L(\Gamma) \cong L(\Lambda)$.

Example 22

(Connes, 1976) ICC amenable groups form a single W*E-class.

$\Gamma \sim_{W^*E} \Lambda$ implies $\Gamma \sim_{vNE} \Lambda$. Given a *-isomorphism $\phi : L(\Gamma) \to R(\Lambda)$, choose

$$(M, \mathrm{Tr}) = (\mathcal{B}(\ell^2(\Lambda)), \mathrm{Tr}), \quad p = q = \mathrm{Proj}(\ell^2(\Lambda) \twoheadrightarrow \mathbb{C}\,\delta_e).$$

Then the action

$$\sigma : \Gamma \times \Lambda \to \mathrm{Aut}(M, \mathrm{Tr}), \quad \sigma_{\gamma,s}(T) = \phi(\lambda_\Gamma(\gamma))\lambda_\Lambda(s) T \lambda_\Lambda(s)^* \phi(\lambda_\Gamma(\gamma))^*$$

gives a vNE coupling for Γ and Λ.

vNE invariants

1. (Ishan-Peterson-Ruth, 2019)
   Amenability, a-T-menability, Property (T), and proper proximality.
2. (Ishan, 2021)
   Weak amenability, weak Haagerup property, and AP.
3. (B., 2022)
   Exactness and M_d type approximation properties.

Proof of the main theorem

Theorem 23 (B., 2023)

Suppose that $\Gamma \sim_{vNE} \Lambda$ and that Λ has M_d-AP. Then Γ has M_d-AP too.

Suppose that $\sigma : \Gamma \times \Lambda \to \mathrm{Aut}(M, \mathrm{Tr})$ gives a vNE-coupling.
Let p be a finite trace f.d. for Λ-action. Denote

$$\theta_p : f \in \ell^\infty(\Lambda) \to \sum_{s \in \Lambda} f(s)\sigma_s^{-1}(p) \in M.$$

The induction map is defined as

$$\Phi : \varphi \in \ell^\infty(\Lambda) \mapsto \widehat{\varphi} \in \ell^\infty(\Gamma), \quad \widehat{\varphi}(\gamma) = \mathrm{Tr}(\sigma_\gamma(p)\theta_p(\varphi)).$$

Proof of the main theorem

Lemma 24 (Ishan, 2021)

The map $\varphi \in A(\Lambda) \to \widehat{\varphi} \in A(\Gamma)$ *is a well defined contraction.*

Lemma 25

The map $\varphi \in M_d(\Lambda) \to \widehat{\varphi} \in M_d(\Gamma)$ *is a w*-continuous contraction.*

Proof of the main theorem.

Since Λ has M_d-AP, there is a net (φ_i) in $A(\Lambda)$ such that $\varphi_i \to 1$ in $\sigma(M_d(\Lambda), X_d(G))$-topology.
Lemma 24 shows that $\widehat{\varphi}_i \in A(\Gamma)$.
Lemma 25 shows that $\widehat{\varphi}_i \to \widehat{1} = 1$ in $\sigma(M_d(\Gamma), X_d(\Gamma))$-topology. □

Proof of Lemma 25

Suppose that $\varphi \in M_d(\Lambda)$, $\xi_i : \Lambda \to \mathcal{B}(\mathcal{H}_i, \mathcal{H}_{i-1})$ are such that

$$\varphi(s_1 \cdots s_d) = \xi_1(s_1) \cdots \xi_d(s_d)(1), \quad (s_1, \ldots, s_d \in \Lambda).$$

Define $\widehat{\mathcal{H}}_i$ and $\widehat{\xi}_i : \Gamma \to \mathcal{B}(\widehat{\mathcal{H}}_i, \widehat{\mathcal{H}}_{i-1})$ as

$$\widehat{\mathcal{H}}_0 = \widehat{\mathcal{H}}_d = \mathbb{C}, \quad \widehat{\mathcal{H}}_{2i} = (L^2(M, \mathrm{Tr})p) \overline{\otimes} \mathcal{H}_{2i}, \quad \widehat{\mathcal{H}}_{2i+1} = (pL^2(M, \mathrm{Tr})) \overline{\otimes} \mathcal{H}_{2i+1},$$

$$\widehat{\xi}_{2i}(\gamma)(xp \otimes u) = \sum_{s \in \Lambda} p\sigma_{\gamma,s}(xp) \otimes \xi_{2i}(s)(u),$$

$$\widehat{\xi}_{2i+1}(\gamma)(px \otimes v) = \sum_{s \in \Lambda} \sigma_{\gamma,s}(px)p \otimes \xi_{2i+1}(s)(v),$$

$$\widehat{\xi}_1(\gamma)(px \otimes w) = \sum_{s \in \Lambda} \mathrm{Tr}\left(\sigma_{\gamma,s}(px)p\right) \xi_1(s)(w).$$

Proof of Lemma 25

$$\widehat{\xi}_1(\gamma_1)\cdots\widehat{\xi}_d(\gamma_d)(1) =$$

$$= \sum_{s_d\in\Lambda}\dots\sum_{s_1\in\Lambda}\mathrm{Tr}\left(\Pi\left(\sigma_{\gamma_1\cdots\gamma_d,s_1\cdots s_d}(p),\dots,\sigma_{\gamma_1,s_1}(p),p\right)\right)\xi_1(s_1)\cdots\xi_d(s_d)(1)$$

$$= \sum_{s_d\in\Lambda}\dots\sum_{s_1\in\Lambda}\mathrm{Tr}\left(\Pi\left(\sigma_{\gamma_1\cdots\gamma_d,s_1}(p),\dots,\sigma_{\gamma_1,s_1s_d^{-1}\cdots s_2^{-1}}(p),p\right)\right)\varphi(s_1)$$

$$= \sum_{s_d\in\Lambda}\dots\sum_{s_2\in\Lambda}\mathrm{Tr}\left(\Pi\left(\sigma_{\gamma_1\cdots\gamma_d}(p),\dots,\sigma_{\gamma_1,s_d^{-1}\cdots s_2^{-1}}(p),\left[\sum_{s_1\in\Lambda}\varphi(s_1)\sigma_{s_1^{-1}}(p)\right]\right)\right)$$

$$= \sum_{s_d\in\Lambda}\dots\sum_{s_2\in\Lambda}\mathrm{Tr}\left(\Pi\left(\sigma_{\gamma_1\cdots\gamma_d}(p),\dots,\sigma_{\gamma_1,s_d^{-1}\cdots s_2^{-1}}(p),\theta_p(\varphi)\right)\right)$$

$$= \mathrm{Tr}\left(\sigma_{\gamma_1\cdots\gamma_d}(p)\theta_p(\varphi)\right) = \widehat{\varphi}(\gamma_1\cdots\gamma_d).$$

Thus $\widehat{\varphi}\in M_d(\Gamma)$ with $\|\widehat{\varphi}\|_{M_d}\leq\prod\|\widehat{\xi}_i\|_\infty\leq\prod\|\xi_i\|_\infty\approx\|\varphi\|_{M_d}$.
The w*-continuity follows from normality of $\Phi:\ell^\infty(\Lambda)\to\ell^\infty(\Gamma)$.

Thank you for your attention!

Amenability and vNE

Let $\Gamma \sim_{vNE} \Lambda$. Let $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ be a unitary representation.$\rightsquigarrow$
Then $\widehat{\pi} : \Gamma \to \mathcal{U}((\overline{M \otimes \mathcal{H}})^{\Lambda})$ is a unitary representation.

Theorem 26 (Ishan-Peterson-Ruth, 2019)

Suppose $\Gamma \sim_{vNE} \Lambda$. Let π and α be unitary representations of Λ.

1. *If $\pi \prec \alpha$, then $\widehat{\pi} \prec \widehat{\alpha}$.*
2. $\widehat{\lambda}_{\Lambda} \cong \lambda_{\Gamma} \otimes 1_{\mathcal{H}}$.

Corollary 27 (Ishan-Peterson-Ruth, 2019)

Suppose that $\Gamma \sim_{vNE} \Lambda$ and that Λ is amenable. Then Γ is amenable.

Proof.

$1_{\Gamma} \leq \widehat{1}_{\Lambda} \prec \widehat{\lambda}_{\Lambda} \cong \lambda_{\Gamma} \otimes 1_{\mathcal{H}} \sim \lambda_{\Gamma}$. □

Proof of Lemma 24

Take $\varphi \in A(\Lambda)$. We have $\varphi(s) = \langle \lambda_\Lambda(s)\xi, \eta\rangle$ for some $\xi, \eta \in \ell^2(\Lambda)$. Define

$$\hat{\xi} = \sum_{s\in\Lambda} \xi(s)\sigma_s^{-1}(p) \quad \text{and} \quad \hat{\eta} = \sum_{t\in\Lambda} \eta(t)\sigma_t^{-1}(p) \quad \text{in} \quad L^2(M, \mathrm{Tr}).$$

Then $\widehat{\varphi}(\gamma) = \langle \sigma_\Gamma^0(\gamma)\hat{\xi}, \hat{\eta}\rangle$, so

$$\varphi \in A_{\sigma_\Gamma^0}(\Gamma) = A_{\lambda_\Gamma \otimes \mathrm{id}}(\Gamma) = A_{\lambda_\Gamma}(\Gamma) = A(\Gamma).$$