A categorical Connes' $\chi(M)$

A UBTC from a II_1-factor

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A categorical Connes' $\chi(M)$ (arXiv: 2111.06378)
Gauging theory for categorical Connes' $\chi(M)$ (in preparation, 2023)

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Tensor category

A tensor category $\mathcal{C}$ is a $\mathbb{C}$-linear category with a bilinear tensor structure $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$.

Examples:

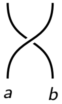
- Vec, Vec(G), Rep(G)
- Bim(M) with $\boxtimes_M$

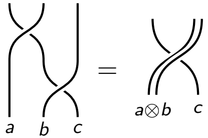
$$\begin{aligned} \alpha : G &\to \mathrm{Aut}(M) \\ g &\mapsto \alpha_g, \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} F_\alpha : \mathrm{Hilb}(G) &\to \mathrm{Bim}(M) \\ \mathbb{C}_g &\mapsto L^2 M_{\alpha_g}. \end{aligned}$$

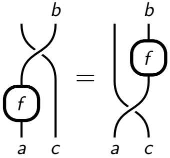
Let $N \subset M$ be a finite depth finite index II_1-subfactor, which means its even part of standard invariant $\mathrm{Std}(N \subset M) := \langle {}_N M_N \rangle \subset \mathrm{Bim}(N)$ under $\boxtimes_N, \oplus, \subset$ is a unitary fusion category.

Braided tensor category

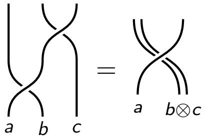
A braided tensor category is a tensor category with a braiding:

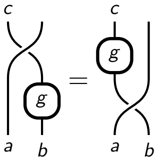
$\beta_{a,b} =$  $\in \mathcal{C}(a \otimes b \to b \otimes a)$





$f \in \mathcal{C}(a \to b)$





$g \in \mathcal{C}(b \to c)$

Connes' Original $\chi(M)$

[Connes 75] Let M be a II_1-factor, $\alpha \in \mathrm{Aut}(M)$ is

- Approximately Inner: if $\exists \{u_n\} \subset U(M)$ such that $\|\alpha(x) - \mathrm{Ad}\, u_n(x)\|_2 \to 0$ for all $x \in M$. $\rightsquigarrow$ group $\overline{\mathrm{Int}(M)}$
- Centrally Trivial: if for all central sequences $\{a_n\} \subset M$, $\|\alpha(a_n) - a_n\|_2 \to 0$ $\rightsquigarrow$ group $\mathrm{Ct}(M)$

$$\chi(M) := \left(\overline{\mathrm{Int}(M)} \cap \mathrm{Ct}(M)\right) / \mathrm{Int}(M) \text{ is an abelian group.}$$

[Jones 80] Quadratic form $\kappa : \chi(M) \to U(1)$.

$\rightsquigarrow$ by (Eilenberg–MacLane), $(\chi(M), \kappa)$ gives a UBTC.

Categorical $\tilde{\chi}(M)$

Notice that $\alpha \in \mathrm{Aut}(M) \iff L^2(M)_\alpha \in \mathrm{Bim}(M)$

[Popa 94] Let $H \in \mathrm{Bim}(M)$ be a right finite Hilbert bimodule.

- Approximately Inner: if there exists an approximate inner Pimsner-Popa basis $\{b_i^n\}_{\substack{1\le i\le K \\ n\in\mathbb{N}}} \in H_M^\circ$, such that
  1. $\sup_{i,n} \| L_{b_i^n} \| < \infty$
  2. $\sum_{i=1}^{K} L_{b_i^n} L_{b_i^n}^* - \mathrm{id}_H \to 0$ SOT $\rightsquigarrow$ UTC $\mathrm{Bim}_{\mathrm{ai}}(M)$
  3. $\| a b_i^n - b_i^n a \|_H \to 0$ for all $a \in M$.
- Centrally Trivial: if for all central sequences $\{a_n\} \in M$ such that $\| a_n \xi - \xi a_n \|_H \to 0$ for all $\xi \in H$. $\rightsquigarrow$ UTC $\mathrm{Bim}_{\mathrm{ct}}(M)$

$$\tilde{\chi}(M) = \mathrm{Bim}_{\mathrm{ai}}(M) \cap \mathrm{Bim}_{\mathrm{ct}}(M) \text{ is a UTC.}$$

Categorical $\tilde{\chi}(M)$

Theorem (C, Jones, Penneys 21)

Let $X \in \mathrm{Bim}_{\mathrm{ai}}(M)$ with approximately inner P-P basis $\{b_i^n\}$, and $Y \in \mathrm{Bim}_{\mathrm{ct}}(M)$ with P-P basis $\{c_j\}$

$$u_{X,Y} := \lim_{n\to\infty}(L_{c_j} \otimes L_{b_i^n}) \circ (L_{b_i^n}^* \otimes L_{c_j}^*) : X \boxtimes_M Y \to Y \boxtimes_M X$$

exists and is a unitary centralizing structure satisfying all the coherences.

Therefore, $\tilde{\chi}(M)$ is a UBTC, which is a full subcategory of $\mathrm{Bim}(M)$.

- $\chi(M) \subset \tilde{\chi}(M)$ as a braided subcategory.
- If M and N are Morita equivalent, then $\tilde{\chi}(M) \cong \tilde{\chi}(N)$ as UBTCs.

Examples

- Let R be the hyperfinite II_1-factor and N be a non-gamma II_1-factor, $\tilde{\chi}(R) \cong \tilde{\chi}(N) \cong \tilde{\chi}(R\overline{\otimes}N) \cong \mathrm{Hilb}$
- Let $N \subset M$ be a finite depth finite index non-gamma II_1-subfactor. Let $M_0 = N \subset M = M_1 \subset M_2 \subset \cdots$ be the Jones Tower and

$$M_\infty := \varinjlim M_n = \overline{\bigcup_n M_n}^{\mathrm{SOT}}$$

Suppose standard invariant $\mathcal{C} := \mathcal{C}(N \subset M)$, we show that

Theorem (C, Jones, Penneys 21)

$\tilde{\chi}(M_\infty) \cong \mathcal{Z}(\mathcal{C})$ *as UBTCs*

Conjecture: If II_1 factor $M \cong M \otimes R$ and $\tilde{\chi}(M) \cong \mathrm{Hilb}$ then $M \cong R \otimes N$ for some non-gamma II_1 factor N.

Corollaries

Corollary 1: Let $N_1 \subset N_2$ and $M_1 \subset M_2$ be non-gamma II_1-subfactors. If $\mathrm{Std}(N_1 \subset N_2)$ and $\mathrm{Std}(M_1 \subset M_2)$ are not Morita equivalent, then N_∞ and M_∞ are not isomorphic.

Corollary 2: By Popa's subfactor reconstruction theorem/GJS construction, every unitary fusion category can be realized as a standard invariant of non-gamma II_1-subfactor. Therefore, every Drinfeld center of a UFC can be realized as $\tilde{\chi}(M)$ for some II_1-factor.

Connes' short exact sequence

Let G be a finite group and $\psi : G \to \mathrm{Aut}(M)$ be an outer action on a II_1 factor M. Suppose ψ is neither approximately inner nor centrally trivial.

Theorem (Connes)

There is a short exact sequence:

$$0 \to \mathrm{Char}(G) \to \chi(M \rtimes_\psi G) \to \bigcup_{g \in G} g\,\mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)} \to 0$$

where $\alpha \in g\,\mathrm{Ct}(M)$ *if* $\psi_{g^{-1}}\alpha \in \mathrm{Ct}(M)$.

$$0 \to \mathrm{Char}(G) \to \chi(M \rtimes_\psi G) \to \bigcup_{g \in G} g\,\mathrm{Ct}(M) \cap \overline{\mathrm{Int}(M)} \to 0$$

$$\Downarrow$$

$$\mathrm{Rep}(G) \qquad \tilde{\chi}(M \rtimes_\psi G) \qquad \bigoplus_{g \in G} g\mathrm{Bim}_{\mathrm{ct}}(M) \cap \mathrm{Bim}_{\mathrm{ai}}(M)$$

Theorem (Muger 04)

$$\left\{ \begin{array}{l} \text{braided tensor categories} \\ \text{containing } \mathrm{Rep}(G) \end{array} \right\} \underset{Equivariantization}{\overset{De-equivariantization}{\rightleftarrows}} \left\{ \begin{array}{l} G\text{-crossed braided} \\ \text{tensor categories} \end{array} \right\}$$

Gauging theory for $\tilde{\chi}(M)$

Suppose ψ is neither centrally trivial nor approximately inner.

- $\tilde{\chi}_G(M) := \bigoplus_{g \in G} g\mathrm{Bim}_{\mathrm{ct}}(M) \cap \mathrm{Bim}_{\mathrm{ai}}(M)$ is a G-crossed braided unitary tensor category (with faithful grading).
- $\mathrm{Rep}(G) \subset \tilde{\chi}(M \rtimes_\psi G)$ as fully braided subcategory.
- $\mathrm{Fun}(G)$-mod in $\tilde{\chi}(M \rtimes_\psi G)$ is equivalent to $\tilde{\chi}_G(|\mathrm{Fun}(G)|)$ as G-crossed braided unitary tensor categories.

Note that $M \subset M \rtimes_\psi G \subset |\mathrm{Fun}(G)|$ is Jone's basic construction, so $|\mathrm{Fun}(G)|$ is Morita equivalent to M, $\tilde{\chi}_G(|\mathrm{Fun}(G)|) \cong \tilde{\chi}_G(M)$.

Theorem (C, Jones, Penneys 23)

$\tilde{\chi}(M \rtimes_\psi G) \cong (\tilde{\chi}_G(M))^G$.

Thank you!