Wreath-like product groups and rigidity aspects of their von Neumann algebras

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Group von Neumann algebras (Murray-von Neumann '43)

Def: G- countable discrete group and $u : G \to \mathcal{U}(\ell^2 G)$ left regular rep.
Group von Neumann algebra

$$\mathrm{L}(G) = \overline{\mathbb{C}[G]}^{\mathrm{sot}} = \overline{\mathrm{span}\{u_g : g \in G\}}^{\mathrm{sot}} \subset \mathcal{B}(\ell^2 G)$$

- $\mathrm{L}(G)$ admits a faithful, normal, tracial state: $\tau(u_g) = \delta_{g,1}$, $\forall g \in G$.
- A von Neumann algebra $\mathcal{M}$ that admits a trace and cannot be decomposed as a direct sum (trivial center) is called a II_1 factor.
- $\mathrm{L}(G)$ is a II_1 factor iff G is icc ($|\{hgh^{-1} : h \in G\}| = \infty, \quad \forall g \neq 1$).
- If $\mathcal{M}$ is II_1 factor so is $p\mathcal{M}p$ for any projection $0 \neq p \in \mathcal{M}$; the isom class of $p\mathcal{M}p$ depends only on $\tau(p) = t$ and is denoted by $\mathcal{M}^t$.

Central problems: (a) Classify $\mathrm{L}(G)$ in terms of G!
(b) Compute: $\mathrm{Out}(\mathrm{L}(G)) = \mathrm{Aut}(\mathrm{L}(G))/\mathrm{Inn}(\mathrm{L}(G))$
$\mathcal{F}(\mathrm{L}(G)) = \{t \in \mathbb{R}_+ : \mathrm{L}(G)^t \cong \mathrm{L}(G)\}$

1. $\exists$ unique approx. finite dimensional II_1 factor $\mathcal{R} = \overline{\cup_n \mathbb{M}_{2^n}(\mathbb{C})}^{\mathrm{sot}}$.
2. For any locally finite icc group G (e.g. $\mathfrak{S}_\infty$) we have $\mathrm{L}(G) \cong \mathcal{R}$.

Theorem (Connes '76): $\forall G$ icc amenable we have $\mathrm{L}(G) \cong \mathcal{R}$.

$\rightsquigarrow$ G-amenable if $\exists\ (\xi_n)_n \subset (\ell^2 G)_1$ so that $\lim_n \|u_g(\xi_n) - \xi_n\| = 0,\ \forall g \in G$

Examples: abelian, solvable, loc. finite, closed under ext/subgr.

$\rightsquigarrow$ in this case all algebraic information on G (rank, torsion, gen/rel) is lost when passing to $\mathrm{L}(G)$.

$\rightsquigarrow$ Thus $\mathrm{Out}(\mathrm{L}(G))$ is "huge" and $\mathcal{F}(\mathrm{L}(G)) = \mathbb{R}_+^*$, $\quad \forall G$ icc amenable

Theorem (Connes '80): For every G icc property (T) group, $\mathrm{Out}(\mathrm{L}(G))$ and $\mathcal{F}(\mathrm{L}(G))$ are countable.

Def: (Kazhdan '67) G has prop. (T) if any unitary rep. of G that has almost invariant vectors must have a nonzero invariant vector.

Examples: (a) $SL_n(\mathbb{Z}), \quad PSL_n(\mathbb{Z}), \quad n \geq 3$

(b) unif. lat. $\Gamma < Sp(n,1) = \{A \in M_{n+1}(\mathbb{H}) : A^* J A = J\}$, $J = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}$

(c) prop. (T) passes to quotients

(d) (Shalom '00, Olivier-Wise '05, de Cornulier '05) Every prop. (T) group is a quotient of a torsion free, word hyperbolic prop. (T) group

Factors of property (T) groups

Connes Rigidity Conjecture '82: Whenever $G \ncong H$ are icc prop (T) groups we have $\mathrm{L}(G) \ncong \mathrm{L}(H)$.

$\rightsquigarrow$(Cowling-Haagerup '89) holds $\forall$ lat. $G < Sp(n,1), H < Sp(m,1)$ $n \neq m$.

$\rightsquigarrow$(Ozawa '02) $\exists$ uncountably many nonisom. prop. (T) group factors.

$\rightsquigarrow$(Popa '06) the map $G \to \mathrm{L}(G)$ is countable-to-one.

Problem (Connes '94): If G is icc prop (T) compute $\mathcal{F}(\mathrm{L}(G))$.

Outer Automorphisms Conjecture (VFR Jones '00, Popa '06): If G is icc prop (T) then $\mathrm{Out}(\mathrm{L}(G)) = \mathrm{Char}(\mathrm{G}) \rtimes \mathrm{Out}(\mathrm{G})$, (i.e. $u_g \to \rho(g) u_{\delta(g)}$).

Popa's strengthening of Connes Rigidity Conjecture '06:
Let G be any icc prop (T) group and let H be any group.
If $\Theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ is any $*$-isomorphism then $t = 1$ and there is a group isomorphism $\delta : G \to H$, a character $\rho \in \mathrm{Char}(G)$, and a unitary $w \in \mathrm{L}(H)$ so that $\Theta(u_g) = w(\rho(g) v_{\delta(g)}) w^*$, $\forall g \in G$.

Deformation/rigidity theory

Popa's deformation/rigidity theory ($\approx$ '01) led to huge progress towards the classification of group factors and computation of their invariants.

Def: $A \wr_I B = (\oplus_I A) \rtimes B$ generalized wreath product of A and $B \curvearrowright I$. When $I = B$ we get the wreath product $A \wr B$.

$\rightsquigarrow$ (Popa '03) $\forall B, D$ icc prop (T) st $\mathrm{L}(\mathbb{Z} \wr B) \cong \mathrm{L}(\mathbb{Z} \wr D)$ then $B \cong D$.

$\rightsquigarrow$ (Popa '01; Popa-Vaes '06) Examples of $G = \mathbb{Z}_2 \wr_I B$ with $\mathcal{F}(\mathrm{L}(G)) = 1$ and $\mathrm{Out}(\mathrm{L}(G)) = \mathrm{Char}(\mathrm{G}) \rtimes \mathrm{Out}(\mathrm{G})$; $\forall C$ fin. pres. $\mathrm{Out}(\mathrm{L}(G)) \cong C$.

$\rightsquigarrow$ (Ioana-Popa-Vaes '10) Certain $G = \mathbb{Z}_2 \wr_I B$ are W^*-superrigid.

None of these results apply to property (T) groups !!!

$\rightsquigarrow$ (C-Das-Houdayer-Khan '19-'20) $\mathcal{F}(\mathrm{L}(G)) = 1$, where G is prop (T) fibered version of the Rips construction (Belegradek-Osin '06).

Wreath-like product groups

Def: Let A, B be groups and let $B \curvearrowright I$ be an action. Then G is a generalized wreath-like product of A and $B \curvearrowright I$ ($G \in \mathscr{W}\mathscr{R}(A, B \curvearrowright I)$) if there is a s.e.s.

$$1 \to \oplus_{i \in I} A \hookrightarrow G \overset{\varepsilon}{\twoheadrightarrow} B \to 1$$

such that $gA_i g^{-1} = A_{\varepsilon(g) \cdot i}$, where A_i is the i-labeled copy of A in $\oplus_{i \in I} A$. When $I = B$, we denote by $G \in \mathscr{W}\mathscr{R}(A, B)$ - regular wreath-like products.

Obs: Let $G = A * B$. Then normal closure $\langle\langle A \rangle\rangle = *_{b \in B} A^b$ and $G = \langle\langle A \rangle\rangle B$. If $S = \langle [A^b, A^c] : b \neq c \in B \rangle$ then $S \triangleleft G$ and $G/S \cong A \wr B$.

$\rightsquigarrow$ $A < G$ is CL subgroup iff $\langle\langle A \rangle\rangle = *_{t \in T} A^t$ for T a transversal of $\langle\langle H \rangle\rangle \triangleleft G$.

$\rightsquigarrow$(Cohen-Lyndon '63) $\forall C < \mathbb{F}_k$ maximal cyclic is a CL subgroup.

Prop: Let $A < G$ be a CL subgroup. Then $S = \langle [A^b, A^c] : b \neq c \rangle < G$ is a normal subgroup of G and $G/S \in \mathscr{W}\mathscr{R}(A, G/\langle\langle A \rangle\rangle)$.

Proof: Follows because $\langle\langle A \rangle\rangle / S \cong \oplus_{t \in T} A^t \cong \oplus_{G/\langle\langle A \rangle\rangle} N$ and we have the short exact sequence $1 \to \langle\langle A \rangle\rangle / S \to G/S \to G/\langle\langle A \rangle\rangle \to 1$.

Theorem (Osin '06, Dahmani-Guirardel-Osin '11, Sun '19)

If $H < W$ with W hyper. rel. to H, $\forall A \triangleleft H$ "sufficiently deep" we have:

- $\langle\langle A \rangle\rangle = *_{t \in T} A^t$ where T is a left transversal for $H\langle\langle A \rangle\rangle < W$; and
- $W/\langle\langle A \rangle\rangle$ is hyperbolic relative to H/A.

$\rightsquigarrow$ In this case it follows that $G/S \in \mathscr{W}\mathscr{R}(A, W/\langle\langle A \rangle\rangle \curvearrowright W/H\langle\langle A \rangle\rangle)$.

$\rightsquigarrow$ $\forall$ W icc hyperbolic and $\forall$ $n \in \mathbb{N}$, there is $\mathbb{F}_n \cong H < W$ such that W is hyperbolic relative to H. Using this in combination with the prior result we get $\exists\, \mathbb{F}_n \cong H < W$ CL-subgroup and the prior quotienting technique yields:

Theorem (C-Ioana-Osin-Sun '21)

Let W be an icc, hyperbolic group. For every finitely generated A there is G a quotient of W so that $G \in \mathscr{W}\mathscr{R}(A, B)$ where B is icc hyperbolic. In particular, if W has property (T) then so does $G \in \mathscr{W}\mathscr{R}(A, B)$. p

$\rightsquigarrow$ As a consequence , if $A = \mathbb{Z}$ then $\mathrm{L}(G) \cong \mathrm{L}^\infty(\mathbb{T}^B) \rtimes_{\sigma, c} B$ where $B \curvearrowright \mathbb{T}^B$ is Bernoulli action and $c \in \mathrm{Z}^2(\sigma, \mathbb{T})$. Hence $\mathrm{H}^2(\sigma, \mathbb{T}) \neq \mathrm{H}^2(B, \mathbb{T})$ answering a question of Popa and recovering (Jiang '15).

W*-superrigidity results

Theorem (C-Ioana-Osin-Sun '21)

Let A be abelian and let B be an icc subgroup of a hyperbolic group. Then any property (T) group $G \in \mathscr{W}\mathscr{R}(A, B)$ is W^*-superrigid.

Theorem (C-Ioana-Osin-Sun '21)

Let G be an icc hyperbolic property (T) group and let $g \in G$ be an element of infinite order. Then there is $d \in \mathbb{N}$ such that for all $k \in \mathbb{N}$ the quotient $G/[\langle\langle g^{dk} \rangle\rangle, \langle\langle g^{dk} \rangle\rangle]$ is a property (T) W^*-superrigid group.

$\rightsquigarrow$ First examples of prop (T) groups satisfying Connes Rigidity Conjecture; in fact one can construct $2^{\aleph_0}$ many such groups.

$\rightsquigarrow$ Our approach combines von Neumann alg. methods with techniques on equivalence relations. Recently, we found another method that yields many W*-superrigid prop (T) groups $G \in \mathscr{W}\mathscr{R}(A, B)$ where A is non-amenable and B is a special type of relative hyperbolic group.

Computations of invariants of prop (T) factors

Theorem (CIOS '21-'22)

Let A, C be abelian or icc. Let B, D be non-parabolic icc subgroups of groups which are hyperbolic relative to a finite family of finitely generated, residually finite groups.

Let $G \in \mathcal{WR}(A, B \curvearrowright I)$, $H \in \mathcal{WR}(C, D \curvearrowright J)$ be any prop (T) groups where $B \curvearrowright I$ and $D \curvearrowright J$ are faithful actions with infinite orbits.

Let $\Theta : \mathrm{L}(G)^t \to \mathrm{L}(H)$ for $t > 0$ be any $*$-isomorphism. Then $t = 1$ and one can find a group isomorphism $\delta : G \to H$, a character $\rho : G \to \mathbb{T}$ and a unitary $w \in \mathrm{L}(H)$ such that for all $g \in G$ we have

$$\Theta(u_g) = w\left(\rho(g) v_{\delta(g)}\right) w^*.$$

p

⇝ Yields $\mathcal{F}(\mathrm{L}(G)) = 1$, providing additional examples to the prior work (C-Das-Houdayer-Khan '20) confirming Popa's conjecture.

⇝ Implies $\mathrm{Out}(\mathrm{L}(G)) = \mathrm{Char}(G) \rtimes \mathrm{Out}(G)$, giving the first examples of prop (T) groups satisfying VFR Jones's conjecture.

⇝ Outer automorphisms of prop (T) groups in general can be very wild (Ollivier-Wise '04, Belegradek-Osin '06). Developing a new approach based on prior work of (Wise '04, Haglund-Wise '08, Agol '13), Rips constructions (Belegradek-Osin '06), and quotienting techniques involving Cohen-Lyndon subgroups we showed the following:

Theorem (CIOS '22)

$\forall$ countable group C, $\exists$ group $G \in \mathscr{W}\mathscr{R}(A, B \curvearrowright I)$ so that:

(a) B is a non-parabolic icc subgroup of a relatively hyperbolic group with finitely generated, ressidually finite peripheral subgroups.

(b) A is abelian and $B \curvearrowright I$ is a faithful action with infinite orbits.

(c) G has prop (T), $[G, G] = G$, and $\mathrm{Out}(G) \cong C$. p

⇝ In fact there is a continuum of G's satisfying the statement.

Corollary - **a converse to Connes' result**

$\forall$ countable group C, $\exists$ prop (T) group G so that $\mathrm{Out}(\mathrm{L}(G)) \cong C$.

⇝ Similar results hold for reduced group C*-algebras. p

Other invariants: $\mathcal{F}_s(\mathcal{M}) = \{t \in \mathbb{R}_+ \mid \exists \Theta : \mathcal{M} \to \mathcal{M}^t$ $*$-homomorphism$\}$
$\mathcal{I}_{\mathcal{M}} = \{r \in [1,\infty] \mid \exists \mathcal{N} \subseteq \mathcal{M}$ subfactor so that $[\mathcal{M} : \mathcal{N}] = r\}$

Theorem (CIOS 21)

Let G be a prop (T) group, $A \triangleleft G$ abelian, $|g^A| = \infty$, $\forall g \in G \smallsetminus A$.
Let $H \in \mathscr{W}\mathscr{R}(C, D \curvearrowright I)$, C-abelian, D-icc subgroup of a hyperbolic group, $C_D(g)$ is virtually cyclic $\forall 1 \neq g \in D$, $D \curvearrowright I$ amenable stabilizers.

Let $t > 0$ and let $\Theta : \mathrm{L}(G) \to \mathrm{L}(H)^t$ be any $*$-homomorphism.

Then $t_1 + \cdots + t_q = t \in \mathbb{N}$ with $t_i \in \mathbb{N}$ and $\exists$ finite index subgroup $K \leqslant G$, a monomorphism $\delta_i : K \to H$, a unitary rep $\rho_i : K \to \mathscr{U}_{t_i}(\mathbb{C})$, $1 \leq i \leq q$ after conjugating by a unitary $w \in \mathrm{L}(H)^t = \mathrm{L}(H) \otimes \mathbb{M}_t(\mathbb{C})$ we have that

$$\Theta(u_g) = \mathrm{diag}(v_{\delta_1(g)} \otimes \rho_1(g), \ldots, v_{\delta_q(g)} \otimes \rho_q(g)), \quad \forall g \in K.$$

If $t = 1$ we can take $K = G$ and hence $\exists\ \delta : G \to H$ monomorphism, $\rho \in \mathrm{Char}(G)$, $w \in \mathscr{U}(\mathrm{L}(H))$ so that $\Theta(u_g) = w(\rho(g) v_{\delta(g)})w^*$, $\forall g \in G$.

$\rightsquigarrow$ we constructed prop (T) wreath-like products G with $\mathrm{End}(G) = \mathrm{Inn}(G)$

Cor: $\mathrm{End}(\mathrm{L}(G)) = \mathrm{Inn}(\mathrm{L}(G))$; $\quad \mathcal{F}_s(\mathrm{L}(G)) = \mathbb{N}$; $\quad \mathcal{I}_{\mathrm{L}(G)} \subset \mathbb{N} \cup \{\infty\}$.

Embedding universality for prop (T) factors

Theorem (C-Drimbe-Ioana '22)

Let $\mathcal{M}$ be a separable II_1 factor. Then the following hold:

1. For any hyperbolic group H there is a representation $\pi : H \to \mathscr{U}(\mathcal{Q})$ with $\pi(H)'' = \mathcal{Q}$ and $\mathcal{M} \subset \mathcal{Q}$.
2. There is a prop (T) II_1 factor $\mathcal{P}$ with $\mathrm{Out}(\mathcal{P}) = \{1\}$ and $\mathscr{F}(\mathcal{P}) = \{1\}$ such that $\mathcal{M} \subset \mathcal{P}$. p

$\rightsquigarrow$ II_1 factor analogue to SQ-universality of hyperbolic groups (Delzant,'96; Ol'shanskii,'95).

$\rightsquigarrow$ Cocompact lattices $H < Sp(n, 1)$, $n \geq 2$ are prop (T) groups whose representations are embbedding universal.

$\rightsquigarrow$ Contrasts (Bekka '06; Peterson '14; Boutonnet-Houdayer '19): If G is an icc lattice in a higher rank simple Lie group (eg $SL_n(\mathbb{R})$, $n \geq 3$), then $\mathrm{L}(G)$ is the only II_1 factor generated by a rep. of G.

$\rightsquigarrow$ Combining this with (Ji-Natarajan-Vidick-Wright-Yuen '20) we obtain prop (T) factors that are not $\mathcal{R}^\omega$-embeddable.

Popa's Factorial Relative Commutant Problem:
Let $\mathcal{M}$ be a separable $\mathcal{R}^\omega$-embeddable II_1 factor. Is there an embedding $\pi : \mathcal{M} \hookrightarrow \mathcal{R}^\omega$ such that $\pi(\mathcal{M})' \cap \mathcal{R}^\omega$ is a factor? **eg:** $SL_3(\mathbb{Z})$ (Popa '13)

Theorem (Farah-Goldbring-Hart-Sherman, '16)

$\exists$ a class $\mathcal{G}$ of separable II_1 factors (infinitely generic) which is model complete, i.e. maximal class satisfying

a) $\mathcal{G}$ is embedding universal;
b) If $\mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{G}$, any embedding $\pi : \mathcal{Q}_1 \hookrightarrow \mathcal{Q}_2$ extends to an isomorphism $\mathcal{Q}_1^\omega \cong \mathcal{Q}_2^\omega$.

$\rightsquigarrow$ (Goldbring '20) showed that any property (T) factor $\mathcal{M}$ admits an embedding into $\mathcal{Q}^\omega$, for any infinitely generic factor $\mathcal{Q}$.

Theorem (C-Drimbe-Ioana, '22)

Let $\mathcal{Q}$ be any infinitely generic II_1 factor. Then any full II_1 factor $\mathcal{M}$ admits an embedding in $\mathcal{Q}^\omega$ with factorial relative commutant.

Def: A II_1 factor $\mathcal{M}$ is called super-McDuff iff $\mathcal{M}' \cap \mathcal{M}^\omega$ is a II_1 factor.

Examples: $\rightsquigarrow \mathcal{R}$; $\mathcal{N}\overline{\otimes}\mathcal{R}$, where $\mathcal{N}$ is a full II_1 factor

$\rightsquigarrow$ (Popa '17) $\overline{\otimes}_{n\in\mathbb{N}}\mathcal{N}_n$, where $\mathcal{N}_n$ are full II_1 factors

Open problems (Atkinson-Goldbring-Kunnawalkam-Elayavalli, '20)

$\exists$ e.c. factors that are super-McDuff? Are all e.c. factors super-McDuff?

$\rightsquigarrow$ (C-Drimbe-Ioana '22) Every infinitely generic factor is super-McDuff.

$\rightsquigarrow$(Goldbring-Jekel-Kunnwalkam-Elayavalli-Pi '23) these are uniformly super-McDuff; thus any factor in their e.e. class is super-McDuff

Conjecture: Any e.c. factor $\mathcal{M}$ satisfies $\mathcal{M} \ncong \mathcal{P}\overline{\otimes}\mathcal{Q}$, $\forall$ $\mathcal{Q}$ a full factor.

$\rightsquigarrow$ (C-Drimbe-Ioana '22) confirmed this for an embedding universal class of e.c. factors, which are inductive limits $\mathcal{Q} = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{N}_n}^{\mathrm{sot}}$ where

- $(\mathcal{N}_n)_{n\in\mathbb{N}}$ is an increasing sequence of prop (T) s-prime II_1 factors
- $[\mathcal{N}_m : \mathcal{N}_n] = \infty, \quad \forall m > n.$

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THANK YOU !!!

⇝ M is a compact unoriented 3-manifold with toric ∂M. Topologically distinct way to attach a solid torus to ∂M are parametrized by slopes of ∂M. For a slope σ, the Dehn filling $M(\sigma)$ of M is obtained by attaching a solid torus $\mathbb{D}^2 \times \mathbb{S}^1$ so its meridian $\partial\mathbb{D}^2$ goes to a curve of slope σ.

⇝ (Thurston '82) showed that if $M \setminus \partial M$ has complete finite volume then $M(\sigma)$ has a hyperbolic structure for all but finitely many slopes.

⇝ If $\pi_1(\partial M) \leqslant \pi_1(M)$ is rel. hyper. then $\exists F \subset \pi_1(M)$ finite so that $\pi_1(M(\sigma)) = \pi_1(M)/\langle\langle x \rangle\rangle$ is hyperbolic $\forall x \in H \setminus F$.

Osin '06, Dahmani-Guirardel-Osin '11, Sun '19

Let $H < G$ with G hyperbolic relative to H. $\exists F \subset H \setminus \{1\}$ finite such that $\forall N \triangleleft H$ with $N \cap F = \varnothing$ we have:

- $\langle\langle N \rangle\rangle = *_{t \in T} N^t$ where T is a left transversal for $H\langle\langle N \rangle\rangle < G$; and
- $G/\langle\langle N \rangle\rangle$ is hyperbolic relative to H/N.

back

Theorem (C-Ioana-Osin-Sun '21)

Let W- icc, hyperbolic group. For every finitely generated A there is G a quotient of W so that $G \in \mathscr{W}\mathscr{R}(A,B)$ where B is icc hyperbolic. In particular, if W has property (T) then so is G.

$\rightsquigarrow$ Using [DGO11] $\exists\ \mathbb{F}_{7n} = K < W$ with W hyp. rel. to K. Using Dehn filling $\exists\ \mathcal{F} \Subset W \setminus \{1\}$ so that $\forall N \lhd K$ with $N \cap \mathcal{F} = \varnothing$ then (W, K, N) is CL-triple and $W/\langle\langle N \rangle\rangle$ is hyp. rel. to K/N.

$\rightsquigarrow$ Using a high-power elements, $\mathbb{F}_n * L = K$ with $\langle\langle \mathbb{F}_n \rangle\rangle^K \cap \mathcal{F} = \varnothing$. Hence $(W, K, \langle\langle \mathbb{F}_n \rangle\rangle^K)$ is CL-triple and $W/\langle\langle \mathbb{F}_n \rangle\rangle^W$ is hyp. rel. to $K/\langle\langle \mathbb{F}_n \rangle\rangle^K = L$.

$\rightsquigarrow$ As $(W, K, \langle\langle \mathbb{F}_n \rangle\rangle^K)$, $(K, \mathbb{F}_n, \mathbb{F}_n)$ are CL-triple then $(W, \mathbb{F}_n, \mathbb{F}_n)$ is CL-triple. Since L is free then $W/\langle\langle \mathbb{F}_n \rangle\rangle = B$ is hyperbolic. Thus,

$$G_0 = W/\langle S \rangle \in \mathscr{W}\mathscr{R}(\mathbb{F}_n, B).$$

$\rightsquigarrow$ If $G_0 \in \mathscr{W}\mathscr{R}(\mathbb{F}_n, B)$ then $\forall\ H \lhd \mathbb{F}_n \Rightarrow G = G_0/\langle\langle H \rangle\rangle \in \mathscr{W}\mathscr{R}(\mathbb{F}_n/H, B)$.

back

If $G \in \mathscr{W}\mathscr{R}(A, B)$ with A abelian then:

a) action $G \curvearrowright^{\sigma} \mathrm{L}(A^{(B)})$ by conjugation $\sigma_g = \mathrm{ad}(u_g)$ is a gen. Bernoulli;
b) Eq. rel. $\mathscr{R}(\mathrm{L}(A^{(B)}) \subset \mathcal{M})$ is the OE rel. of Bernoulli action $B \curvearrowright \hat{A}^B$.

Fix $G \in \mathscr{W}\mathscr{R}(A, B)$, $H \in \mathscr{W}\mathscr{R}(C, D)$ with $\mathrm{L}(G) = \mathrm{L}(H) =: \mathcal{M}$.

I: If $\mathcal{P} = \mathrm{L}(A^{(B)})$, $\mathcal{Q} = \mathrm{L}(C^{(D)}) \Rightarrow \exists u \in \mathscr{U}(\mathcal{M})$ so that $u\mathcal{P}u^* = \mathcal{Q}$.

As $\mathcal{P}, \mathcal{Q} \subset \mathcal{M}$ regular, B, D-rel hyp follows from (Popa-Vaes '12, Ioana '12, C-Ioana-Kida '13).

II: $\exists$ maps $\zeta : G \to \mathscr{U}(\mathcal{P})$ and $\delta : G \to H$ with $\zeta_g u_g = v_{\delta(g)}$, $\forall g \in G$.

Using b) to identify the eq. rel. of $\mathcal{P} \subset \mathcal{M}$ in two ways $\Rightarrow$ an OE between $B \curvearrowright \hat{A}^B$ and $D \curvearrowright \hat{C}^D$. Using Popa's CSR Thm $\Rightarrow \mathscr{U}(\mathcal{P})G = \mathscr{U}(\mathcal{P})H$.

III: Let $\Delta : \mathcal{M} \to \mathcal{M}\overline{\otimes}\mathcal{M}$ defined $\Delta(v_h) = v_h \otimes v_h$, $\forall h \in H$. Then $\exists$ $w \in \mathscr{U}(\mathcal{M}\overline{\otimes}\mathcal{M})$, $\eta \in \mathrm{Char}(G)$ with $w\Delta(u_g)w^* = \eta(g)u_g \otimes u_g$, $\forall g \in G$.

$\Delta(u_g) = \Delta(\zeta_g^* v_{\delta(g)}) = \Delta(\zeta_g^*)v_{\delta(g)} \otimes v_{\delta(g)} = (\Delta(\zeta_g^*)\zeta_g \otimes \zeta_g)(u_g \otimes u_g)$
$\Rightarrow g \to \omega_g \in \mathscr{U}(\mathcal{P}\overline{\otimes}\mathcal{P})$ is a 1-cocycle for $\sigma \otimes \sigma$; Using CSR results $\Rightarrow$ **III**.
Using a height technique of (Ioana-Popa-Vaes '10) we get conclusion. back

$\rightsquigarrow$ Using the work of ;(Wise '04, Haglund-Wise '08, Agol '13)
$\forall$ countable group $C < S/M$, where S fin. gen., torsion free, res. finite.
$\rightsquigarrow$ Rips construction (Belegradek-Osin '06) one can find

$$\begin{array}{ccccl} S & < & G & \leftarrow & \text{torsion free, hyp rel to } S \\ & & \triangledown & & \\ & & N & \leftarrow & \text{prop (T), trivial abelianization} \end{array}$$

with $C < S/M \cong G/N$
$\rightsquigarrow$ CL-subgroup $\to \langle x \rangle < N \triangleleft C_0 = \pi^{-1}(C) < G$. If $H = \langle\langle x \rangle\rangle^G = *_b b\langle x \rangle b^{-1}$

$$\begin{array}{ccccccccc} 1 \to & \oplus_{G/H}\mathbb{Z} & \hookrightarrow & G/[H,H] & \twoheadrightarrow & G/H & \to 1 \\ & & & \vee & & \vee & \\ 1 \to & \oplus_{C_0/H} A & \hookrightarrow & C_0/[H,H] & \twoheadrightarrow & C_0/H & \to 1 \end{array}$$

$\rightsquigarrow$ The group yielding the conclusion is the image $N/[H,H]$ in a suitable quotient of $C_0/[H,H]$ ($\Leftarrow$ a generalized wreath-like product with infinite, untwisted stabilizers). back

Theorem (CIOS '21-'22)

Let A-icc, Haagerup group with trivial amenable radical and B-icc subgroup of a hyperbolic group. Let $G \in \mathscr{W}\mathscr{R}(A,B)$ be a torsion free prop (T) group. Then for any H and any $*$-isomorphism $\Theta : \mathrm{C}^*_r(G) \to \mathrm{C}^*_r(H)$ there is a group isomorphism $\delta : G \to H$, a character $\rho \in \mathrm{Char}(G)$, and a unitary $w \in \mathrm{L}(H)$ such that $\Theta(u_g) = w(\rho(g) v_{\delta(g)}) w^*$, $\forall g \in G$.

⇝ The proof uses von Neumann algebra techniques and $\mathrm{C}^*_r(G)$ having unique trace and being projectionless (G satisfies Baum-Connes conjecture (Higson-Kasparov '97, Mineyev-Yu '01, Oyono-Oyono '01))

⇝ If G is non-inner amenable, with trivial amenable radical then

$$1 \to \mathrm{swInn}(\mathrm{C}^*_r(G)) \to \mathrm{Out}(\mathrm{C}^*_r(G)) \to \mathrm{sOut}(\mathrm{C}^*_r(G)) \to 1$$

Corollary: Let A- icc group with trivial amenable radical, B- icc subgroup of a hyperbolic group. $\forall$ prop (T) group $G \in \mathscr{W}\mathscr{R}(A,B)$ we have

$$\mathrm{sOut}(\mathrm{C}^*_r(G)) = \mathrm{Char}(G) \rtimes \mathrm{Out}(G).$$

Thus, $\forall$ finitely presented C there is such G with $\mathrm{sOut}(\mathrm{C}^*_r(G)) \cong C$.

back

Ideas behind the proof of part 2):

- Can assume that $\mathcal{M}$ is generated by 3 unitaries by considering $\mathcal{M} \subset \mathcal{M}\overline{\otimes}\mathcal{R}$, where R is the hyperfinite II_1 factor (Ge-Popa '98).
- Let $\pi : \mathbb{F}_3 \to \mathscr{U}(\mathcal{M})$ be a homomorphism with $\pi(\mathbb{F}_3)'' = \mathcal{M}$.
- $\exists$ a property (T) group $G \in \mathscr{W}\mathscr{R}(\mathbb{F}_3, B)$ with no non-trivial characters for some icc hyperbolic group B with $\mathrm{Out}(B) = 1$, (CIOS'21).

$G \in \mathscr{W}\mathscr{R}(A, B) \Leftrightarrow \exists\, \rho : B \to A^B$ with $v_{b,c} = \rho_b \sigma_b(\rho_c)\rho_{bc}^{-1} \in A^{(B)}, \forall\, b, c \in B$, and letting $\alpha_b := \mathrm{Ad}(\rho_b)\sigma_b \in \mathrm{Aut}(A^{(B)})$ we have $G \cong A^{(B)} \rtimes_{\alpha,v} B$. def

- π extends to a homomorphism $\tilde{\pi} : G \to \mathscr{U}(\mathcal{P})$ with $\tilde{\pi}(G)'' = \mathcal{P}$, $\mathcal{P} \supset \mathcal{M}$ (where $G = \mathbb{F}_3^{(B)} \rtimes_{\alpha,v} B$ and $\mathcal{P} = \mathcal{M}^B \rtimes_{\beta,w} B$).
- $\mathcal{P}$ has prop (T) since G has prop (T).

back

Definition

A cocycle action $B \curvearrowright^{\alpha,v} A$ is a pair $\alpha : B \to \mathrm{Aut}(A)$, $v : B \times B \to A$ with

1. $\alpha_b \alpha_c = \mathrm{Ad}(v_{b,c})\alpha_{bc}$, for every $b, c \in B$,
2. $v_{b,c} v_{bc,d} = \alpha_b(v_{c,d}) v_{b,cd}$, for every $b, c, d \in B$, and
3. $v_{b,1} = v_{1,b} = 1$, for every $b \in B$.

The cocycle semidirect product $A \rtimes_{\alpha,v} B$ is the group $A \times B$ endowed with the unit $(1,1)$ and the multiplication $(x, b) \cdot (y, c) = (x\alpha_b(y) v_{b,c}, bc)$.

Definition

A cocycle action $B \curvearrowright^{\beta,w} (M, \tau)$ is a pair $\beta : B \to \mathrm{Aut}(M)$, $w : B \times B \to \mathscr{U}(M)$ with

1. $\beta_b \beta_c = \mathrm{Ad}(w_{b,c})\beta_{bc}$, for every $b, c \in B$,
2. $w_{b,c} w_{bc,d} = \beta_b(w_{c,d}) w_{b,cd}$, for every $b, c, d \in B$, and
3. $w_{b,1} = w_{1,b} = 1$, for every $b \in B$. back

The cocycle crossed product $M \rtimes_{\beta,w} B$ is a tracial vN algebra generated by a copy of M and unitaries $\{u_b\}_{b \in B}$ such that $u_b x u_b^* = \beta_b(x)$, $u_b u_c = w_{b,c} u_{bc}$ and $\tau(x u_b) = \tau(x)\delta_{b,e}$, for every $b, c \in B$ and $x \in M$.