Equivalence relation of group actions and Kesten's criteria for topological groups

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Borel equivalence relations

Definition

Let (X, μ) be a standard probability space. We say that an equivalence relation R on X is a countable Borel equivalence relation on X if R is a Borel subset of $X \times X$ and equivalence classes of R are countable.

Definition

Let R be a countable Borel equivalence relation on a standard probability space (X, μ). The full group $[R]$ is defined as the group of all Borel automorphisms $\phi \in Aut(X, \mu)$ such that $graph(\phi) \subseteq R$ on a subset of full measure.

μ is **invariant (respectively, quasi-invariant)** if μ (respectively, equivalence class of μ) is preserved under the action of $[R]$ on (X, μ). A quasi-invariant probability measure μ is called R**-ergodic** if every R-invariant Borel set is either null or co-null.

Liouville measures

Let a countable discrete group G act on a set X. A probability measure on G is called *non-degenerate* if its support generates G as a semigroup. Let μ be a symmetric non-degenerate probability measure on G. A function $f : X \to R$ is called

μ-**harmonic**, if the equality

$$f(x) = \sum_{g \in G} f(gx)\mu(g)$$

holds for every $x \in X$. An action is called μ**-Liouville** if every bounded μ-harmonic function is constant. We will say the action $G \curvearrowright X$ is **Liouville**, if it is μ-Liouville for some symmetric non-degenerate probability measure μ on G.

Theorem (Kaimanovich-Vershik)

A discrete group G is amenable if and only if the left multiplication action of G on itself is Liouville.

Generalization to locally compact second countable groups was obtained by Rosenblatt, and second-countable topological groups by Schneider and Thom.

If the left multiplication action of G on itself is μ-Liouville, then any transitive action of G is also μ-Liouville.

Theorem (Chaudkhari, J, Schneider, ’22)

Assume that R is a countable Borel equivalence relation on (X, μ) *such that* μ *is R-quasi-invariant and non-atomic. If G is a countable dense subgroup of* $[R]$*, then the following statements are equivalent:*

1. *R is* μ*-amenable.*
2. *There exists a symmetric non-degenerate measure* ν *on G, such that the action of G on almost every orbit in X is* ν*-Liouville.*

Theorem (Kesten, ’59)

Let Γ *be a finitely generated discrete group and let* μ *be a finitely supported symmetric generating measure on* Γ*. Let* ρ *be the spectral radius of the* μ*-random walk on* Γ*. Then* Γ *is amenable if and only if* $\rho = 1$*.*

Schneider and Thom: Følner criterion and Kaimanovich-Vershik theorem for topological groups.

What assumptions on topological group would guarantee that $\limsup_{n\to\infty} \mathbb{P}(X_n \in U)^{1/n} = 1$?

Theorem

For a topological group G the following properties are equivalent.

1. *G is amenable.*
2. *Every continuous affine action of G on a non-empty compact subset of a locally convex topological vector space has a fixed point.*
3. *For every non-empty compact space X and any action of G on X there exists a G-invariant Borel probability measure on X.*

A topological group G has **small invariant neighborhoods** (or G is a SIN group) if for every neighborhood $U \in \mathcal{U}(G)$ we have

$$\cap_{g \in G} gUg^{-1} \in \mathcal{U}(G)$$

Theorem (Chaudkhari, J, Schneider, '22)

Assume that G is a Hausdorff, amenable and SIN group, and ν is a symmetric probability measure with at most countable support on G. Then, for any neighborhood U of the identity, a lazy ν-random walk X_n started at the identity satisfies

$$\lim_{n\to\infty} \mathbb{P}(X_n \in U^n)^{1/n} = 1$$

G is SIN implies U is invariant under conjugation and $U = U^{-1}$.

Schenider, Thom '17: there exists $\alpha : G \to Sym(G)$ such that the action of the group generated by $\alpha(G)$ is amenable, and for any $g, h \in G$ there exists $u(g, h) \in U$ such that $\alpha(g)(h) = u(g, h)gh$.

Notice that invariance of U under taking inverses and conjugation implies that in this case for any $g, h \in G$, there exists $u'(g, h) \in U$ such that $\alpha(g)^{-1}(h) = u'(g, h)g^{-1}h$.

Let $S = supp(\nu)$. Let Γ be the subgroup of $Sym(G)$ generated by $\alpha(S)$. Consider a symmetric random walk on G induced by the random walk on Γ defined by the probability measure supported on $\alpha(S)$ (treated as a multiset) which assigns to the elements of this multiset the weights equal to the ν-weights of corresponding elements of S. Denote the resulting probability measure on the multiset $\alpha(S)$ by ν', and for $s \in S \cup S^{-1}$ we denote by α_s the element $\alpha(s)$ if $s \in S$ or the element $\alpha(s^{-1})^{-1}$ if $s \in S^{-1}$.

If for some tuple $(s_n, \dots, s_1) \in (S \cup S^{-1})^n$ and some $x \in G$ one has

$$\alpha_{s_n} \circ \alpha_{s_{n-1}} \circ \dots \circ \alpha_{s_1}(x) = x,$$

we can conclude that there are $u_1, ..., u_n \in U$ such that

$$u_n s_n u_{n-1} s_{n-1} \dots u_1 s_1 = id_G,$$

which implies that

$$X_n = s_n s_{n-1} \dots s_1 = \prod_{i=1}^{n} (u_i^{-1})^{s_{i+1}..s_n} \in U^n.$$

Thus, the invariance of U under taking inverses and conjugation implies that

$$\sup_{x \in G} \mathbb{P}_{(\nu')^n}(x, x) \leq \mathbb{P}_{\nu^n}(X_n \in U^n).$$

Since the action of Γ on G admits an invariant mean, for any $\epsilon > 0$ and any finite subset E of Γ, G admits an (E, ϵ)-Følner set. Such a set can always be selected from the same orbit of the action of Γ on G. Therefore, the infimum of isoperimetric constants of ν'-random walks on Γ-orbits on G is equal to 0.

Mohar's isoperimetric inequality implies that the supremum of the spectral radii of ν'- random walks on Γ-orbits on G is equal to 1, hence $\sup_{x \in G} \mathbb{P}_{(\nu')^n}(x, x)$ decays subexponentially, and

$$\lim_{n \to \infty} \mathbb{P}(X_n \in U^n)^{1/n} = 1.$$

Bad news: the reverse does not hold.

A topological group G is called **bounded** if, for every neighborhood U of $e \in G$, there exist a finite subset $F \subseteq G$ and a natural number n such that $G = FU^n$.

It is well known, that G is bounded if and only if every right-uniformly continuous real-valued function on G is bounded.

We will say that G is **power-bounded** if, for every neighborhood U of $e \in G$, there exists a natural number n such that $G = U^n$.

Let X be a compact Hausdorff space with a regular Borel probability measure μ. A map $f\colon X \to Y$ into a topological space Y is called *μ-almost continuous* if, for every $\epsilon > 0$, there exists a closed subset $A \subseteq X$ with $\mu(X \setminus A) \leq \epsilon$ such that $f|_A\colon A \to Y$ is continuous. If the target space is metrizable, then μ-almost continuity is equivalent to μ-measurability.

Consider the Lebesgue probability measure λ on the closed real interval $[0, 1]$. Given a topological group G, we define $L^0(G)$ to be the set of all (λ-equivalence classes of) λ-almost continuous maps from $[0, 1]$ to G. Equipped with the group structure inherited from G and the topology of convergence in measure, $L^0(G)$ is a topological group. The sets of the form

$$N(U, \epsilon) := \{f \in L^0(G) \mid \lambda(\{x \in [0, 1] \mid f(x) \notin U\}) < \epsilon\}$$
$$(\epsilon > 0,\ U \subseteq G \text{ open with } e \in U)$$

constitute a neighborhood basis at the neutral element of $L^0(G)$.

Let G be a topological group.

(1) The topological group $L^0(G)$ is power-bounded. Let U be any identity neighborhood in $L^0(G)$. Then we find some $n \in \mathbb{N} \setminus \{0\}$ as well as an open identity neighborhood V in G such that $N\left(V, \frac{1}{n}\right) \subseteq U$. We claim that $L^0(G) = U^n$. To see this, let $f \in L^0(G)$. For each $i \in \{0, \ldots, n-1\}$, consider the element $f_i \in L^0(G)$ defined by

$$f_i|_{[i/n,(i+1)/n)} = f|_{[i/n,(i+1)/n)}, \qquad f_i|_{[0,1]\setminus[i/n,(i+1)/n)} \equiv e,$$

and note that $f_i \in N\left(V, \frac{1}{n}\right)$. Hence, as desired,

$$f = f_1 \cdot \ldots \cdot f_n \in \left(N\left(V, \tfrac{1}{n}\right)\right)^n \subseteq U^n.$$

(2) The topological group $L^0(G)$ is (extremely) amenable if and only if G is amenable (Pestov, Schneider, 2017).

(3) If G is Polish, then so is $L^0(G)$ due to (Moore, '76). Since G is topologically isomorphic to a closed subgroup of $L^0(G)$, the converse holds as well.

(4) It is straightforward to verify that $L^0(G)$ is SIN if and only if G is SIN.

Conclusion: $L^0(F_2)$ is a power-bounded, non-amenable, SIN, Polish group. Hence, the condition

$$\lim_{n\to\infty} \mathbb{P}(X_n \in U^n)^{1/n} = 1$$

does not imply amenability of a topological group.

Good news: characterizes amenability for locally compact groups

If G is an amenable locally compact group and λ is its left Haar measure, and ν is a symmetric measure with countable support with $\nu(id) \geq 1/2$, then the norm of the Markov operator M_ν on $L^2(G, \lambda)$ is equal to 1, and it is equal to the

$$\limsup_{n\to\infty} (\nu^n(V))^{1/n}$$

for any compact neighborhood V of the identity. On the other hand, since non-amenability of a locally compact group is witnessed by its compactly generated subgroups, we have that a non-amenable locally compact group fails the condition (Quint's Lecture notes).

Lamplighter action in the measurable settings.
Define measure M_l on the subsets of R as follows. For any Borel $A \subset R$

$$M_l(A) = \int_X |A_x| d\mu(x), \text{where } A_x = \{y \in X : (x, y) \in A\}.$$

The sets A with $M_l(A) < \infty$ form a group C with respect to the symmetric difference operation. C is equipped with a distance derived from the distance between indicator functions in $L^1(R, M_l)$. Now take $[R]$ (or any G which is dense in $[R]$) and consider its action on $X \times X$ defined by

$$g(x, y) = (x, gy).$$

This construction induces an action of $[R]$ on C by isometries. This action defines $C \rtimes [R]$ as a topological group.

Proposition

Assume that R is an ergodic amenable countable Borel equivalence relation on a non-atomic standard probability space (X, μ). Let $[R]$ be endowed with the uniform topology, and C with the topology induced by the distance in $L^1(R, M_I)$. Then the following statements are true.

1. *$C \rtimes [R]$ with the product topology is a topological group.*
2. *$C \rtimes [R]$ with the product topology is amenable.*
3. *$C \rtimes [R]$ does not have SIN property*

Applications to inverted orbits.

Definition

Let G be a discrete group acting on a set X. For a sequence $h = \{h_1, h_2, ..., h_n\}$ of elements of G (one may think of its elements as of the increments defining the trajectory of a random walk) and a point $x \in X$ an inverted orbit of x under h is the set $\{x, h_n x, h_n h_{n-1} x, \dots, h_n h_{n-1} \dots h_1 x\}$. We will sometimes use the notation $O_h(x)$ for the inverted orbit of a point x under the action of h.

Theorem

If R is amenable, then an affirmative answer to Topological Kesten for $C(G) \rtimes G$ or $C \rtimes [R]$ implies that

$$\int_X \mathbb{P}(|O_n(x)| \leq \epsilon n) d\mu(x)$$

decays subexponentially for each $\epsilon > 0$.

Thank you!