Discrete inclusions of C*-algebras

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joint with Brent Nelson

arXiv: 2305.05072

UNIVERSITY OF WATERLOO

10 May 2023,
NCGOA. Vanderbilt University, Nashville, TN
slides: https://www.math.uwaterloo.ca/~r5hernan/

Overview

Main theme:

♦ Transfer subfactor techniques to C*-algebras.

Goals:

- ♠ Obtain class of C*-algebra inclusions admitting standard invariant,
- ♡ Characterize C*-discreteness explicitly,
- ♣ Galois Correspondence for C*-discrete extensions,
- ♢ Applications and examples.

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▷ [Ocn88, Pop95a, Jon99]

$$N \subset M \rightsquigarrow \underbrace{\text{Standard invariant}}_{\text{higher relative commutants}}$$

[Pop95b] Reconstruct amenable $\mathcal{R} \subset \mathcal{R}$ from standard invariant.

vN Algs & subfactors chronology

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▶ [Müg03] $\left({}_N L^2 M_N =: Q \in \mathcal{C} = \langle {}_N L^2 M_N,\ \oplus,\ \overline{\cdot},\ \boxtimes_N \rangle\right)$
$\rightsquigarrow$ Classification small index subfactors $N \subset M$.

Unitary tensor categories in Nature

- UTC: $\mathcal{C} = (\mathcal{C},\ \circ,\ \oplus,\ \dagger,\ \overline{\cdot},\ \otimes,\ 1_{\mathcal{C}},\ \text{data/axioms})$.

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Higher relative commutants $\rightsquigarrow \mathcal{C}_{N\subset M} \overset{F}{\hookrightarrow} \mathsf{Bim}_{\mathrm{fgp}}(N)$.

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$\therefore$ $\mathcal{C}$ acts on operator algebra A via generalized fiber functor:

$$F : \mathcal{C} \xrightarrow{\otimes} \mathrm{Bim}_{\mathrm{fgp}}(A).$$

Discrete subfactors & quantum dynamics

Discrete subfactors: [ILP98]

$$\underbrace{(N,\tau)\overset{E}{\subset} M}_{\text{discrete}} \;\Leftrightarrow\; {}_N L^2(M)_N \;\cong \bigoplus_{K\in \mathrm{Irr}(\mathrm{Bim}_{\mathrm{fgp}}(N))} K^{\oplus n_K},\;\; n_K \in \mathbb{N}\cup\{0\}$$

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Construction/classification discrete subfactors

- Construct/classify quantum dynamics: $\mathcal{C}\curvearrowright N$,
- Describe $W^*\mathrm{Alg}(\mathcal{C})$. (Internal to $\mathcal{C}$)

Quantum symmetries for C*-algebras

- [GLN14, EGLN15, TWW17] Elliott Program: K-theory & traces

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$$\mathrm{Ind}_W \left(A \overset{E}{\subset} B \right) < \infty \Leftrightarrow \exists \{(u_i, v_i)\}_1^n \subset B \times B$$

$$\forall b \in B \quad \sum u_i E(v_i b) = b = \sum E(b u_i) v_i .$$

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Infinite index C*-subalgebras?

$$\left(A \overset{E}{\subset} B\right) \leftrightarrow \left({}_B B_B,\ \underbrace{{}_A\mathcal{B}_A}_{\in \mathrm{C^*Alg}(\mathcal{C})},\ \underbrace{\mathcal{C} \overset{F}{\curvearrowright} A}_{\text{outer}}\right)$$

Infinite index inclusions in practice

- The canonical B-A correspondence ${}_B\mathcal{B}_A$:

$$\underbrace{A \overset{E}{\subset} B}_{\text{faithful}} \rightsquigarrow \langle b_1 | b_2 \rangle_A := E(b_1^* b_2) \rightsquigarrow \mathcal{B} = \overline{B\Omega}^{\|\cdot\|_A},$$

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Example (Reduced crossed products)

Given action $F : \mathcal{C} \to \mathrm{Bim_{fgp}}(A)$ and $\mathcal{C}$-graded C*-algebra:

$$A \rtimes_{F,r} \mathbb{B} = \mathrm{C}^*_r\left(\bigoplus_{c\in \mathrm{Irr}(\mathcal{C})} F(c) \otimes \mathbb{B}(c)\right)$$

Synthetic examples

Reduced crossed products by outer group actions are C*-discrete:

$$\Gamma \overset{\alpha}{\curvearrowright} A \ \rightsquigarrow\ \left\{ A \overset{E}{\subset} A \rtimes_{\alpha,r} \Lambda \right\}_{\Lambda \leq \Gamma} \subset \mathrm{C}^*\mathrm{Disc}.$$

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- *Genuine* W*/C*-algebra objects are abundant:

•• $\mathbb{G}$: CQG $\overset{\text{T-K}}{\rightsquigarrow}$ $\mathrm{Rep}_f(\mathbb{G}) \curvearrowright \mathrm{Hilb}_f$ $\overset{\text{[JP17]}}{\rightsquigarrow}$ $\mathbb{C}[\mathbb{G}]$,

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The C*-discrete family

$$\underbrace{\{A \rtimes_{\alpha,r} \Gamma\}}_{\text{discrete groups}} \subset \underbrace{\{\mathrm{Ind}_W < \infty\}}_{\text{Q-systems}} \subset \underbrace{\{A \rtimes_{F,r} \mathbb{B}\}}_{C^*\text{-alg objects}} \subseteq \mathrm{C}^*\mathrm{Disc}$$

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Theorem ([HPN23])

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sketch:

$(\Leftarrow)$: take reduced crossed product $A \overset{E'}{\subset} A \rtimes_{F,r} \mathbb{B}$.
$(\Rightarrow)$: $F : \mathcal{C}_{A \subset B} \hookrightarrow \mathrm{Bim_{fgp}}(A)$,

$$B^{\Diamond} \cong \underbrace{\bigoplus_{K \in \mathrm{Irr}(\mathrm{Bim_{fgp}}(A))} K \otimes \mathrm{Hom}_{A-A}(K \to B\Omega) \subset B}_{*\text{-subalgebra!}}.$$

Eg: Cores of Cuntz algebras

Example

$$n \in \mathbb{N},\ \underbrace{\mathbb{T} \curvearrowright \mathcal{O}_n}_{\text{gauge}} \rightsquigarrow \mathcal{O}_n^{\mathbb{T}} \cong \mathrm{UHF}_{n^\infty} =: A$$

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Proof.

$$\underbrace{b \overset{\lambda_1}{\mapsto} s_1 b s_1^*}_{\text{one-sided Bernoulli}} \rightsquigarrow F_1 : \mathrm{Hilb}(\mathbb{Z}) \to \mathrm{Bim_{fgp}}(A),\ k \mapsto (s_1 s_1^*)_{\lambda_1}[A]_A.$$

$$\mathcal{O}_n^{\diamondsuit} = * - \mathrm{Alg}(\{s_i\}_1^n) \overset{\text{dense}}{\subset} \mathcal{O}_n.$$

$$\therefore\ \mathcal{O}_n \cong \mathrm{UHF}_{n^\infty} \rtimes_{F_1, r} \mathbb{C}[\mathbb{Z}].$$

□

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- Some examples from free probability:

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$$\left\{\eta_{ij} : A \overset{\text{c.p.}}{\to} A\right\}_{i,j\in I} \rightsquigarrow \underbrace{\eta = \{\eta_{ij}\} : A \to A \otimes \mathrm{End}(\ell^2(I))}_{\text{covariance matrix}}.$$

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- $\exists\tau$ faithful compatible trace: $\tau(\eta_{ij}(x)y) = \tau(x\eta_{ij}(y))$ and $A \otimes_{\eta_{i,j}} A \in \mathrm{Bim}_{\mathrm{fgp}}(A)$
  $\Rightarrow \underbrace{A \overset{E}{\subset} \hat{\Phi}(A,\eta)}_{\text{C*-discrete}} := \mathrm{C}^*\left(A \cup \{\ell(\xi_i) + \ell(\xi_i)^*\}_i\right) \subset \mathrm{End}^\dagger(\mathcal{F}(\eta)_A).$

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- Assuming $\#I \in \mathbb{N}$:
  $A \subset \hat{\Phi}(A,\eta)$ irreducible $\Leftrightarrow \mathcal{F}(\eta)^A \cong \mathbb{C}$.

$\therefore$ $\hat{\Phi}(A,\eta)$ yields irreducible C*-discrete extension realized as a crossed product!

Galois Correspondence

Theorem ([HPN23])

Given $A \overset{E}{\subset} B$ irreducible C-discrete:*

$$\left\{ D \,\middle|\, \begin{array}{c} A \subseteq D \subseteq B \\ A \overset{E|_D}{\subseteq} D \in C^*\mathrm{Disc} \end{array} \right\} \leftrightarrow \left\{ \mathbb{D} \,\middle|\, \begin{array}{c} \mathcal{C}_{A\subset B}\text{-graded } C^*\text{-alg obj} \\ \mathbb{1} \subseteq \mathbb{D} \subseteq \mathbb{B} \end{array} \right\}$$

$$\leftrightarrow \left\{ D \,\middle|\, \begin{array}{c} A \subseteq D \subseteq B \\ \exists E_D^B : B \twoheadrightarrow D,\ E_D^B \circ E = E \end{array} \right\}.$$

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- Our Theorem categorifies [CS19].
- If A is simple, C-S show correspondence is tight.

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Cores of Cuntz Algebras

- [Rø21] shows Correspondence is tight and *C*-irreducible*.

C*-discrete extensions & simplicity

Free UTC-action:

$F : \mathcal{C} \to \mathrm{Bim}_{\mathrm{fgp}}(A)$ is <u>free</u> iff $\forall c \in \mathcal{C}, \forall \xi \in F(c)$:

$$\inf \left\{ \left\| \sum_{1}^{n} a_i^* \triangleright \xi \triangleleft a_i \right\| \;\middle|\; \{a_i\}_1^n \subset A,\ \sum_{1}^{n} a_i^* a_i = 1 \right\} = 0.$$

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Theorem ([HPN23])

Let $1\in A$ be simple, $\mathcal{C}\overset{F}{\curvearrowright} A$ be free and outer, and $\mathbb{B}\in \mathrm{C^*Alg}(\mathcal{C})$. Then $A\rtimes_{F,r}\mathbb{B}$ remains simple.

- ◦ Is freeness automatic for outer UTC-actions?
- • When is the Galois correspondence tight?
- •• Characterize C*-discrete extensions by TLJ(δ) GJS-actions by K-theory.
- • • • Approximation properties for C*-discrte inclusions in terms of their C*-algebra objects.
- • • •• Are C*-discrete extensions of classifiable C*-algebras classifiable?

Thank you!

- Quan Chen, Roberto Hernández Palomares, and Corey Jones, *K-theoretic classification of inductive limit actions of fusion categories on AF-algebras*, arXiv e-prints (2022), arXiv:2207.11854.
- Quan Chen, Roberto Hernández Palomares, Corey Jones, and David Penneys, *Q-system completion for* C^* *2-categories*, J. Funct. Anal. **283** (2022), no. 3, Paper No. 109524.
- A. Connes, *Classification of injective factors. Cases* II_1, II_∞, III_λ, $\lambda \neq 1$, Ann. of Math. (2) **104** (1976), no. 1, 73–115, MR0454659.
- Jan Cameron and Roger R. Smith, *A Galois correspondence for reduced crossed products of simple* C^**-algebras by discrete groups*, Canad. J. Math. **71** (2019), no. 5, 1103–1125.
- George A. Elliott, Guihua Gong, Huaxin Lin, and Zhuang Niu, *On the classification of simple amenable C*-algebras with finite decomposition rank, II*, arXiv e-prints (2015), arXiv:1507.03437.

Samuel Evington and Sergio Girón Pacheco, *Anomalous symmetries of classifiable* C^**-algebras*, Studia Math. **270** (2023), no. 1, 73–101.

Michael Frank and Eberhard Kirchberg, *On conditional expectations of finite index*, J. Operator Theory **40** (1998), no. 1, 87–111, MR1642530.

Guihua Gong, Huaxin Lin, and Zhuang Niu, *Classification of finite simple amenable* $\mathcal{Z}$*-stable* C^**-algebras*, arXiv e-prints (2014), arXiv:1501.00135.

Michael Hartglass and Roberto Hernández Palomares, *Realizations of rigid* C^**-tensor categories as bimodules over GJS* C^**-algebras*, J. Math. Phys. **61** (2020), no. 8, 081703, 32, MR4139893 DOI:10.1063/5.0015294.

Masaki Izumi, Roberto Longo, and Sorin Popa, *A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras*, J. Funct. Anal. **155** (1998), no. 1, 25–63, MR1622812.

- Masaki Izumi, *Subalgebras of infinite C^*-algebras with finite Watatani indices. II. Cuntz-Krieger algebras*, Duke Math. J. **91** (1998), no. 3, 409–461, MR1604162, DOI:10.1215/S0012-7094-98-09118-9.
- Vaughan F. R. Jones, *Index for subfactors*, Invent. Math. **72** (1983), no. 1, 1–25, MR696688, DOI:10.1007/BF01389127.
- Vaughan F. R. Jones, *Planar algebras, I*, arXiv Mathematics e-prints (1999), math/9909027.
- Corey Jones, *Remarks on anomalous symmetries of C*-algebras*, Comm. Math. Phys. **388** (2021), no. 1, 385–417.
- Corey Jones and David Penneys, *Operator algebras in rigid C^*-tensor categories*, Comm. Math. Phys. **355** (2017), no. 3, 1121–1188, MR3687214 DOI:10.1007/s00220-017-2964-0 arXiv:1611.04620.

- ______, *Realizations of algebra objects and discrete subfactors*, Adv. Math. **350** (2019), 588–661, MR3948170 DOI:10.1016/j.aim.2019.04.039 arXiv:1704.02035.
- Tsuyoshi Kajiwara and Yasuo Watatani, *Jones index theory by Hilbert C^*-bimodules and K-theory*, Trans. Amer. Math. Soc. **352** (2000), no. 8, 3429–3472, MR1624182, DOI:10.1090/S0002-9947-00-02392-8.
- Michael Müger, *From subfactors to categories and topology. I. Frobenius algebras in and Morita equivalence of tensor categories*, J. Pure Appl. Algebra **180** (2003), no. 1-2, 81–157, MR1966524 DOI:10.1016/S0022-4049(02)00247-5 arXiv:math.CT/0111204.
- Adrian Ocneanu, *Quantized groups, string algebras and Galois theory for algebras*, Operator algebras and applications, Vol. 2, London Math. Soc. Lecture Note Ser., vol. 136, Cambridge Univ. Press, Cambridge, 1988, MR996454, pp. 119–172.

Sorin Popa, *An axiomatization of the lattice of higher relative commutants of a subfactor*, Invent. Math. **120** (1995), no. 3, 427–445, MR1334479 DOI:10.1007/BF01241137.

______, *Classification of subfactors and their endomorphisms*, CBMS Regional Conference Series in Mathematics, vol. 86, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1995, MR1339767.

Mikael Rørdam, *Irreducible inclusions of simple C^*-algebras*, arXiv e-prints (2021), arXiv:2105.11899.

Aaron Tikuisis, Stuart White, and Wilhelm Winter, *Quasidiagonality of nuclear C^*-algebras*, Ann. of Math. (2) **185** (2017), no. 1, 229–284.

Yasuo Watatani, *Index for C^*-subalgebras*, Mem. Amer. Math. Soc. **83** (1990), no. 424, vi+117 pp., MR996807, (preview at google books).