Oligomorphic Groups, Measures, and the Delannoy Category

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Work of Harman-Snowden partially joint with me.

Slides at nsnyder1.pages.iu.edu

Outline

- Rep(G)-like $\otimes$-categories
- The Delannoy Category HSS
- $\otimes$-categories from measures on Oligomorphic groups HS
- Examples HS + HSS

Tensor Categories

<u>Ex</u> $\mathrm{Rep}(G)$

compact group

Objs: reps

Morphs: G-covariant maps

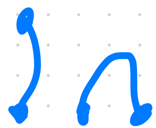
Other structure: $\otimes$, $\mathbb{1}$, $*$

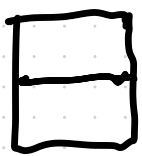
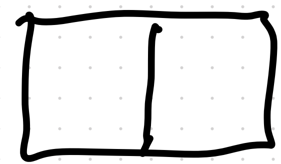
<u>Ex</u> Finite index R-mod-R

hyperfinite II_1

Planar Algebra Example

Objs 

Mors  $+ i\cdot$  mod $\bigcirc = d\cdot$

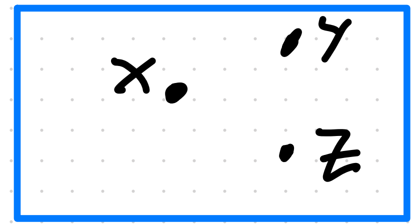
Compose  Tensor 

Temperley-Lieb-Jones

Commutativity

Ordinary algebra

non-comm.



comm.

Includes R-mod-R

Tensor Categories

2-dim diagrams non-comm.

Includes TLJ

3-dim diagrams braided

Includes Rep(G)

4-dim diagrams symmetric

Pre-Tannakian Categories

"Looks like" Rep(G)

Algebraic Group

- Symmetric monoidal (including rigid) duals
- Finite dimensional k-linear Hom spaces usually $\mathbb{C}$ in this talk
- Abelian. Objects have finite length.
- $\mathbb{1}$ is absolutely simple $\mathrm{End}(\mathbb{1}) = k$

Problem 1

Classify pre-Tannakian categories.

Additional Source: "essentially Tannakian"

Ex Representations of supergroups

Ex Algebraic groups in Ver_p Ostrik et al.

Thm (Deligne) If $\mathcal{C}$ is pre-Tannakian over $\mathbb{C}$ and has moderate growth then $\mathcal{C} \simeq \mathrm{Rep}(G)$

length$(X^{\otimes n}) \leq C^n$

supergroup

See Coulembier-Etingof-Ostrik for a related thm in characteristic p and Benson-E.-O. and C. for additional examples

Deligne Interpolation Categories

$K = \mathbb{C}$

$\underline{GL_t}$: Universal sym. $\otimes$-cat with an object of dimension t

$\underline{O_t}$: ——————— symmetrically self-dual obj of dimension t.

i.e., Subfactor

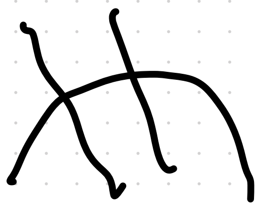
$\underline{S_t}$: ——————— Frobenius algebra obj of dimension t.

Diagram description

GL_t

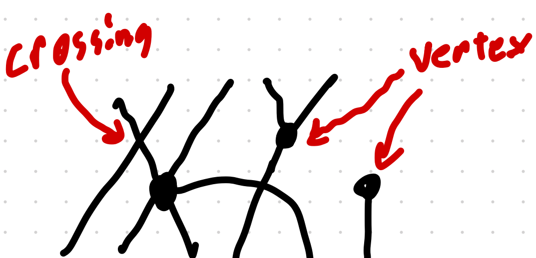
mod $\bigcirc = t$

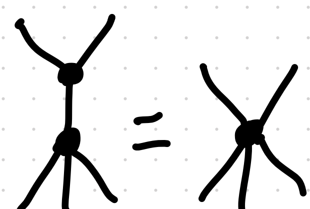
O_t



mod $\bigcirc = t$

S_t





etc., $\bigcirc = t$

Warning

These are not abelian!

- If $t \notin \mathbb{Z}$ or $\mathbb{N}$ then take Cauchy completion
- If $t \in \mathbb{Z}$ or $\mathbb{N}$ work harder.

Even more interesting in characteristic p!

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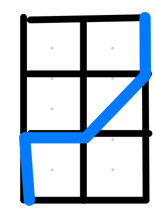
New Example (HSS)

Cauchy completion of:

Objects $\mathcal{C}(\mathbb{R}^{(n)})$ for each $n \in \mathbb{Z}_{\geq 0}$

Morphisms $\mathrm{Hom}(\mathcal{C}(\mathbb{R}^{(n)}), \mathcal{C}(\mathbb{R}^{(m)}))$ spanned by

n-by-m Delannoy paths



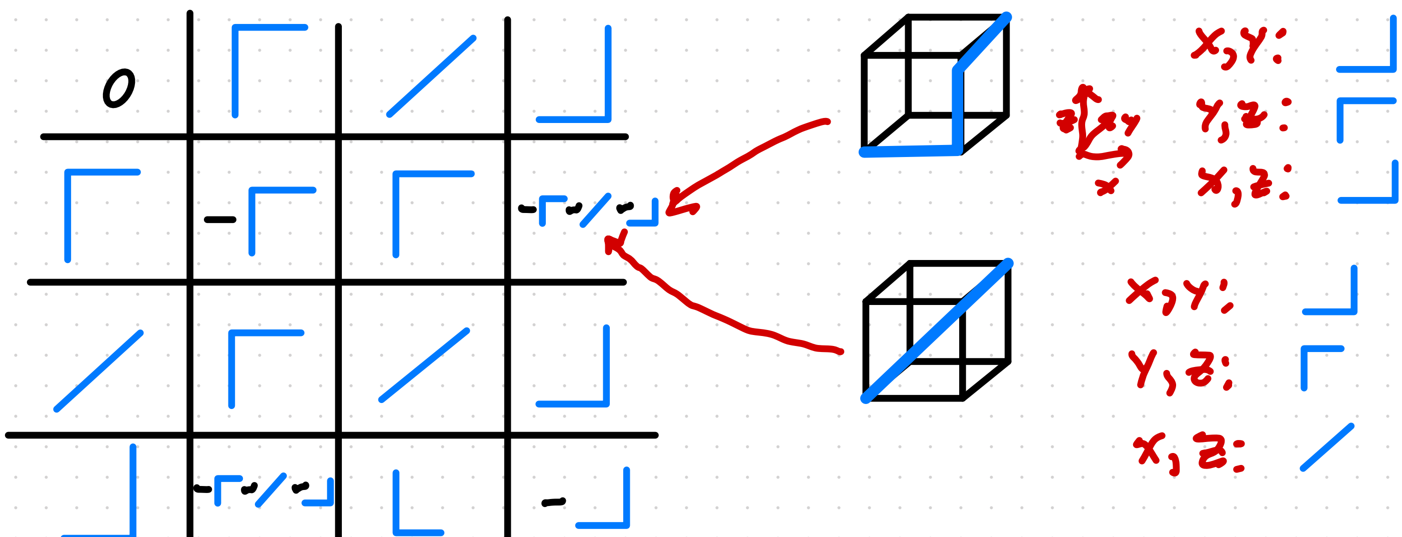
steps of $(1,0)$, $(0,1)$ or $(1,1)$

So we call it the Delannoy Category

<u>Composition:</u> $p \circ q = \sum_{\gamma} (-1)^{\cdots} r$

Where γ is a 3-dim Delannoy path which projects down to p, q, and r

<u>Ex</u>



Tensor product is more complicated.

Can describe $\mathrm{Hom}(\mathcal{C}(\mathbb{R}^{(a)}) \underline{\otimes} \mathcal{C}(\mathbb{R}^{(b)}), \mathcal{C}(\mathbb{R}^{(c)}))$ via (a,b,c)-Delannoy paths. But then composing these uses 4-dim Delannoy paths.

<u>Aside:</u> Could give a Planar Algebraic description using

is the span of $(a_1, \dots, a_5)$-paths

Structure

Lots more in paper!

- "New stuff" in $\mathcal{C}(\mathbb{R}^{(n)})$ is mult. free with simple summands $L_{{\color{red}\bullet}{\color{red}\bullet}{\color{blue}\bullet}{\color{red}\bullet}}$

  length n word.

- Tensor product is a modified shuffle product

$$L_{{\color{blue}\bullet}{\color{red}\bullet}} \otimes L_{{\color{blue}\bullet}} \cong \underbrace{L_{{\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}} \oplus L_{{\color{blue}\bullet}{\color{blue}\bullet}{\color{red}\bullet}} \oplus L_{{\color{blue}\bullet}{\color{red}\bullet}{\color{blue}\bullet}}}_{\text{shuffle part}}$$

$$\oplus \underbrace{L_{{\color{blue}\bullet}{\color{red}\bullet}}}_{{\color{blue}\bullet}\ \&\ {\color{blue}\bullet}\ \text{collide}} \oplus \underbrace{L_{{\color{blue}\bullet}{\color{red}\bullet}} \oplus L_{{\color{blue}\bullet}{\color{blue}\bullet}} \oplus L_{{\color{blue}\bullet}}}_{{\color{red}\bullet}\ \&\ {\color{blue}\bullet}\ \text{collide}}$$

Novel Properties (HS)

- Doesn't come from (super)groups <u>or</u> from Deligne interpolation.

- In characteristic p it's the first example of a semisimple pre-Tannakian cat that does not have finite growth.

But where does it come from?

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Concreteness

Concrete: $\mathrm{Rep}(G)$, $R\text{-mod-}R$, etc.

Objects have internal mathematical structure

Abstract TLJ_d etc.

Objects are formal.

From abstract to Concrete

Thm (Jones) For $d = \zeta + \zeta^{-1}$ there's a subcategory of R-mod-R equivalent to TLJ_d.

Ocneanu + Popa generalize this to unitary fusion categories and amenable tensor categories.

Problem 2

Can Deligne Categories be made concrete?

Warning: Can't use operator algebras because negative dims!

1) Harman-Snowden answer this question for S_t in characteristic zero.

2) Their answer leads to several new examples, and the Delannoy Category is the nicest.

Idea:

1) Start with a group Exs S_∞ or $\mathrm{Aut}(\mathbb{R},>)$

2) Define a category $\mathrm{Perm}(G,\mu)$ of permutation representations. Uses a measure on G.

3) Take an abelian envelope

Sometimes this envelope is $\mathrm{Rep}(G,\mu)$ ← Concrete!

Most general version still open.

Under strong conditions can just take Cauchy completion

No time to fit in this talk!

Oligomorphic Permutation Groups

Group G acting on a set Ω.

G acts on Ω^n with finitely many orbits

n-tuples

finitary

G has a topology where the basic open subgroups are stabilizers of pts. in Ω^n

Closely connected to Model Theory

Oligomorphic groups are the automorphism groups of ω-categorical theories

$\exists!$ countable model

These can be understood via Fraïssé's Thm.

<u>Ex</u> $\mathrm{Aut}(\Gamma)$ Rado graph

Key idea: Want to also understand restrictions to open subgroups.

Def A $\hat{G}$-set is a set with an action of some open subgroup. Shrinking subgroup doesn't change the $\hat{G}$ set.

$\underline{\mathrm{Perm}(G, \mu)}$

Objects are Vec_X where X is formal symbol

a finitary and smooth G-set

finitely many orbits

stabilizers are open subgroups

Ex Ω^n, $\Omega^{(n)}$ n element subsets

Morphisms $\mathrm{Hom}(X,Y)$ are G-covariant

"$X \times Y$ matrices"

Schwartz functions on $X \times Y$

smooth and finitary support

How to compose?

$$B \circ A(x,z) = \int_Y B(x,y)\, A(y,z)\, dy$$

needs a measure

not required to be positive

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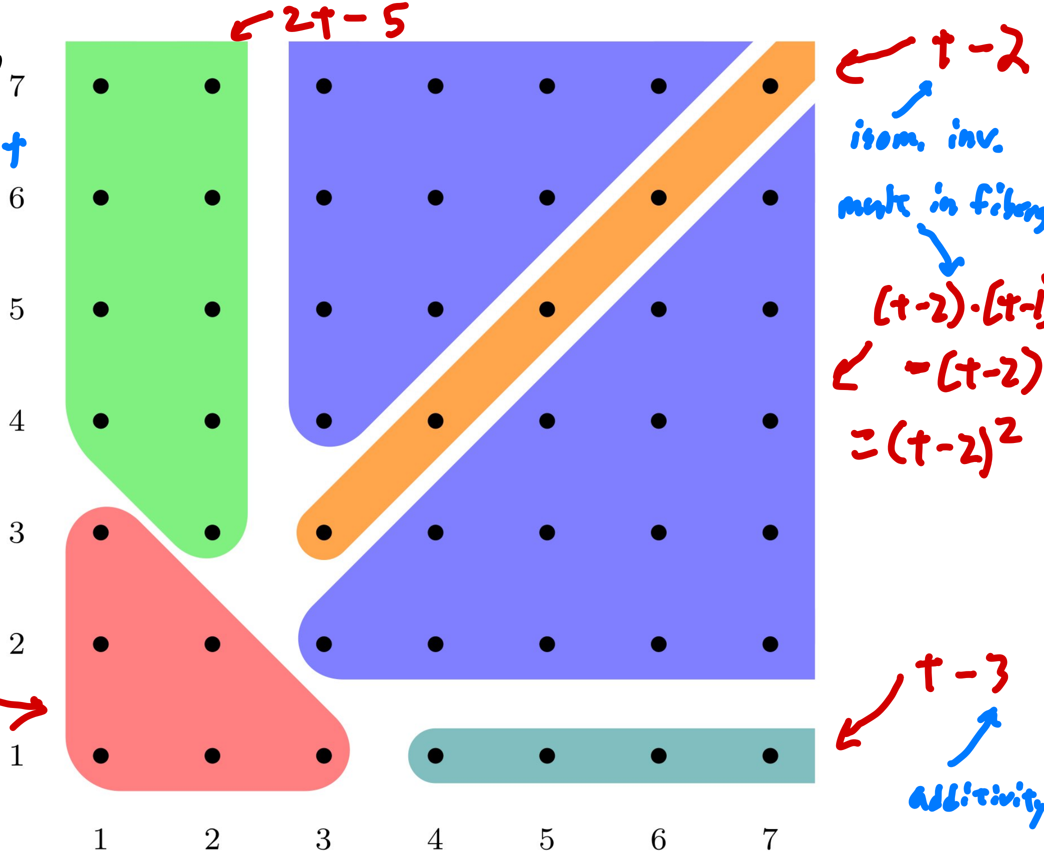
Ex S_∞

$\mu(\Omega) = t$



Thm (HS) $\exists!$ μ_t with $\mu_t(\Omega) = t$ given by $\mu_t(X) = P_X(t)$ where

$$P_X(n) = \# \underbrace{X^{S(n)}}_{\text{fixed pts}} \quad \text{for } n \gg 0.$$

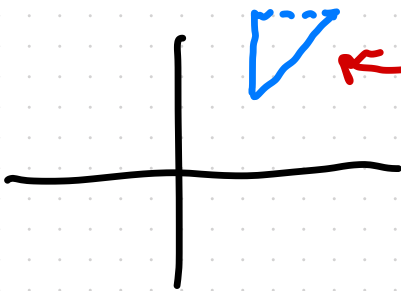
Thm (HS) The abelian envelope of Perm(S_∞, μ_t) is Deligne's S_t and can be realized a category of modules for a completed group algebra.

$\mathrm{Vec}_X \longrightarrow \mathcal{C}(X)$ Schwartz functions on X

<u>Ex</u> $\mathrm{Aut}(\mathbb{R}, >)$

Open subsets are $G(A)$ fixing a finite $A \subseteq \mathbb{R}$

There's a measure give by Euler char.



measure $1-2+1=0$.

Has nice properties so again get a pre-Tannak

Concrete category <u>Rep</u>(G).

$\underline{\text{Ex}}$ $\quad \mathrm{Hom}(\mathrm{Vec}_{\mathbb{R}}, \mathrm{Vec}_{\mathbb{R}}) \simeq \mathbb{C}^3$

Spanned by $\quad \chi_{x<y} =: A, \quad \chi_{x=y} =: I, \quad \chi_{x>y} =: B$

Characteristic function

These correspond combinatorially to $\lrcorner$, $/$, $\ulcorner$

$\underline{\text{Check}}$ $\quad A \circ B(x,z) = \int_y A(x,y) \cdot B(y,z)\, dy$

$= \mu(\{y : y > x, z\}) = \mu(\circ\!\!-\!\!-\!\!-) = 0 - 1 + 0 = -1$

$= -A - B - I.$