On the story of matrices of high finite order: Then (1942) and now (2022)

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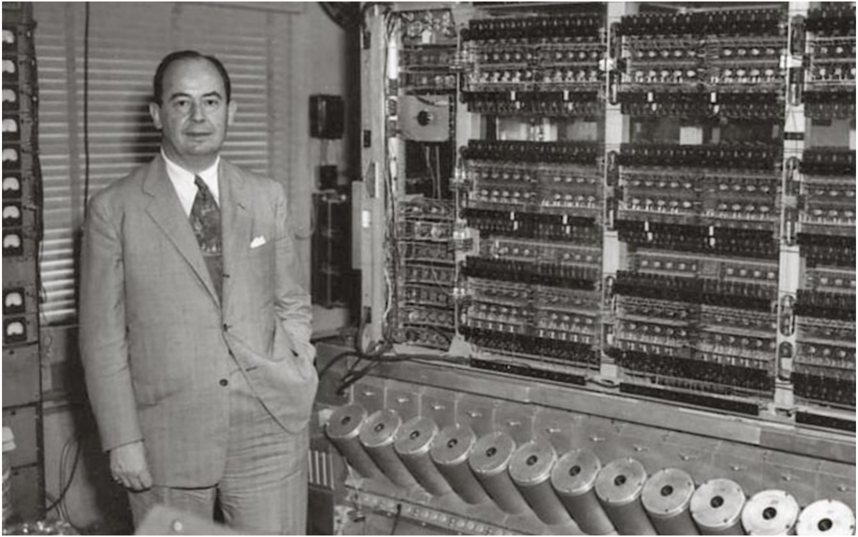

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APPROXIMATIVE PROPERTIES OF MATRICES OF HIGH FINITE ORDER

by JOHN VON NEUMANN
PRINCETON, N. J.

(Recebido em 1941, Setembro, 10)

INTRODUCTION

4. Our interest will be concentrated in this note on the conditions in $\mathfrak{E}_n$ and $\mathbf{M}_n$ —mainly $\mathbf{M}_n$— when n is *finite*, but *very great*. This is an approach to the study of the infinite dimensional, which differs essentially from the usual one. The usual approach consists in studying an actually infinite dimensional unitary space, i. e. the Hilbert space $\mathfrak{E}_\infty$, as done loc. cit. [3]. We wish to investigate instead the *asymptotic* behavior of $\mathfrak{E}_n$ and $\mathbf{M}_n$ for finite n, when $n \to \infty$.

We think that the latter approach has been unjustifiably neglected, as compared with the former one. It is certainly not contained in it, since it permits the use of the notions $\|A\|$ and $t(A)$, which, owing to

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3. Powers '67: continuum many non isomorphic hyperfinite type III factors!
4. Connes '76: hyperfiniteness is equivalent to injectivity (amenability for group von Neumann algebras: Schwartz '63).

10. The present investigations were undertaken with a definite aim, as mentioned at the end of Paragraph 4. The results of F. J. Murray and the author on operator rings [19] led to various problems, of which one seems to be of particular importance: Whether all those operator rings in Hilbert space, which — in the terminology of the papers mentioned in [8] [19] — belong to the «finite, continuous dimensional class», i. e. to the «class (II_1)», are isomorphic to each other or not. [20] We surmise that they are not, and that the results of this paper may be helpful in establishing this fact. This and other related aspects of the subject will be dealt with in forthcoming publications of F. J. Murray and the author. *

Tracial Ultraproducts:

Let N_i ($i \in \mathbb{N}$) be separable II_1 factors and let $\omega \in \beta\mathbb{N} \setminus N$. Denote the ultraproduct by

$$\prod_{i \to \omega} N_i = \frac{\{(x_i)_{i\in\mathbb{N}} \mid \sup_i \|x_i\| < \infty\}}{\{(x_i)_{i\in\mathbb{N}} \mid \lim_{i\to\omega} \|x_i\|_2 = 0\}}.$$

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(see also Ocneanu, Udea, Ando-Haagerup and other works in the type III setting which is more subtle!)

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Problem 1: The Uniqueness Problem

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Problem 3: The Pseudocompactness Problem

Classify II_1 factors N such that $N^\omega \cong \mathcal{M}$.

Theorem (von Neumann '42):

For any $\epsilon > 0$ there is $\delta > 0$ and a sequence of matrices $A_i \in \mathbb{M}_i(\mathbb{C})$ such that if $B \in \mathbb{M}_n(\mathbb{C})^{sa}$ with $\|[B, A_n]\|_2 < \delta$ then $\|B - \tau(B)\|_2 < \epsilon$.

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Corollary:

Let N be any II_1 factor with property Gamma. Then $N^\omega \not\cong \mathcal{M}_f$ for any f.

11. The method by which our results — i. e. those enumerated in Paragraph 9 — will be obtained consists mainly in computing and comparing volumes. The space $\mathbf{M}_n$ of all n-th order matrices may be looked at as a $2n^2$-dimensional real Euclidean space: The matrix $A=(a_{\varkappa\lambda})$, $\varkappa, \lambda = 1, \cdot\cdot, n$, being identified with a point whose $2n^2$ (real) coordinates are the $\Re a_{\varkappa\lambda}$ and $\Im a_{\varkappa\lambda}$,[21] $\varkappa, \lambda = 1, \cdots, n$. Then volumes can be defined in $\mathbf{M}_n$ in the usual way. Now if we could prove that the volume of the set of those A in $\mathbf{M}_n$ which do not possess the desired properties (cf. Paragraph 9), is smaller than the Volume of the *pseudo sphere* $|||A||| \leqq 1$, then this would constitute a proof of the existence of elements A in $\mathbf{M}_n$ with $|||A||| \leqq 1$ and of the desired kind.

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Does there exist a separable non Gamma factor N that is not pseudocompact.

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Theorem (C-I-KE '22):

There exists a separable non Gamma factor N that is not pseudocompact. Moreover, $N^\omega \not\cong L(\mathbb{F}_2)^\omega$ giving the first two explicit examples of non-elementary equivalent non Gamma factors.


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Let (M, τ) be a diffuse tracial von Neumann algebra, and $X \subset M_{\text{sa}}$ finite such that $\|x\| \leq R$ for all $x \in X$. For each weak* neighborhood $\mathcal{O}$ of ℓ_X and $n \in \mathbb{N}$, we define

$$\Gamma_R^{(n)}(X; \mathcal{O}) = \{A \in \mathbb{M}_n(\mathbb{C})_{\text{sa}}^{|X|} : \ | \ \ell_A \in \mathcal{O}, \|A_x\| \leq R\}.$$

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Crucial property:

If $N_1, N_2 \subset M$ and $N_1 \cap N_2$ is diffuse then $h(N_1 \vee N_2) \leq h(N_1) + h(N_2)$.

The invariant: Property $\widetilde{Gamma}$

For every $\epsilon > 0$, and every $u_1, u_2 \in \mathcal{U}(M)$ such that $u_1^2 = 1$, $u_2^3 = 1$ and $\{u_1\}'' \perp \{u_2\}''$ there exists Haar unitaries $v_1, v_2 \in \mathcal{U}(M)$ satisfying $\|u_i, v_i\|_2 \leq \epsilon$ for each $i = 1, 2$ and $\|v_1, v_2\|_2 \leq \epsilon$.

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In progress and/or future work:

1. Connes embeddability of our N!
2. Finite index subfactors of matrix ultraproducts.
3. Finding three full factors with non isomorphic ultrapowers!
4. $L(\mathbb{F}_2)^\omega \cong \mathcal{M}_f$?

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