Local Topological Order and Boundary Algebras

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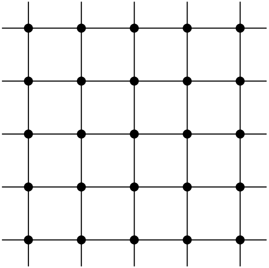
Joint with Corey Jones, Pieter Naaijkens and David Penneys

Spin systems!



(2 + 1)D spin systems: consider infinite planar lattice, associate $\mathbb{C}^d$ to each vertex or edge

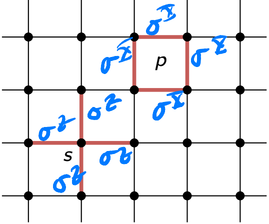
For each finite region Λ, have tensor product Hilbert space $\bigotimes_\Lambda \mathbb{C}^d$

Get UHF *quasilocal algebra* by taking inductive limit: $\mathfrak{A} := \bigotimes M_d(\mathbb{C})$

Simple example: toric code—associate $\mathbb{C}^2$ to each edge

Toric code

$\sigma^Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\sigma^X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



Star terms and plaquette terms:

$$A_s := \bigotimes_{j \in s} \sigma_j^Z \qquad B_p := \bigotimes_{j \in p} \sigma_j^X$$

Local Hamiltonians: For finite regions Λ

$$H_\Lambda := -\sum_{s \subseteq \Lambda} A_s - \sum_{p \subseteq \Lambda} B_p$$

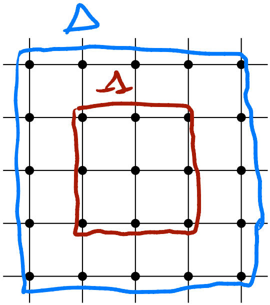
Topological order à la BHM '10

For Λ rectangle in the lattice, let

$$p_\Lambda = \prod_{s \subseteq \Lambda} \frac{I + A_s}{2} \prod_{p \subseteq \Lambda} \frac{I + B_p}{2}$$

(projection onto ground state space)
Get *net of projections*: $p_\Delta \leq p_\Lambda$ if $\Lambda \subseteq \Delta$
Net of projections for toric code satisfies TQO of Bravyi-Hastings-Michalakis '10:

> For x supported on Λ and Λ completely surrounded by Δ, $p_\Delta x p_\Delta \in \mathbb{C} p_\Delta$



The canonical state

If x is supported on Λ which is completely surrounded by Δ, then

$$p_\Delta x p_\Delta = \psi(x) p_\Delta,$$

where $\psi(x)$ is independent of Δ!
Get a canonical state $\psi\colon \mathfrak{A} \to \mathbb{C}$, which is the unique translation-invariant ground state for toric code (Alicki-Fannes-Horodecki '07)

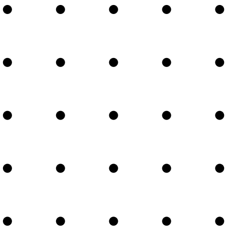
Boundary algebras: The why

Boundary algebras are net of algebras corresponding to line of sites in the lattice (a boundary!)

Using machinery by Jones '23, we get category of DHR bimodules for the boundary algebras

This gives us information about excitations for the bulk—a bulk-boundary correspondence!

The setup



Have $\mathbb{Z}^k$ lattice

A *net of algebras* is an assignment of an algebra $\mathfrak{A}(\Lambda)$ to each rectangle Λ of the lattice satisfying

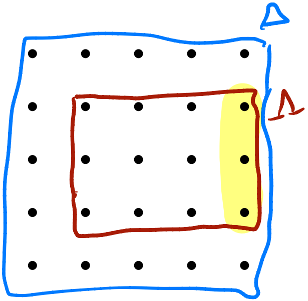
$$\mathfrak{A}(\emptyset) = \mathbb{C}$$

$$\Lambda \subseteq \Delta \text{ implies } \mathfrak{A}(\Lambda) \subseteq \mathfrak{A}(\Delta)$$

$$\text{If } \Lambda_1 \cap \Lambda_2 = \emptyset, \text{ then } [\mathfrak{A}(\Lambda_1), \mathfrak{A}(\Lambda_2)] = 0$$

(Think $\mathfrak{A}(\Lambda)$ operators supported on Λ for a spin system!)

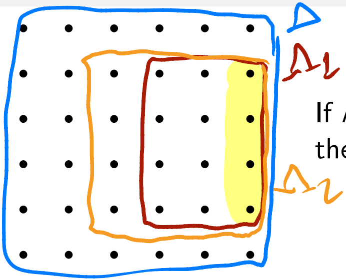
The setup: Part 2



A *net of projections* is an assignment of a projection $p_\Lambda \in \mathfrak{A}(\Lambda)$ for each rectangle Λ satisfying $p_\Delta \leq p_\Lambda$ for $\Lambda \subseteq \Delta$

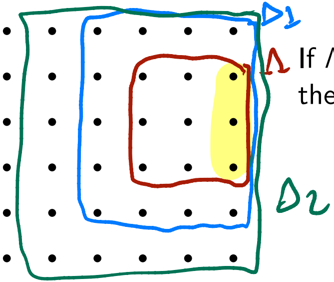
Given two rectangles Λ, Δ, say $\Lambda \Subset \Delta$ if $\Lambda \subseteq \Delta$ and $\partial\Lambda \cap \partial\Delta$ is either empty or a $(k-1)$-dimensional rectangle.

The local topological order axioms



If $\Lambda_1 \subseteq \Lambda_2 \Subset \Delta$ with $\partial\Lambda_1 \cap \partial\Delta = \partial\Lambda_2 \cap \partial\Delta$, then

$$p_\Delta \mathfrak{A}(\Lambda_1) p_\Delta = p_\Delta \mathfrak{A}(\Lambda_2) p_\Delta$$

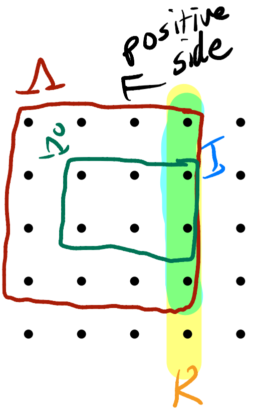


If $\Lambda \Subset \Delta_1 \subseteq \Delta_2$ with $\partial\Lambda \cap \partial\Delta_1 = \partial\Lambda \cap \partial\Delta_2$, then for $x \in \mathfrak{A}(\Lambda)$,

$p_{\Delta_1} x p_{\Delta_1}$ commutes with p_{Δ_2}

$x p_{\Delta_2} = 0$ implies $x p_{\Delta_1} = 0$

The boundary algebras



Fix $\mathbb{Z}^{k-1}$ hyperplane $\mathcal{K}$ inside $\mathbb{Z}^k$ and a positive side of $\mathcal{K}$

Pick subrectangle $I \subseteq \mathcal{K}$

Choose Λ a rectangle on the positive side of $\mathcal{K}$ with $\partial\Lambda \cap \mathcal{K} = I$, and let Λ° be the largest rectangle such that $\Lambda^\circ \Subset \Lambda$ and $\emptyset \neq \partial\Lambda^\circ \cap \partial\Lambda \subseteq I$

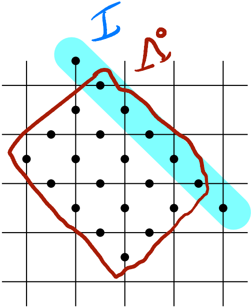
Get net of *boundary algebras*

all isomorphic

$$I \mapsto \mathfrak{B}(I) := \varinjlim_{\Lambda} p_\Lambda \mathfrak{A}(\Lambda^\circ) p_\Lambda$$

Get a canonical state ψ_∂ on $\mathfrak{B} = \varinjlim \mathfrak{B}(I)$ by using TQO of BHM '10 to get state on $\mathfrak{A} = \varinjlim \mathfrak{A}(\Lambda)$ and extending to $\mathfrak{B}$

Example: Toric code!



$\mathfrak{B}(I)$ generated as an algebra by operators

$$\sigma^X \otimes \sigma^X \otimes I \otimes I \otimes I \qquad I \otimes \sigma^Z \otimes \sigma^Z \otimes I \otimes I$$
$$I \otimes I \otimes \sigma^X \otimes \sigma^X \otimes I \qquad I \otimes I \otimes I \otimes \sigma^Z \otimes \sigma^Z$$

Get two families of self-adjoint commuting unitaries X_i, Z_j satisfying that

$$Z_j X_i = \begin{cases} -X_i Z_j & \text{if } j = i \pm 1, \\ X_i Z_j & \text{otherwise} \end{cases}$$

Bratteli diagram for net $\mathfrak{B}(I)$: $B = \varinjlim B(I) = M_{2^\infty}$

What do we get for toric code?

Canonical state on $\mathfrak{B}$ is the normalized trace!
The DHR bimodule theory gives $\mathcal{D}(\mathbb{Z}/2\mathbb{Z})$, exactly the category of excitations for toric code!

But what about other models?

Can also do Levin-Wen model corresponding to a unitary fusion category $\mathcal{C}$!

Net of boundary algebras is $\mathfrak{B}(I) = \mathrm{End}_{\mathcal{C}}(X^{\otimes n})$, where $X = \bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} c$ and $n = |I| - 2$

DHR bimodules gives $\mathcal{Z}(\mathcal{C})$, the category of excitations for Levin-Wen

The canonical state on $\mathfrak{B}$ is

$$\psi_\partial(f) = \frac{1}{D_{\mathcal{C}}^n} \sum_{c_1,\ldots,c_n \in \mathrm{Irr}(\mathcal{C})} d_{c_1} \cdots d_{c_n} \, \mathrm{tr}_{\mathcal{C}}(f p_{c_1 \cdots c_n})$$

ψ_∂ is a trace iff $\mathcal{C}$ is pointed!