Limit Multiplicities and Von Neumann Dimensions

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Contents

$$\lim_{n\to\infty} \frac{\text{multiplicity}}{\text{von Neumann dimension}} = 1$$

1. An Example:
   $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ and Discrete Series.
2. From Discrete Series
   to Bounded Subsets of the Unitary Dual
3. Limits for Cocompact Lattices
4. Limits for $SL(n,\mathbb{R})$ and Its Arithmetic Subgroups

An Example: $SL(2,\mathbb{Z}) \subset SL(2,\mathbb{R})$ and Discrete Series

- a lattice $\Gamma = SL(2,\mathbb{Z})$ in $G = SL(2,\mathbb{R})$;
- $L^2(SL(2,\mathbb{Z})\backslash SL(2,\mathbb{R})) \curvearrowleft G$:

$$(R(g)\phi)(x) = \phi(xg),\ \phi \in L^2(\Gamma\backslash G),\ g \in G.$$

Question 1: What is the decomposition of R?

- What are the unitary irreps of G?
  or the unitary dual $\widehat{G} = \{\text{unitary irreps of } G\}$?
  1. the discrete series $\{\pi_k^{\pm} | k \geq 2\}$;
  2. the principal series $\{\pi_{it}^{\pm} | t \in \mathbb{R}\}$;
  3. the complementary series $\{\sigma_s | s \in (0,1)\}$;
  4. the limits of discrete series δ_1^+, δ_1^-;
  5. the trivial rep $\mathbb{C}$.

Theorem (Selberg 1950s)

$$L^2(\Gamma\backslash G) = \underbrace{L^2_{disc}(\Gamma\backslash G)}_{\textit{discrete spectrum}} \oplus \underbrace{L^2_{cont}(\Gamma\backslash G)}_{\textit{continuous spectrum}}$$

- $L^2_{\text{cont}}(\Gamma\backslash G) \overset{G-\text{mod}}{=} \int^{\oplus}_{(0,+\infty)} \pi^+_{it}\,dt.$

Theorem

$L^2_{disc}(\Gamma\backslash G) = \oplus_\pi m_\Gamma(\pi)\cdot\pi$ *with each multiplicity* $m_\Gamma(\pi) < \infty$.

Question 2: $m_\Gamma(\pi) = ?$ Unknown in general.

Theorem (Gelfand et al. 1960s)

For π_k, $m_\Gamma(\pi_k) = \dim S_k(\Gamma)$= *dim of cusp forms of weight* k.

- $\Gamma(n) :=\{g \in \mathrm{SL}(2,\mathbb{Z}) | g \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod n\}$.
- Decomposition of $L^2_{\text{disc}}(\Gamma(n)\backslash G)$?
- $m_{\Gamma(n)}(\pi_k) = \dim S_k(\Gamma(n)) =$

$$(k - 1 \pm \tfrac{6}{n}) \cdot \tfrac{n^3}{24} \cdot \prod_{p|n}(1 - p^{-2}),$$

by Riemann-Roch theorem for modular curves.

An Example: $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$ and Discrete Series

- a discrete series $(\pi, H) :=$ an irrep $\leq L^2(G)$.
- the matrix coefficient $c^{\pi}_{u,v}(g) = \langle \pi(g)u, v\rangle$

Lemma

There is a constant $d(\pi) \in \mathbb{R}_{\geq 0}$, called formal dimension, s.t.

$$d(\pi) = \frac{\langle u,x\rangle_{H_k} \cdot \overline{\langle v,y\rangle_{H_k}}}{\langle c^{\pi}_{u,v}, c^{\pi}_{x,y}\rangle_{L^2(G)}}, \text{ for all } u, v, x, y \in H_k \backslash \{0\}.$$

- $L\Gamma :=$ the group von Neumann algebra of Γ.
- The discrete series (π_k, H_k) of $\mathrm{SL}(2,\mathbb{R})$ is a $L\Gamma$-module.

Theorem (Atiyah & Schmid, 1970s)

$\dim_{L\Gamma} H_k = \mathrm{vol}(\Gamma \backslash G) \cdot d(\pi_k)$.

- $\mathrm{vol}(\Gamma\backslash G), d(\pi_k)$ depends on the Haar measure, but $\dim_{L\Gamma}$ does NOT.
- $\dim_{L\Gamma(n)}(H_k) = (k-1) \cdot \frac{n^3}{24} \cdot \Pi_{p|n}(1 - p^{-2})$.

An Example: $\mathrm{SL}(2,\mathbb{Z}) \subset \mathrm{SL}(2,\mathbb{R})$ and Discrete Series

- $m_{\Gamma(n)}(\pi_k) = (k-1 \pm \frac{6}{n}) \cdot \frac{n^3}{24} \cdot \prod_{p|n}(1-p^{-2})$.
- $\dim_{L\Gamma(n)}(H_k) = (k-1) \cdot \frac{n^3}{24} \cdot \prod_{p|n}(1-p^{-2})$.

Theorem (Limit multiplicities of d.s. for arithmetic lattices in $\mathrm{SL}(2,\mathbb{R})$)

$$\lim_{n\to\infty} \frac{m_{\Gamma(n)}(\pi_k)}{\dim_{L\Gamma(n)}(H_k)} = \lim_{n\to\infty} \frac{k-1 \pm \frac{6}{n}}{k-1} = 1.$$

Question 3: Other irreps of $\mathrm{SL}(2,\mathbb{R})$?

- If (π, H) is NOT a discrete series,
- Almost all $m_\Gamma(\pi)$ are unknown.
- H is not a $L\Gamma$-module $\Rightarrow$ No $\dim_{L\Gamma} H$.

Question 4: How about other Lie groups?

- $m_\Gamma(\pi)$ are more complicated even π is a d.s.
- Most Lie groups have NO discrete series.
- G has a d.s. iff $\operatorname{rank} G = \operatorname{rank} K$ ($K=$ a max cpt subgrp).
- $\mathrm{SL}(n,\mathbb{R})$ has no d.s. if $n \geq 3$.

$G = \mathrm{SL}(2, \mathbb{R})$

$$\widehat{G} = \underbrace{\{\pi_k^{\pm} | k \geq 2\}}_{\text{discrete series}} \bigsqcup \underbrace{\{\pi_{it}^{\pm} | t \in \mathbb{R}\}}_{\text{principal series}} \bigsqcup \underbrace{\{\sigma_s | s \in (0,1)\} \bigsqcup \{\delta_1^{\pm}\} \bigsqcup \{\mathbb{C}\}}_{\text{the remaining irreps}}$$

$$\approx \underbrace{\sqcup_{1,2}\{k | k \geq 2\}}_{\text{discrete series}} \bigsqcup \underbrace{\sqcup_{1,2}\mathbb{R}}_{\text{principal series}} \bigsqcup \underbrace{(0,1) \bigsqcup \{\pm 1\} \bigsqcup \{1\}}_{\text{the remaining irreps}}$$

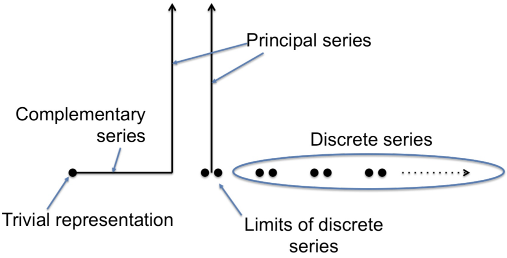

Figure: The unitary dual of $\mathrm{SL}(2, \mathbb{R})$ by P. Hochs

Bounded Subsets of the Unitary Dual

- G : a semisimple Lie group

Theorem (Harish-Chandra, Knapp, Vogan, .etc...)

$\widehat{G} \subset \bigsqcup_{finite} \mathbb{R}^{\text{rank } G}$ *(as a set).*

- $X \subset \widehat{G}$ is bounded if it is bounded in $\bigsqcup_{\text{finite}} \mathbb{R}^{\text{rank } G}$.
- $\Leftrightarrow$ relatively compact in the Fell topology (not Hausdorff).

Definition

For a bounded $X \subset \widehat{G}$, $m_\Gamma(X) := \sum_{\pi \in X} m_\Gamma(\pi)$.

Question 4: Is $m_\Gamma(X)$ finite?

Theorem (Borel & Garland 1980s)

For a bounded X, only finitely many $\pi \in X$ occur in $L^2_{disc}(\Gamma \backslash G)$

$\Longrightarrow m_\Gamma(X)$ is finite!!

Bounded Subsets of the Unitary Dual

- A measure on $\widehat{G}$:

Theorem (Harish-Chandra, for Lie groups)

There is a measure ν_G on $\widehat{G}$ (Plancherel measure) such that

$$L^2(G) \overset{G-G-\text{bimod}}{\cong} \int_{\widehat{G}}^{\oplus} H_\pi \otimes H_{\overline{\pi}} \, d\nu_G(\pi)$$

1. $X \subset \widehat{G}$ bounded $\Rightarrow \nu(X) < \infty$.
2. π is a d.s. iff it is an atom $\nu(\{\pi\}) > 0$. $\nu(\{\pi\}) = d(\pi)$.
3. $\mathrm{supp}(\nu_G) =$ tempered irreps $:= \{\pi | c^\pi_{u,v} \in L^{2+\varepsilon}(G), \forall \varepsilon > 0\}$.
4. $\widehat{G} = \widehat{G}_{\text{temp}} \bigsqcup \widehat{G}_{\text{untemp}}$.
5. $\widehat{\mathrm{SL}(2,\mathbb{R})}_{\text{temp}} = \{$ discrete seires, principal series $\}$.

Bounded Subsets of the Unitary Dual

- $X =$ a bounded subset of $\widehat{G}$
$$H_X := \int_X^{\oplus} H_\pi d\nu(\pi)$$
- $\Rightarrow H_X$ is a module over G, Γ and also $L\Gamma$.

Theorem (Y, 2022)

Given a lattice $\Gamma \subset G$,
$$\dim_{L\Gamma} H_X = \mathrm{vol}(\Gamma \backslash G) \cdot \nu(X).$$

- $X = X_{\mathrm{temp}} \bigsqcup X_{\mathrm{untemp}}$, only X_{temp} contributes to $\nu(X)$.
- works for any Lie group;
- reduces to the Atiyah-Schmid Thm if $X = \{\pi\} =$ a d.s.
- G may also be a p-adic group, an adelic group, etc. ...

Limits for Cocompact Lattices

- If $\Gamma\backslash G$ is compact $\Rightarrow$ only discrete spectrum
$$L^2(\Gamma\backslash G) = \bigoplus m_\Gamma(\pi)\cdot\pi \text{ with all } m_\Gamma(\pi) < \infty.$$
- $\Gamma_1 \supset \Gamma_2 \supset \cdots$ with $\cap_n \Gamma_n = \{1\}$, $\Gamma_n \lhd \Gamma_1$, $[\Gamma_1 : \Gamma_n] < \infty$.
- $m_\Gamma(X) := \sum_{\pi\in X} m_\Gamma(\pi)$ for a bounded $X \subset \widehat{G}$.

Corollary (Y, 23)

For a tower of cocompact lattices, $\lim_{n\to\infty} \dfrac{m_{\Gamma_n}(X)}{\dim_{L\Gamma_n}(H_X)} = 1$

- G always has such a tower (Borel & Harder 1977).
- G has a cocompact arithmetic lattice Γ iff $\operatorname{rank}_{\mathbb{Q}} G(\mathbb{Q}) = 0$.
- does NOT work with most arithmetic subgroups, i.e., $\mathrm{SL}(n,\mathbb{Z})$

Limits for $SL(n, \mathbb{R})$ and Its Arithmetic Subgroups

- $G = SL(n, \mathbb{R})$ and $\Gamma = SL(n, \mathbb{Z})$.
- principal congruence subgroups:

$$\Gamma_n = \ker\{SL(n, \mathbb{Z}) \to SL(n, \mathbb{Z}/n\mathbb{Z})\}.$$

Corollary (Y, 23)

$$\lim_{n\to\infty} \frac{m_{\Gamma_n}(X)}{\dim_{L\Gamma_n}(H_X)} = 1.$$

- $\Rightarrow$ only X_{temp} contributes to $\dim_{L\Gamma_n}(H_X)$.

Lemma (Finis-Lapid-Müller, 2015)

$$\lim_{n\to\infty} \frac{m_{\Gamma_n}(X_{temp})}{m_{\Gamma_n}(X)} = 1.$$

- may NOT hold for a general G.

- $\lim_{n\to\infty} \frac{m_{\Gamma_n}(X_{\text{temp}})}{m_{\Gamma_n}(X)} = 1.$

Question 5 Why $L^2(\Gamma\backslash G)$?

- classical $G(\mathbb{Z})\backslash G(\mathbb{R}) \sim$ adelic $G(\mathbb{Q})\backslash G(\mathbb{A})$.

Conjecture (Ramanujan Conjecture)

If π occurs in $L^2_{disc}(\Gamma\backslash G)$ ($\Leftrightarrow m_\Gamma(\pi) \geq 1$), then $\pi \in \widehat{G}_{temp}$.

- if true, $m_\Gamma(X_{\text{untemp}}) = 0$.
- a counterexample by Howe & Piatetski-Shapiro.

Conjecture (Langlands Correspondence of GL(n))

There is a one-to-one correspondence between

1. $\{$*irreps of G in $L^2_{cusp}(\Gamma\backslash G)$*$\}$;
2. $\{$*n-dimensional reps of* $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\}$

Questions?

Thank you!