Name ____________________ Date __________ Partners ________________

TA ________________ Section _______ ________________

Measurement, Uncertainty, and Uncertainty Propagation

Objective: To understand the importance of reporting both a measurement and its uncertainty and to address how to properly treat uncertainties in the lab.

Equipment: meter stick, 2-meter stick

DISCUSSION

Understanding nature requires measuring things, be it distance, time, acidity, or social status. However, measurements cannot be "exact". Rather, all measurements have some uncertainty associated with them.[1] Thus all measurements consist of *two* numbers: the *value* of the measured quantity and its *uncertainty*[2]. The uncertainty reflects the reliability of the measurement. The range of measurement uncertainties varies widely. Some quantities, such as the mass of the electron $m_e = (9.1093897 \pm 0.0000054) \times 10^{-31}$ kg, are known to better than one part per million. Other quantities are only loosely bounded: there are 100 to 400 billion stars in the Milky Way.

Note that **we not talking about "*human error*"!** We are **not talking about mistakes!** Rather, uncertainty is inherent in the instruments and methods that we use *even when perfectly applied*. The goddess Athena cannot not read a digital scale any better than you.

Significant Figures

The electron mass above has eight significant figures (or digits). However, the measured number of stars in the Milky Way has barely one significant figure, and it would be misleading to write it with more than one figure of precision. The number of significant figures reported should be consistent with the uncertainty of the measurement. In general, uncertainties are usually quoted with no more significant figures than the measured result; and the last significant figure of a result should match that of the uncertainty. For example, a measurement of the acceleration due to gravity on the surface of the Earth might be given as $g = 9.7 \pm 1.2$ m/s^2 or $g = 9.9 \pm 0.5$ m/s^2 but

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[1] Possible exceptions are counted quantities. "There are exactly 12 eggs in that carton."

[2] Sometimes this is also called the *error* of the measurement, but *uncertainty* is the modern preferred term.

not as g = 9.7 ± 1.25 m/s^2 or g = 9.92 ± 0.5 m/s^2. In the last two cases, the last significant figure of the result and uncertainty do not match.

When multiplying or dividing two numbers that have the same number of significant figures but differ in magnitude (e.g. 125 × 5.25), the final results should be quoted to the same number of significant figures (656). When adding and subtracting a two numbers, the lowest number of decimal places is used to specify the number of significant figures. (10.3 + 11.256 = 21.6)

To minimize errors in calculations due to round off during intermediate calculations, you should generally keep at least one additional significant figure than is warranted by the uncertainties in each number.

Work out the following examples in your lab write-up:

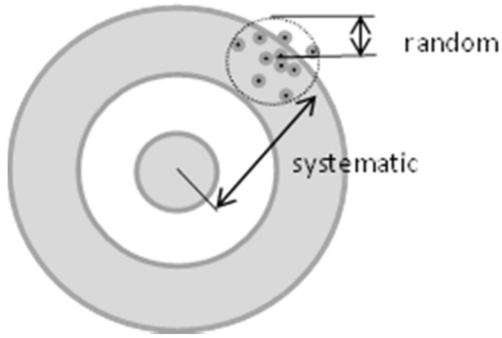
1. The length of the base of a large window is measured in two steps. The first section has a length of ℓ_1=1.22 m and the length of the second section is ℓ_2=0.7 m. What is the total length of the base of the window?

2. A student going to lunch walks a distance of x = 102 m in t = 88.645 s. What is the student's average speed?

Types of uncertainties

A *systematic uncertainty* occurs when all of the individual measurements of a quantity are biased by the same amount. These uncertainties can arise from the calibration of instruments or by experimental conditions such as slow reflexes on a stopwatch.

Random uncertainties occur when the result of repeated measurements vary due to truly random processes. For example, random uncertainties occur due to small fluctuations in experimental conditions or due to variations in the stability of measurement equipment. These uncertainties can be estimated from the distribution of values in repeated measurements.



Mistakes can be made in any experiment, either in making the measurements or in calculating the results. However, by definition, mistakes can also be avoided. Such blunders and major systematic errors can only be avoided by a thoughtful and careful approach to the experiment.

Estimating uncertainty

By eye or reason: Measurement uncertainty can often be reasonably estimated from properties of the measurement equipment. For example, using a meter stick (with marks every millimeter), a straight line can be easily measured to within half a millimeter. For an irregularly-edged object, the properties of its edges may limit the determination of its length several millimeters. Your reasoned judgment of the uncertainty is quite acceptable.

By repeated observation: If a quantity *x* is measured repeatedly, then the average or *mean* value of the set of measurements is generally adopted as the "result". If the uncertainties are random, the uncertainty in the mean can be derived from the variation in the set of observations. Shortly, we will discuss how this is done. (Oddly enough, truly random uncertainties are the easiest to deal with.)

Useful definitions

Here we define some useful terms (with examples) and discuss how uncertainties are reported in the lab.

Absolute uncertainty: This is the magnitude of the uncertainty assigned to a measured physical quantity. It has the same units as the measured quantity.

Example 1. Suppose we need 330 ml of methanol to use as a solvent for a chemical dye in an experiment. We measure the volume using a 500 ml graduated cylinder that has markings every 25 ml. A reasonable estimate for the uncertainty in our measurements is ½ of the smallest division. Thus we assign an absolute uncertainty to our measurement of $\Delta V = \pm 12$ ml. Hence, we state the volume of the solvent (before mixing) as $V = 330 \pm 12$ ml.

Relative uncertainty: This is the ratio of the absolute uncertainty and the value of the measured quantity. It has no units, that is, it is dimensionless. It is also called the *fractional uncertainty* or, when appropriate, the *percent uncertainty*.

Example 2. In the example above the fractional uncertainty is

$$\frac{\Delta V}{V} = \frac{12\,ml}{330\,ml} = 0.036 = 3.6\% \qquad (0.13)$$

Reducing random uncertainty by repeated observation

By taking a large number of individual measurements, we can use statistics to reduce the random uncertainty of a quantity. For instance, suppose we want to determine the mass of a standard U.S. penny. We measure the mass of a single penny many times using a balance and interpolate between divisions by eye. The results of 17 measurements on the same penny are summarized in Table 1.

Table 1. Data recorded measuring the mass of a US penny.

	mass (g)	deviation (g)
1	2.43	-0.088
2	2.49	-0.028
3	2.49	-0.028
4	2.58	0.062
5	2.52	0.002
6	2.55	0.032
7	2.52	0.002
8	2.64	0.122
9	2.55	0.032

	mass (g)	deviation (g)
10	2.46	-0.058
11	2.52	0.002
12	2.4	-0.118
13	2.58	0.062
14	2.61	0.092
15	2.49	-0.028
16	2.52	0.002
17	2.46	-0.058

The **mean** value $\overline{m}$ of the measurements is defined to be

$$\overline{m} = \frac{1}{N}\sum_{i=1}^{N} m_i = \frac{1}{17}\left(m_1 + m_2 + \ldots + m_{17}\right) = 2.518g \qquad (0.14)$$

The **deviation** d_i of the ith measurement m_i from the mean value $\overline{m}$ is defined to be

$$d_i = m_i - \overline{m} \tag{0.15}$$

Fig. 1 shows a histogram plot of the data on the mass of a US penny. Also on the graph is a plot of a smooth, bell-shaped curve that represents what the distribution of measured values would look like if we took many, many measurements. The result of a large set of repeated measurements subject only to random uncertainties will always approach a limiting distribution called the *normal* or *Gaussian* distribution. The larger the number of measurements, the closer the data will approach the normal distribution. This ideal curve has the mathematical form:

$$Number(m) = \left(\frac{N}{\Delta m\sqrt{2\pi}}\right) e^{-\frac{1}{2}\left(\frac{m-\overline{m}}{\Delta m}\right)^2} \tag{0.16}$$

where N is the total number of measurements. The normal distribution is symmetrical about $\overline{m}$.

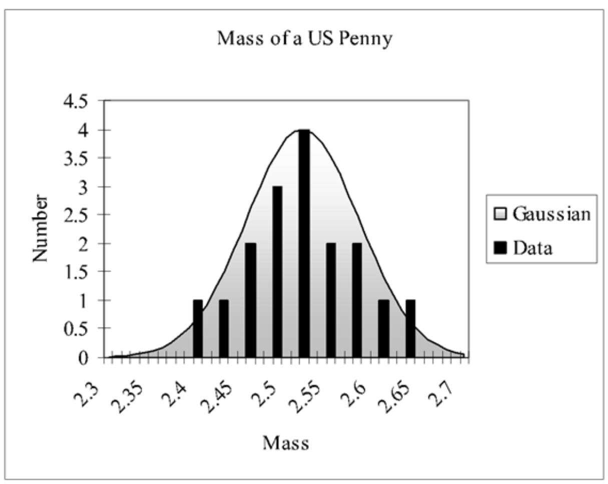


Figure 1. The Gaussian or normal distribution for the mass of a penny N=17, $\overline{m}$ *=2.518 g,* Δm*=0.063 g.*

We now define the **standard deviation** Δm as

$$\Delta m = \sqrt{\sum_{i=1}^{N} \frac{\left(m_i - \overline{m}\right)^2}{\left(N-1\right)}} = \sqrt{\sum_{i=1}^{N} \frac{\left(m_i - \overline{m}\right)^2}{16}} = 0.063g \tag{0.17}$$

For standard distributions, 68% of the time the result of an individual measurement would be within ± Δm of the mean value $\overline{m}$. Thus, Δm is the experimental uncertainty for an individual measurement of m.

But the mean $\overline{m}$ should be better than any individual measurement. How much better? It can be shown that this uncertainty or the **standard deviation of the mean** is

$$\Delta\overline{m} = \frac{\Delta m}{\sqrt{N}} \tag{0.18}$$

With a set of N=17 measurements, our result is

$$\begin{aligned} \text{mass of a penny} &= \overline{m} \pm \Delta\overline{m} = \overline{m} \pm \frac{\Delta m}{\sqrt{N}} \\ &= 2.518g \pm \frac{0.063g}{\sqrt{17}} \\ &= \left(2.518 \pm 0.015\right) g \end{aligned} \tag{0.19}$$

Thus, if our experiment is subject to random uncertainties in an individual measurement of Δm, we can improve the precision of that measurement by doing it repeatedly and taking the mean of those results. Note, however, that the precision improves only as $1/\sqrt{N}$ so that to improve by a factor of say 10, we have to make 100 times as many measurements. We also have to be careful in trying to get better results by letting $N \rightarrow \infty$, because the overall accuracy of our measurements may be limited by systematic errors, which *do not cancel out* the way random errors do.

Combination and propagation of random uncertainties

To obtain a final result, we have to measure a variety of quantities (say, length and time) and mathematically combine them to obtain a final result (speed). How the uncertainties in individual quantities combine to produce the uncertainty in the final result is called the *propagation of uncertainty*.

For all these formulae, it is important that the quantities being combined are the results of truly independent measurements and that the uncertainty Δx assigned to quantity x not be related to the uncertainty Δt assigned to quantity t. For example, we may measure the speed of an object by measuring a distance (using a meter stick) and the time it takes to traverse that distance (using a clock). The measurement of time and distance can be truly independent as they are done with two measurement devices and there is no reason to think that if the time measurement is too large, then the distance measurement is also too large.

Here we summarize a number of common cases. For the most part these should take care of what you need to know about how to combine uncertainties.

Uncertainties in sums and differences:

If several quantities $x_1, x_2, x_{3.}$ are measured with independent, random uncertainties $\Delta x_1, \Delta x_2, \Delta x_{3.}$ then the uncertainty in Q where $Q = x_1 \pm x_2 \pm x_3$ is

$$\Delta Q = \sqrt{\Delta x_1{}^2 + \Delta x_2{}^2 + \Delta x_3{}^2} \qquad (0.20)$$

In other words, the random uncertainties add as the square root of the sum of the squares, whether the terms are all added, subtracted, or some combination of the two.

Uncertainties in products and quotients:

Several quantities *x*, *y*, *z* (with independent, random uncertainties Δ*x,* Δ*y,* Δ*z*,) combine to form Q, where

$$Q = \frac{x\,y}{z} \qquad (0.21)$$

(or any other combination of multiplication and division). Then the uncertainty in Q will be

$$\frac{\Delta \bar{Q}}{\left|\bar{Q}\right|} = \sqrt{\left(\frac{\Delta x}{\bar{x}}\right)^2 + \left(\frac{\Delta y}{\bar{y}}\right)^2 + \left(\frac{\Delta z}{\bar{z}}\right)^2} \qquad (0.22)$$

In other words, the fractional uncertainties combine as the square root of the sum of the squares of the individual fractional uncertainties in the component terms.

EXERCISES:

Neatly tabulate and record your data on a separate sheet. Present your calculations so that it is clear what equations and data you used to find what numbers.

1. In the book of Genesis (Chapter 6) it is recorded that God told Noah to build an ark (that is to say, a box).

"The length of the ark shall be 300 cubits, its breadth 50 cubits, and its height 30 cubits".

A cubit is the length of the forearm from the elbow to the tip of the middle finger.

a . First, determine the mean length of a cubit in meters by measuring the appropriate length on each student in the lab.
b . Calculate the standard deviation of these measurements.
c . What is your “official” value of a cubit and its associated uncertainty?
d . What is your best estimate of the volume of Noah's Ark? (Give both your estimate of the volume of the Ark and the uncertainty in the volume.)
e . What systematic uncertainties might contribute to your estimate of the ark's volume?

2 . Noah must walk around the ark to inspect it.
a . Develop and implement a procedure to measure your walking speed and the associated uncertainty. Briefly describe your procedure, tabulate your data and present your results
b . Calculate the time required to walk around the ark and the associated uncertainty.

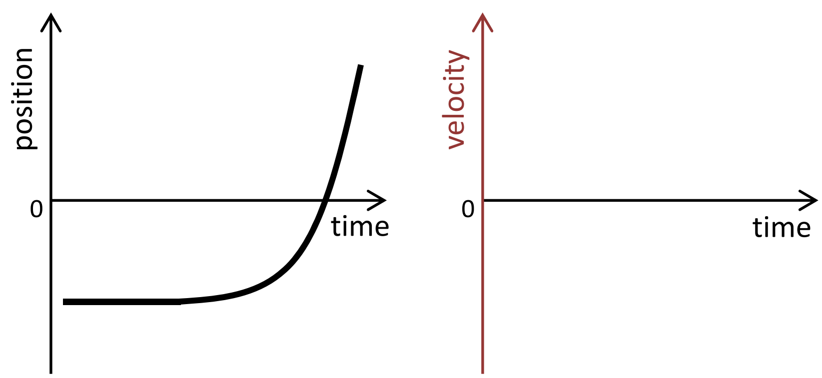
Name________________________ Section _______ Date_____________

PRE-LAB PREPARATION SHEET FOR
Position, Velocity, and Acceleration in one-dimensional motion

(DUE AT THE BEGINNING OF LAB)

Read over the lab and then answer the following questions

1. Given the following position curve, sketch the corresponding velocity curve.

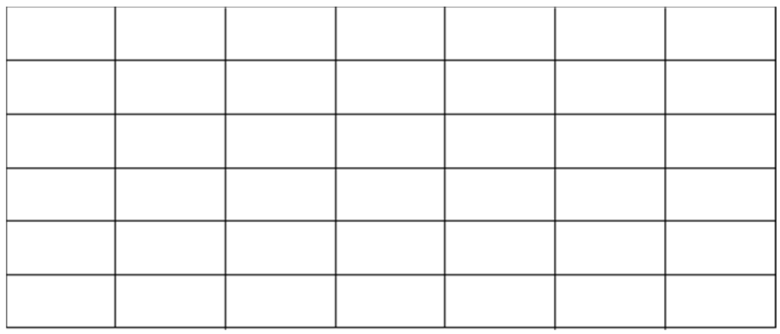


2. Imagine kicking a box across the floor: it suddenly starts moving, slides for a short distance, and comes to a stop. Make a sketch of the position and velocity curves for such motion.

3. Go to the following website and watch the Java applet.

Physics.bu.edu/~duffy/semester1/c01_motion.html

Sketch the position vs. time curve for the lavender ball during constant acceleration.



Name ______________________ Date ___________ Partners _________________

TA _________________ Section _______ _________________

Position, Velocity, and Acceleration in one-dimensional motion

Objectives:

- To understand graphical descriptions of the motion of an object.
- To understand the mathematical and graphical relationships among position, velocity and acceleration

Equipment:

- 2.2-meter track w/ adjustable feet and end stop
- A block to raise one end of the cart
- Motion sensor
- Torpedo level
- PASCO dynamics cart

DISCUSSION

Velocity is the rate of change or time derivative of position.

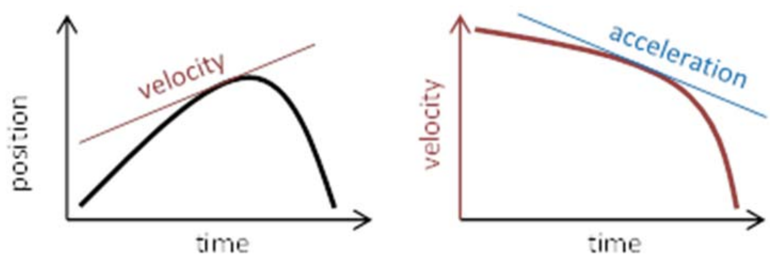
$$\vec{v} = \frac{d\vec{x}}{dt} \tag{2.1}$$

On a Cartesian plot of position vs. time, the slope of the curve at any point will be the instantaneous velocity.

Likewise, acceleration is the rate of change or time derivative of velocity (the 2nd derivative of position).

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} \tag{2.2}$$

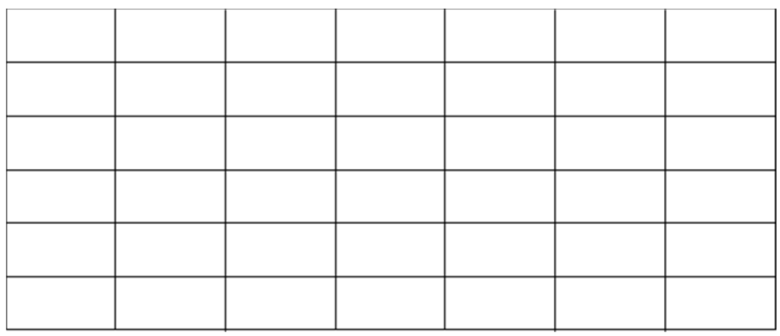
On a Cartesian plot of velocity vs. time, the slope of the curve at any point will be the instantaneous acceleration.



Thus, the shape of any one curve (position, velocity, or acceleration) can determine the shape of the other two.

Exercise 1: Back and Forth

a. Place the friction cart on the track. (That is the one with the friction pad on the bottom.
   Without letting go of the cart, quickly push it toward the detector by about a foot, then stop it for 1 or 2 seconds. Then quickly but smoothly return the cart to the starting point.
   Note the distance it travels, and sketch the position vs. time curve for the block on the plot below.



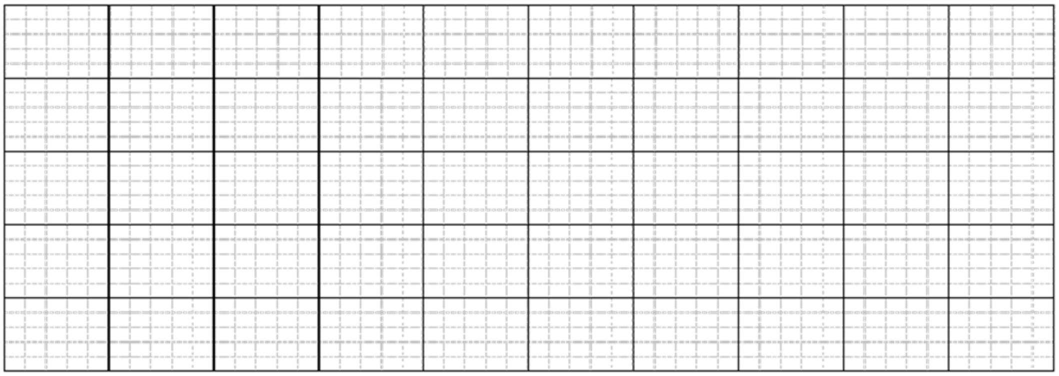
b. Now, open the Labfile directory found on your computer's desktop. Navigate to A Labs/Lab2, and select the program Position. The PASCO DataStudio program should open and present you with a blank position vs. time graph.
c. Click the **Start** button (upper left side of the screen), and repeat the experiment above. Click **Stop** to cease recording data. Note how the PASCO plot compares to yours.

   Note: The cart may bounce or stutter in its motion. If you don't get a smooth curve, delete the data[1] *and repeat the run with more Zen*[2]*.*

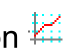
d. By clicking the scaling icon  (top left corner of the Graph window) you can better fill the screen with the newly acquired data.
e. Select the slope icon . A solid black line will appear on the screen. By dragging this line to points along the plot, you can measure the slope of the curve at those points. Using this tool, find the steepest part of the curve (that is, the largest

[1] To delete data: Top bar, Experiment, Delete ALL Data Runs
[2] "This time, let go your conscious self and act on instinct." Obi-Wan Kenobi

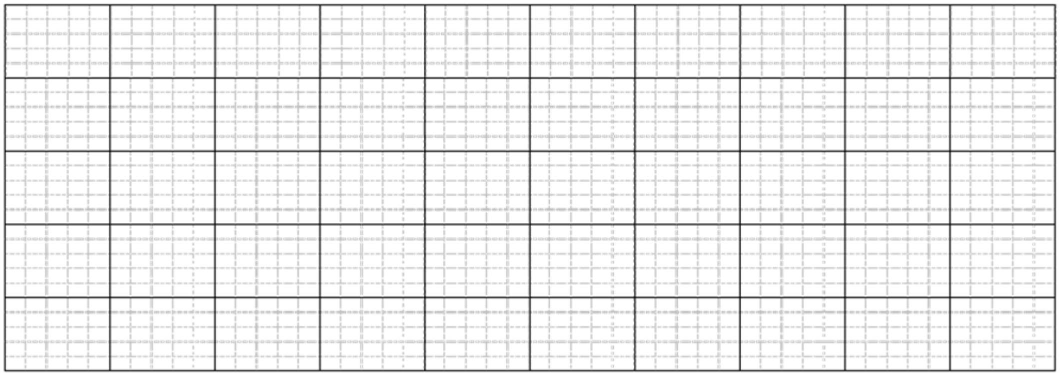
velocity). Then, sketch the velocity curve for the block in the graph below. Add appropriate numbers to the x and y axes.



f. How does the shape of the <u>position curve</u> determine the sign of the <u>velocity curve</u>?

g. Now, let's see how well you drew it! Double-click on the new graph icon  (left side of the screen, lower half) and select **Velocity** for the y-axis. Note the shape and position of the curve and see how well it matches your sketch. Also note how it aligns with the position curve.
h. Use the slope tool to find the changing slope along the velocity curve. With this information, sketch the acceleration curve for the block. Again, appropriately mark the axes.

Modified from: *RealTime Physics*, P. Laws, D. Sokoloff, R. Thornton
and University of VA Physics Labs: S. Thornton



i. Let's see what PASCO says about the acceleration. Again, create a new graph and select **Acceleration** for the y-axis. Compare it to your acceleration curve and PASCO's velocity curve.

j. How does the shape of the <u>position curve</u> determine the sign of the <u>acceleration curve</u>?

k. Print out the three PASCO plots. On these plots, annotate the times when the push began, when the push ended, when it was slowing, and when it stopped. Notice how these times correspond to features on the three curves.

Exercise 2: Skidding to a Stop

Delete your previous runs. (Top bar, Experiment, Delete ALL Data Runs). With a left click of the mouse, you can remove the slope tools.

a. Move the cart to end of the track opposite the detector.
b. Start recording data, then give the cart a quick, firm push so that it slides a few feet before coming to rest. Stop the data acquisition.
c. By clicking the scaling icon , you can better fill the screen with the newly acquired data. Again, if the data is not reasonably smooth, delete the data and repeat the experiment with more Zen.
d. Print out the curves and annotate on the graphs with the times when the push began, when the push ended, and when the cart was sliding on its own.

You should notice that as the cart is slowing down, the acceleration curve is nearly a constant flat line.

e. Given constant acceleration, what mathematical expression describes the velocity?

f. What mathematical expression describes the position?

You can verify that these expressions work by numerically fitting the data.

g. With a click and drag of the mouse, highlight the region of the velocity curve where the cart is slowing down. Then, select the fitting tool  and choose the appropriate expression to describe the data. Record the results of the fit below. (Note the uncertainty provided by the fit.)

h. Similarly, apply a numerical fit to the position data. Record the results below. Are the results consistent with the velocity and acceleration curves?

i. Similarly, find the average acceleration of this region.

j. Are the results of the fit consistent with each other?

Exercise 3: Up and Down

a. Place a block under one of the track stands to form a ramp. The detector must be on the raised end.
b. Place a low friction cart on the track and give it a push so that it rolls a few feet up the incline and then rolls back. After a few practice runs, run the detector and acquire motion data.
c. With a click and drag of the mouse, highlight that section of the data where the cart is freely rolling along the track. Then use the scaling tool to zoom-in on that section of the data.
d. Print out these plots and annotate the graphs with the following information.

   When and where does the velocity of the cart go to zero?
   When and where does the acceleration of the cart go to zero?

e. Find the average acceleration going up the slope and down the slow. Record the the results below.

f. How does the acceleration up the slope compare with the acceleration down the slope? What might account for the difference?

Student checkbox		TA checkbox
☐	**When you have completed the experiment, return all equipment to your lab table as it was when you began lab and initial the box to your left. Then have your TA verify that everything is accounted for and initial the box on the right.**	☐