RESEARCH STATEMENT

1. Introduction

I work in the field of algebraic topology, which uses algebraic tools to study geometric objects. One of these algebraic tools is called *cohomology.* The first cohomology theory a topologist typically learns about is called *singular cohomology.* Topologists have extensively studied this tool over the years, and it is well known how to perform computations in this theory. Singular cohomology takes in the data of a geometric object known as a topological space and assigns algebraic data to it. For the more complex data of a topological space with a group action, there is an analogue of singular cohomology called *Bredon cohomology.* This theory is much less well understood, and computations in this theory are harder. Many are actively working to expand what we know about this theory and extend the list of computations [6, 7, 8, 10, 11, 12, 13, 14, 16, 19].

The work in my thesis contributes to this list of Bredon cohomology calculations for different spaces. In particular, I chose to focus on surfaces with certain group actions. A *surface* is a mathematical object which looks locally like $\mathbb{R}^2$. Some familiar examples of surfaces are the sphere or the torus. We can additionally consider symmetries of such objects. For example, we may rotate a torus 120° clockwise through the axis at the center of its donut hole. This rotation can be considered as an action of the cyclic group of order 3, C_3. In general, we call a surface with an action of a group G an *equivariant surface.* In the case $G = C_3$, we call such a surface a *C_3-surface.* More precisely, we can think of a C_3-surface as a surface X with a map $\sigma\colon X \to X$ with the property that σ^3 is the identity. Another example of such a C_3-surface is the space denoted $S^{2,1}$, which is the sphere S^2 with $\sigma\colon S^2 \to S^2$ defined to be rotation of 120° about the axis passing through its north and south poles. We can similarly consider the notion of a C_p-surface where p is any odd prime. My thesis project accomplishes two goals: to completely classify all C_p-surfaces when p is an odd prime and to compute the Bredon cohomology of these surfaces in the case $p = 3$.

2. Current Work

2.1. **Classifying C_3-surfaces.** In 2019, Dugger used a method called *equivariant surgery* to classify all C_2-surfaces [9]. Equivariant surgery is a method of building new, more complicated equivariant surfaces out of old ones. For example, one could start with the sphere whose action is given by the map $\sigma\colon S^2 \to S^2$ which rotates the sphere by 120°. Recall that this space is denoted $S^{2,1}$. Remove a small disk D from the sphere along with the disks $\sigma(D)$ and $\sigma^2(D)$. Next take a torus, M_1, and remove a small disk from it. We can then glue three copies of this torus to the sphere along the boundaries of the removed disks. This results in a new C_3-space which we will denote $S^{2,1}\#_3M_1$. This process is depicted in Figure 1. In general, given any C_3-surface X and any other surface Y, we can follow this procedure to construct a new C_3-space, denoted $X\#_3Y$, which we call their *equivariant connect sum.*

Let R_3 be the C_3-space depicted in the middle of Figure 2 whose action σ is given by rotation of 120°. Given another C_3-space X, we can define *C_3-ribbon surgery* on X by removing a small disk D from X as well as the disks $\sigma(D)$ and $\sigma^2(D)$. Then attach R_3 to this new space by gluing

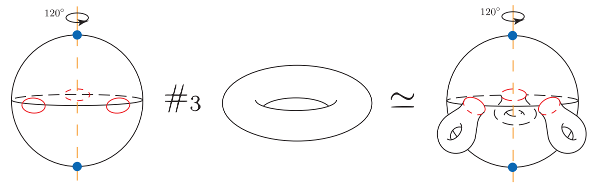

FIGURE 1. The equivariant connect sum of $S^{2,1}$ with a torus, denoted $S^{2,1}\#_3 M_1$.

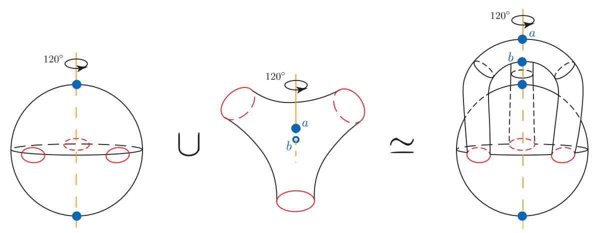

FIGURE 2. The result of C_3-ribbon surgery on $S^{2,1}$ is the space on the right, denoted $S^{2,1} + [R_3]$.

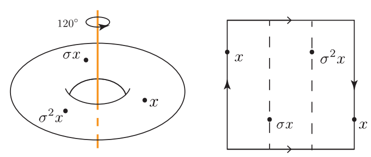

FIGURE 3. On the left is a C_3-action on the torus denoted M_1^{free}. On the right is an action on the Klein bottle denoted N_2^{free}.

together their boundary components. We call the new space $X + [R_3]$. An example of this process is depicted in Figure 2.

Result 2.2. *I prove that any C_3-surface can be built from one of a family of basic "building block" C_3-surfaces using equivariant connect sums and $+[R_3]$-surgery. Given a C_3-surface, its construction is completely determined by a handful of properties of that surface.*

Examples of these "building block" surfaces can be seen in Figures 3, 4, and 5, with fixed points shown in blue. To obtain this result, I adapt Dugger's classification of C_2-surfaces to the case of surfaces with an order 3 symmetry. Many surprising differences arise when considering C_p-surfaces in the case $p = 3$. Fortunately, the adaptations required to prove the $p = 3$ case can be applied to the case of C_p-surfaces, where p is any odd prime. In my thesis project, I proved a more general result which classifies surfaces with an action of C_p up to an action of $\text{Aut}(C_p)$. The constructions used in the generalized classification are all analogous to the ones presented here.

For a prime p, the problem of classifying all actions of C_p on surfaces is classical. Various papers have treated aspects of this classification result, mostly focusing on the orientable case [2, 4, 5, 15, 17, 18]. The new idea presented in my classification and that of [9] is construction of surfaces via equivariant surgeries. This method allows for greater accessibility of the results and has directly aided in cohomology computations.

2.3. **Cohomology computations for C_3-surfaces.** After Dugger's classification of C_2-surfaces, Hazel computed their Bredon cohomology in constant $\mathbb{Z}/2$-coefficients [11]. In my thesis project, I

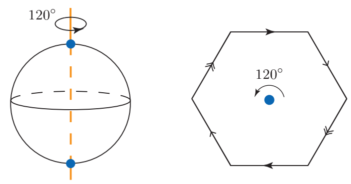

FIGURE 4. On the left is the space $S^{2,1}$. On the right is a C_3-action on $\mathbb{R}P^2$ denoted $N_1[1]$. In this notation, the value in brackets refers to the number of fixed points of the action.

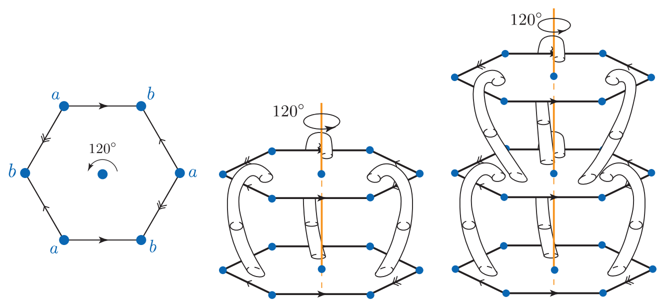

FIGURE 5. From left to right, we have the spaces Poly_1, Poly_2, and Poly_3. More generally, when the prime p is understood, Poly_n denotes a C_p-action on the genus $(3n-2)\frac{p-1}{2}$ orientable surface with $3n$ fixed points.

find an analogy of Dugger's classification of C_2-surfaces in the C_3-case and then compute the cohomology of these surfaces. The geometric presentation of C_3-surfaces given by the new classification provided helpful insight into how to proceed with the cohomology computations.

Result 2.4. *I compute the Bredon cohomology of all C_3-surfaces in constant $\mathbb{Z}/3$-coefficients. The cohomology of a C_3-surface in this theory depends precisely on its construction via the classification in the previous section.*

I am presently working on computing the Bredon cohomology of C_p-surfaces when $p > 3$. This turns out to be significantly more challenging than even the $p = 3$ case. For a topological space with an action of the group G, Bredon cohomology is graded on $RO(G)$, the Grothendieck group of finite-dimensional, real, orthogonal G-representations. Since I am working in the case $G = C_3$, Bredon cohomology is a bigraded theory. However when we consider spaces with an action of $G = C_5$, we are already working in a tri-graded theory. Fortunately, it turns out that $RO(C_p)$-graded Bredon cohomology can still sometimes be reduced to a bigraded theory [1] , allowing us to pursue similar methods as the $p = 3$ case when considering other odd primes.

3. Future Research

Bredon cohomology in constant $\mathbb{Z}/3$-coefficients does not do a perfect job of distinguishing between C_3-spaces. For example the C_3-surface $N_1[1]$ has an interesting group action, but its Bredon cohomology in constant $\mathbb{Z}/3$-coefficients is no different than that of a fixed point. A similar phenomenon occurs at the level of singular cohomology in $\mathbb{Z}/3$-coefficients when you forget about the group action. It is not until you look at other coefficient groups that you can see the difference in cohomology between a point and N_1. I plan to continue these Bredon cohomology calculations in different coefficient systems with the hope of uncovering additional structure in the cohomology of spaces such as N_1[1].

These results on C_p-surfaces can help pave the way for Bredon cohomology computations of some more complicated topological spaces. Given a manifold M, there is an important class of topological spaces known as *configurations of k points on M*, denoted $C^k(M)$, which is defined to be the space of all k-element subsets of M. The singular cohomology of $C^k(M)$ depends on that of M [3]. Given an equivariant surface X with an action of the group G, we could consider the equivariant space $C^k(X)$ which inherits a G-action from the action on X. Now that we have computed the Bredon cohomology of C_p-surfaces, we can use this knowledge to compute the cohomology of the corresponding equivariant configuration spaces.

After computing the Bredon cohomology of a G-space X, we are left with algebraic data which somehow describes X. A natural next step for algebraic topologists is to ask precisely what geometric properties of X its Bredon cohomology is detecting. After computing the cohomology of C_2-surfaces in 2019, Hazel developed a theory to geometrically describe her Bredon cohomology results [10]. Now that these cohomology computations have been completed for C_3-surfaces, I want to explore this theory in the case of $G = C_3$.

I want to teach undergraduates about topology, and surfaces are an excellent introduction to the subject since they are easy to visualize. It would be a fun and engaging project for advanced undergraduates to explore basic examples of equivariant surgery on spaces with an action of more complicated groups. Even exploring the case of non-simple cyclic groups would be interesting. In fact, there are still some surfaces for which all actions of C_4 and $C_2 \times C_2$ have not been classified. Below is an example of a research question I would like to explore with an undergraduate in the case $G = C_4$ or $G = C_2 \times C_2$.

Problem 3.1. *Let X be a surface with an action of a group G and two points $x, y \in X$ which are fixed by the action. When does there exist a path α in X from x to y which does not intersect any of its conjugate paths?*

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