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Costly voting with multiple candidates under plurality rule ☆

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ABSTRACT

We analyze a costly voting model with multiple candidates under plurality rule. In equilibrium, the set of candidates is partitioned into a set of "relevant candidates" (which contains at least two candidates) and the remaining candidates. All relevant candidates receive votes and have an equal chance of winning, independent of their popular support levels. The remaining candidates do not receive any votes. Furthermore, all voters who cast votes do so for their most preferred candidate, i.e., there is no "strategic voting."

1. Introduction

The literature analyzing the decision whether or not to vote when voting is costly generally focuses on settings in which there is an election between exactly two candidates (Palfrey and Rosenthal, 1983, 1985; Ledyard, 1984 and the literature discussed in Section 2 below). The advantage of assuming that only two candidates compete in the election is that there are only two weakly undominated strategies for each citizen: Abstain, or vote for one's favorite candidate.

However, many real-life elections involve competition between more than two parties or candidates. In this paper, we analyze a costly voting model with multiple candidates, in which all citizens have the same structure of preferences (i.e., value their favorite candidate winning at 1, their second-favorite candidate at λ_2 etc., and their least favorite candidate at 0), but differ in how they rank the candidates. Specifically, citizens' preferences over candidates are drawn from a distribution that is common knowledge, but each citizen only knows his own realized type. Citizens decide strategically whether to vote at all (in which case they have to pay a voting cost c), and if they choose to vote, also which candidate to vote for.

This is a potentially very complicated setting because, in addition to the participation decision, we allow for citizens to vote strategically for other candidates than their most preferred one, and it is well known that only voting for one's least-preferred candidate is a dominated strategy in a multi-candidate election. In spite of this, the model remains surprisingly tractable. Our analysis focuses on the case of three candidates, but it will be clear that our qualitative results generalize to the general case of m candidates.

We characterize a quasi-symmetric equilibrium (i.e., one in which all voters with the same type play the same mixed strategy) in which the set of candidates is partitioned in a set of "relevant candidates" (who receive a positive expected number of votes, and all have a positive winning probability), and the remaining "irrelevant" candidates, who do not. The set of relevant candidates can be any subset of size greater or equal to 2 of the set of candidates.

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For the case of three candidates, we can completely characterize the set of quasi-symmetric equilibria, and it is also easiest to describe the set of equilibria intuitively for the case of three candidates. First, there is an equilibrium in which all three candidates receive the same positive expected number of votes, and each wins with probability 1/3. Second, there are equilibria in which the set of relevant candidates is equal to a subset of two candidates, who receive a positive expected number of votes, and each of these relevant candidates wins with probability 1/2; there are three different equilibria of this second type equilibrium, one for each subset of two candidates (i.e., one in which candidates A and B are the relevant candidates; another one in which A and C are the relevant candidates, and a last one in which B and C are the relevant candidates).

Interestingly, in all equilibria, all voters vote sincerely, that is, for their most preferred candidate. This result contrasts with the literature on strategic voting in settings where the set of participating voters is exogenous (see Myerson and Weber, 1993; Messner and Polborn, 2011) and where there are generally many equilibria in which strategic voting occurs.

Intuitively, the reason for our result that "strategic voting" (i.e., voting for a candidate who is not the voter's most preferred one) does not occur in equilibrium when voting is costly is as follows: In a sufficiently large society, those citizen types who vote with positive probability must randomize whether to participate (because, otherwise, their probability of deciding the election would go to zero, and then it would not be worthwhile to incur the participation cost), and therefore must be indifferent between participating and not. A citizen who ranks, say, candidate A highest and votes for A has a higher expected marginal benefit from potentially swinging the election to A than a citizen who ranks A lower and only votes for him "strategically." Because, in equilibrium, the sincere voter is indifferent between voting and not voting, the second type of voter must strictly prefer participation over abstention.

The paper proceeds as follows: Section 2 places our paper in the literature. We present the model in Section 3, and our results in Section 4 and in Section 5. Section 6 concludes.

2. Related literature

Our paper contributes to the literature on costly voting (and, more generally, endogenous participation models) pioneered by Ledyard (1984) and Palfrey and Rosenthal (1983, 1985), and developed further by a large number of papers.

In particular, the costly voting framework has been used and modified to understand stylized facts about participation in elections (Feddersen and Sandroni, 2006; Herrera and Martinelli, 2006; Levine and Palfrey, 2007), as well as the more normative question whether a social planner should encourage citizens to participate in elections (Börgers, 2004; Krasa and Polborn, 2009; Taylor and Yildirim, 2010a,b; Krishna and Morgan, 2012, among others). These models, and – to the best of our knowledge – all other costly voting models assume that citizens have to choose between only two candidates if they vote.

Clearly, focusing on the two candidate case simplifies the analysis because voters effectively only have to decide whether to participate, while their vote decision if they participate is trivial. Focusing on the case of two candidates thus enables these authors to focus on other interesting questions in the framework. We complement this literature by focusing on the basic costly voting model, and analyzing the case of more than two candidates.

Our paper also contributes to another literature, namely the one analyzing strategic voting in multi-candidate elections. It is well known that, in voting games with multiple candidates, the only weakly dominated strategy for voters is to vote for one's least preferred candidate. Even iterated elimination of weakly dominated strategies usually does not narrow down the set of possible equilibrium outcomes (Dhillon and Lockwood, 2004). Myerson and Weber (1993) and Messner and Polborn (2011) consider different trembling refinements, and Messner and Polborn (2007) consider refinements based on coordination between different voter groups. These models primarily aim to increase our understanding of "Duverger's Law" which states that, under plurality rule, most voters vote for one of two "main" candidates, with all other candidates receiving very few votes because those voters who like them expect that a vote for their favorite candidate would be wasted because he has no chance of winning. Therefore, these voters are better off voting for the main candidate whom they like better than his main competitor.

A similar effect is present in those equilibria of our model in which only two candidates receive a positive expected vote share: However, in contrast to the models above, the supporters of the third candidate in such an equilibrium abstain completely, rather than vote for one of the two main candidates. Our model also admits an equilibrium in which all three candidates receive a positive vote share and even have the same probability of winning. In contrast, in Messner and Polborn (2007, 2011), this cannot happen in equilibrium. Myerson and Weber (1993) also obtain an equilibrium in which all three candidates receive votes and tie; however, this equilibrium is based on a reduced form modeling of the pivot event.[1]

3. The model

We consider a game with K candidates. The voting system is plurality rule, i.e., there is a single round of voting, and the candidate who receives the most votes is elected.

[1] The equilibrium concept of Myerson and Weber (1993) is based on a vector of pivot beliefs held by the voters (essentially summarizing their beliefs that any pair of candidates will be tied). Myerson and Weber allow for the pivot belief to take any value between 0 and 1 if two candidates are tied (and not just 1/2), and this feature is essential to support their 3 candidate equilibria.

The players of our game are N *citizens*. To avoid confusion, we reserve the term "voter" to a citizen who chooses to actually vote. Each citizen decides simultaneously whether to abstain or to vote, and if he votes, which candidate to vote for. Those citizens who vote incur a cost $c > 0$.

Voters receive different utilities from different candidates. Specifically, the utility for a citizen of type t from having his ℓth-ranked candidate win is λ_ℓ. We normalize the utility from the top ranked candidate to $\lambda_1 = 1$, and the utility from the least-preferred candidate to $\lambda_K = 0$.

Following Myerson (2000), we assume that the number of citizens of type t is drawn from a Poisson distribution with parameter $N_t = \rho_t N$, for all types $t \in T$. Here, ρ_t is the (expected) proportion of type-t citizens (so $\sum_{t\in T} \rho_t = 1$), and N_t is the expected number of citizens of type t. The analytical advantage of the Poisson game assumption is that, from the point of view of a particular citizen of type t, the number of other citizens of type t is still Poisson distributed with parameter N_t.

Our equilibrium concept is a quasi-symmetric mixed strategy equilibrium, that is, a strategy profile in which all players of preference type t participate with the same probability p_t and, if so, vote for the same candidate (not necessarily their most preferred candidate, although we will show that, in any equilibrium, they will do that). Note that, while we look for an equilibrium in which voters of the same type play the same strategy, the deviations that we consider are, of course, unilateral deviations by one voter, both with respect to participation, and with respect to which candidate to vote for in case of participation.

4. Analysis for three candidates

We start our analysis by analyzing the case of $K = 3$ candidates, whom we denote by A, B and C. In this case, we have 6 citizen preference types, $t \in T = \{AB, AC, BA, BC, CA, CB\}$, where type ij ranks candidate i highest (payoff 1), candidate j next (payoff $\lambda_2 \equiv \lambda$), and the remaining candidate last (payoff 0).

4.1. Equilibria with two relevant candidates

We start by analyzing the possibility of an equilibrium in which all citizens who vote do so for one of two candidates, while no player will ever vote for the third candidate. Without loss of generality, we focus on an "AB-equilibrium" in which the two "relevant" candidates are A and B.

It is intuitively clear that the expectation that candidate C is ignored is self-sustaining: If C is ignored by the other voters, and if the expected number of votes for the two other candidates is not too small (which translates into a very mild condition on the cost of voting in Proposition 1 below), then it follows that C's winning probability with only one vote would be very small.

Assuming that $c \le 1/4$, the following Proposition 1 shows that, if the expected number of citizens is sufficiently large, equilibria with two relevant candidates exist, and then characterizes them. In particular, we show that the only citizen preference types who vote with positive probability are those types who are maximally motivated because they rank the one relevant candidate highest and the other relevant candidate lowest.

Intuitively, in a sufficiently large economy, no voter type can vote with probability 1 because then the probability of being pivotal for the election outcome would go to zero, making it not worthwhile for a citizen to spend the cost of voting. Thus, those voter types who participate with positive probability must be indifferent between voting and not voting.

In an AB-equilibrium, citizens of type AC receive a benefit of 1 if A wins and of 0 if B wins, and are therefore maximally motivated to change the outcome of the election. If they are indifferent, in equilibrium, between voting and not voting, it must be true that citizens of type AB (who also get a payoff of 1 if A wins, but of λ if B wins) have a smaller benefit from swinging the election, and thus strictly prefer to abstain.

Proposition 1. *Suppose that $c \le 1/4$, and fix voter type proportions $\rho_t > 0$ for all $t \in T$. There exists a function $\bar{N}(c)$ such that, for all $N \ge \bar{N}(c)$:*

1. *For each pair of candidates $i, j \in \{A, B, C\}$, there exists a unique equilibrium in which exactly two candidates, i and j receive a positive expected number of votes.*
2. *In every such two-candidate equilibrium, both candidates win with equal probability, and only two of the six preference types vote, namely the ones who rank candidate i highest and candidate j lowest, and vice versa.*

Proof. See Appendix. □

4.2. Equilibria with three relevant candidates

We now turn to the second type of equilibrium, namely one in which all three candidates receive a positive expected number of votes. We call this type of equilibrium an "ABC equilibrium," and will show the following fundamental properties of such an equilibrium. First, the expected number of votes for each of the three candidates is equal, and so are their winning probabilities. Second, all citizen types vote for their most preferred candidates (if they choose to vote at all, of course); in other words, like in the two candidate equilibria discussed above, there is no strategic voting in an ABC equilibrium, either.

The intuition for these results is the following. If the expected number of votes for different candidates was different, different voter types would face different participation incentives; but this cannot be the case because all types face the same cost of participation and must be indifferent between voting and not voting.

Given that all candidates receive the same expected number of votes, the probability that a voter is pivotal is the same for all pairs of candidates. If a citizen votes for his favorite candidate, he may thus replace his second-ranked or third-ranked candidate by his top choice, depending on which type of pivot event occurs. In contrast, if a citizen votes for his second-ranked candidate, he may replace either his third choice (which is good) or his first choice (which is bad). However, it is quite clear that in expectation, voting for one's second-ranked candidate is less attractive than voting for one's top choice in an ABC equilibrium.

The following Proposition 2 formally states these results.

Proposition 2. *Suppose that $c \leq \frac{2}{3}(1 - \frac{\lambda}{2})$, and fix voter type proportions $\rho_t > 0$ for all $t \in T$. There exists a function $\bar{N}(c)$ such that, for all $N \geq \bar{N}(c)$,*

1. *There exist equilibria in which all three candidates receive a positive expected number of votes.*
2. *In every such equilibrium, all candidates win with probability $1/3$, and the expected number of voters for each candidate is the same: $v_A = v_B = v_C \equiv v(c)$, and $v(c)$ is the same in all equilibria. Moreover, $v(\cdot)$ is a decreasing function of c. Furthermore, in every such equilibrium, every citizen who votes does so for his most preferred candidate.*

Proof. See Appendix. □

In contrast to the two-candidate equilibrium characterized in Proposition 1, there is (generically) a continuum of different three-candidate equilibria. However, the number of votes that each candidate gets is the same in all equilibria, both between candidates and across equilibria. In other words, the equilibria differ only in the expected number of votes that each citizen-type contributes to the number of votes that each candidate receives. For example, if there exists an equilibrium in which each candidate receives, in expectation, 100 votes, then there is one equilibrium in which citizen types vote in a way that candidate A receives 90 votes (in expectation) from type AB and 10 votes from type AC, and another equilibrium in which citizen types vote in a way that candidate A receives 50 votes from type AB and 50 votes from type AC (and, of course, many other equilibria as well).

It is interesting to note that an ABC equilibrium may exist even for values of participation costs that are too high for a 2-candidate equilibrium to exist. For example, if λ is close to zero, then an ABC equilibrium exists for all cost values lower or equal to $2/3$; in such an equilibrium, the expected number of votes for each candidate is very small, and so the main pivot event is actually that, among other voters, all three candidates receive no votes. In this case, the benefit of voting is about $2/3$ because there is a chance of $2/3$ that the voter's favorite candidate replaces one of the two candidates that the voter does not like. In contrast, in an equilibrium in which there are only two relevant candidates, the increase of a voter's utility in case of a pivot event is limited to $1/2$.

Observe that the symmetry of the equilibrium (in terms of the expected number of votes received by the candidates) is essential in the last part of the proof because there are, of course, instances in which voting for one's second-ranked candidate is better; for example, if one's second- and third-ranked candidates are tied, while one's top-candidate is more than a vote behind. The fact that the equilibrium is symmetric ensures that these cases are as probable as that one's first- and third-ranked candidates are tied, and that the second-ranked candidate is behind, so that, in expectation, it is strictly better to vote for one's top candidate.

While the symmetry of all preference types with respect to their cost of voting is a necessary condition for the symmetry of equilibrium, it is quite clear that the absence of strategic voting in the ABC-equilibrium is not a knife-edge result: If, say, some voter group has a slightly lower cost of voting than others, the equilibrium will adjust continuously in a way that their favorite candidate receives slightly more expected votes than his competitors, which in turn affects the pivot probabilities in a continuous way. However, since $\lambda < 1$, there is a range in which the different pivot probabilities remain close enough to each other that strategic voting for one's second-ranked candidate remains dominated by voting for one's top candidate.

It is interesting to relate our result that there is no strategic voting in multi-candidate elections when voting is costly to Krishna and Morgan (2012). They analyze a model in which voters are differentially informed and may either be forced to vote, or may decide whether to vote. If voting is mandatory, voters may have to randomize their vote and will sometimes vote strategically for a candidate that their private information does not indicate as the best candidate, because they have to condition on the event that their vote is pivotal for the outcome of the election. This is the famous swing voter's curse (Feddersen and Pesendorfer, 1996). However, if voters have a cost of voting and can decide not to vote, then, rather than voting strategically against one's signal, it is actually better to abstain. Consequently, strategic voting (in the sense of voting against the candidate indicated by one's private information) disappears in Krishna and Morgan (2012).

Similarly, in our model, strategic voting may occur in equilibrium if participation were exogenously imposed, or if the cost of voting was so low that all citizens would have a positive expected net benefit from participation, even if all other citizens participate. However, if there are sufficiently many voters, not all voters will participate in equilibrium, and those who do must be indifferent between voting for their top candidate and abstaining. But if this is true, voting for one's second-ranked candidate must lead to a lower gross utility gain from voting, and is thus less attractive, implying that citizens would rather abstain than vote for their second-ranked candidate.

5. General case

In this section, we now analyze the general case of an arbitrary number of candidates. The results of the previous section suggest the following generalization: 1. In equilibrium, there is a subset of the candidates that is "relevant" (i.e. some voter types vote for the candidates in this set with positive probability), while the other candidates receive zero votes. 2. All relevant candidates win with the same probability. 3. The citizens who vote for a relevant candidate are those types who are maximally-motivated, in the sense that they rank the candidate whom they vote for highest among all candidates, and all other relevant candidates are at the bottom of their preference ranking (in some arbitrary permutation).

The following Proposition 3 confirms that this is indeed an equilibrium.

Proposition 3. *Fix voter type proportions $\rho_t > 0$ for all $t \in T$. Let $\mathcal{K}$ be the set of candidates, and let $R \subseteq \mathcal{K}$ be a subset of size r greater or equal to 2 (i.e., $r = \#R \geq 2$). There exists $c_R > 0$, and a function $\bar{N}(c)$ such that, for all $c < c_R$ and $N \geq \bar{N}(c)$, there exists an equilibrium in which*

1. *Candidate i receives votes with positive probability if and only if $i \in R$;*
2. *Candidate i wins with probability $1/r$ if and only if $i \in R$;*
3. *Citizen type t has a positive probability of voting for Candidate i only if $u_t(C_i) = 1$ and $u_t(C_j) \leq \lambda_{K-r+2}$ for every $j \in R$.*

Note that the equilibria of the previous section for the case of 3 candidates are special cases in which $r = 2$ and $r = 3$. In those special cases, we could characterize the equilibrium in somewhat more detail; for example, we could impose an upper bound on c that would guarantee the existence of such an equilibrium for all lower voting costs. Since that bound differs already between Proposition 1 and 2, it is unsurprising that there is no uniform bound on c that works for all possible sizes of the relevant candidate set.

Likewise, uniqueness of the equilibrium vote distribution for each set of relevant candidates (i.e., the fact that, for each set of relevant candidates, there is only one number of expected votes for each candidate that is compatible with equilibrium) is much harder to prove in this general case, because the function that gives the benefit of voting is much more cumbersome. We conjecture that the benefit of voting is monotonically decreasing in the expected number of other voters, which would be sufficient to conclude that the uniqueness of the equilibrium vote distribution result carries over to the general case, but have not been able to prove this.

6. Discussion and conclusion

In this article, we have developed a theory of voting in plurality rule elections with multiple candidates when voting is costly. We have shown that, if the number of citizens is large so that the equilibrium is in mixed strategies, equilibria look as follows: The candidate set is arbitrarily divided into a set of (at least 2) "relevant" candidates, and all other candidates. Only relevant candidates receive a positive expected number of votes, and all citizens who vote do so "sincerely," that is, for their top-ranked candidate.

In the case of three candidates, this amounts to two different types of equilibria. One in which only two candidates receive votes, and the other one in which all three candidates receive votes. In the three-candidate case, we can also prove a certain "uniqueness" property of the equilibrium. That is, while there may be multiple equilibria, the equilibrium vote distribution is unique for each set of relevant candidates.

We derive our results in a setting where citizens preferences and cost of voting are very symmetric. That is, while citizen types disagree on the ranking of candidates, the schedule of a citizen's payoffs from getting his favorite candidate, his second-most favorite candidate etc., elected, is identical for all voters, as is their cost of voting. If, instead, some citizen types have systematically lower cost of voting or, equivalently, a higher preference intensity, those equilibria in which their favorite candidate is a relevant candidate would feature a higher chance of winning for this candidate (as in Campbell, 1999).

Of course, voting rules differ between societies, and many countries use proportional representation[2] or runoff rule instead of plurality rule. Our model should provide useful tools for the analysis of costly voting in the first stage of a runoff rule system, which, however, is significantly more involved than the plurality rule election that we analyze: First, except in equilibria in which there are only two relevant candidates already in the first round, the main motivation to vote in the first round of a runoff system is to influence who proceeds to the second round (rather than to determine the ultimate winner, as in our model). Second, the voters' payoffs from first-round voting outcomes need to be derived from their fundamental preferences over ultimate outcomes (i.e., overall election winners); moreover, a voter's intermediate payoff from having a

[2] For an analysis of endogenous participation under proportional representation, see Herrera et al. (2016).

candidate proceed into the second round may vary by the identity of other candidate who also proceeds to the second round.

While the number and "position" of candidates (i.e., the voters' preference rankings) are exogenously given, our model may also be useful in richer models where first potential candidates decide whether or at which position to enter, and then citizens decide whether and for whom to. Since there are many equilibria at the voting stage, and which of these equilibria is played can be conditioned on the entry behavior of potential candidates in the first stage, it appears likely that many entry profiles can be sustained as equilibria.

7. Appendix

7.1. Proof of Proposition 1

Proof. 1. Observe first that, for any fixed c, as $N \to \infty$, there is no equilibrium in which the voting probability of any type t is 1, because otherwise, the pivot probability of every voter would go to zero (and thus be less than the cost of voting, c). Thus, any citizen of a voter type that votes with positive probability in equilibrium must be indifferent between voting and not voting.

Without loss of generality, consider a mixed strategy equilibrium in which the relevant candidates i and j are A and B. The voter types AB, AC, and CA prefer A over B, so if they vote in an AB-equilibrium, they vote for candidate A. Similarly, if types BA, BC, and CB vote in an AB-equilibrium, they vote for candidate B. The equilibrium is characterized by a vector p_{ij} denoting, for each type ij, the probability that a voter of the respective type votes. However, it is notationally more convenient to focus rather on the expected number of *votes* from preference group ij, ν_{ij}, given by $\nu_{ij} = p_{ij}N_{ij}$. By Myerson (1998) (see also Johnson et al., 2005), the number of votes from group ij is Poisson distributed with parameter ν_{ij}. Denote the realized number of A-voters by V_A, and that of B-voters by V_B. Note that V_A is Poisson distributed with parameter $\nu_A \equiv \nu_{AB} + \nu_{AC} + \nu_{CA}$,[3] and analogously for V_B.

The expected benefit of voting for candidate A for an AC type is

$$\sum_{x=0}^{\infty}\left[\frac{1}{2}e^{-(\nu_A)}\frac{(\nu_A)^x}{x!}e^{-(\nu_B)}\frac{(\nu_B)^x}{x!}+\frac{1}{2}e^{-(\nu_A)}\frac{(\nu_A)^x}{x!}e^{-(\nu_B)}\frac{(\nu_B)^{(x+1)}}{(x+1)!}\right]=c. \tag{1}$$

Here, $e^{-(\nu_A)}\frac{(\nu_A)^x}{x!}e^{-(\nu_B)}\frac{(\nu_B)^x}{x!}$ is the probability that both A and B receive x votes from the other voters, so that, if the citizen we consider votes, he breaks the tie and thus increases A's winning probability by $1/2$, from $1/2$ to 1. The second term is the probability that, among the other voters, A is one vote behind B, and thus an additional vote for A gets A into a tie, and thus increases A's winning probability again by $1/2$ (from 0 to $1/2$). In a mixed strategy equilibrium, AC types are indifferent between voting or not, i.e., the left-hand side of (1) is equal to cost c.

Similarly, the expected benefit of voting for candidate B for a BC type is

$$\sum_{x=0}^{\infty}\left[\frac{1}{2}e^{-(\nu_B)}\frac{(\nu_B)^x}{x!}e^{-(\nu_A)}\frac{(\nu_A)^x}{x!}+\frac{1}{2}e^{-(\nu_B)}\frac{(\nu_B)^x}{x!}e^{-(\nu_A)}\frac{(\nu_A)^{(x+1)}}{(x+1)!}\right]=c. \tag{2}$$

We first claim and prove that there is a unique solution to equation (1) and (2).

Lemma 1. *For each $c \in (0, 0.5)$, there is exactly one pair (ν_A, ν_B) that satisfies equations (1) and (2). Moreover $\nu_A = \nu_B$ for every value of c, and ν_A and ν_B are decreasing in c.*

Proof. The formulas for expected benefits are given by (1) and analogous condition for the BC types (see (10) below). Using formula 9.6.10 from Abramowitz and Stegun (1964), these equations are equivalent to

$$\frac{I_0(2\sqrt{\nu_A\nu_B})}{e^{\nu_A+\nu_B}}+\frac{\sqrt{\nu_B}}{\sqrt{\nu_A}}\frac{I_1(2\sqrt{\nu_A\nu_B})}{e^{\nu_A+\nu_B}}=:\Sigma_{11}+\Sigma_{21} \tag{3}$$

and

$$\frac{I_0(2\sqrt{\nu_A\nu_B})}{e^{\nu_A+\nu_B}}+\frac{\sqrt{\nu_A}}{\sqrt{\nu_B}}\frac{I_1(2\sqrt{\nu_A\nu_B})}{e^{\nu_A+\nu_B}}=:\Sigma_{12}+\Sigma_{22}, \tag{4}$$

where I_0 and I_1 are modified Bessel functions of order 0 and 1. Because the expressions (3) and (4) are equal to $2c$, it follows that

[3] This follows because, if X and Y are independent Poisson random variables with parameters θ_1 and θ_2, respectively, then $X+Y$ has a Poisson distribution with parameter $\theta_1+\theta_2$.

$$\frac{I_0(2\sqrt{v_A v_B})}{e^{v_A+v_B}} + \frac{\sqrt{v_B}}{\sqrt{v_A}}\frac{I_1(2\sqrt{v_A v_B})}{e^{v_A+v_B}} = \frac{I_0(2\sqrt{v_A v_B})}{e^{v_A+v_B}} + \frac{\sqrt{v_A}}{\sqrt{v_B}}\frac{I_1(2\sqrt{v_A v_B})}{e^{v_A+v_B}}.$$

Since, for positive values, modified Bessel functions are strictly positive, and since $\Sigma_{11} = \Sigma_{12}$, it follows that $\Sigma_{21} = \Sigma_{22}$, which implies that

$$\frac{\sqrt{v_B}}{\sqrt{v_A}} = \frac{\sqrt{v_A}}{\sqrt{v_B}},$$

and this immediately implies that $v_A = v_B$; denote this value by v. The expected benefit of voting for AC and BC types is the same and is equal to

$$\sum_{x=0}^{\infty}[\frac{1}{2}e^{-2v}\frac{v^{2x}}{x!x!} + \frac{1}{2}e^{-2v}\frac{v^{2x+1}}{x!(x+1)!}] = c. \tag{5}$$

Equation (5) can be simplified to this expression using formula 9.6.10 from Abramowitz and Stegun (1964) to yield

$$\sum_{x=0}^{\infty}[e^{-2v}\frac{v^{2x}}{x!x!} + e^{-2v}\frac{v^{2x+1}}{x!(x+1)!}] = \frac{I_0(2v)+I_1(2v)}{e^{2v}} = 2c.$$

Denote

$$f(v) = \frac{I_0(2v)+I_1(2v)}{e^{2v}}$$

Using formula 9.7.1 in Abramowitz and Stegun (1964), we have

$$\lim_{v\to 0} f(v) = 1, \quad \text{and} \quad \lim_{v\to\infty} f(v) = 0. \tag{6}$$

Since modified Bessel functions are strictly positive and continuous in $(0,\infty)$, this proves the existence of v for each cost value $c < 0.5$.

To show uniqueness, it is enough to prove that $f(v)$ is strictly monotone. Using formula 9.6.26 in Abramowitz and Stegun (1964), it follows that

$$\frac{d}{dv}I_0(2v) = 2I_1(2v), \tag{7}$$

$$\frac{d}{dv}I_1(2v) = 2I_0(2v) - \frac{I_1(2v)}{v}. \tag{8}$$

The derivative of the $f(v)$ function, using these two formulas, is

$$f'(v) = \frac{(2I_1(2v)+2I_0(2v)-\frac{I_1(2v))}{v})e^{2v} - 2e^{2v}(I_0(2v)+I_1(2v))}{e^{4v}} = -\frac{I_1(2v)}{ve^{2v}} < 0, \tag{9}$$

which shows that f is strictly decreasing. Thus, for each cost value $c < 0.5$, there exists a unique solution to (5). □

Lemma 1 thus shows that there is a unique pair (v_A, v_B) that satisfies (1) and (2), a necessary condition for an equilibrium. Since this pair satisfies $v_A = v_B$, it follows immediately that the analogous necessary condition, namely that the expected benefit from voting for a BC-type voter is equal to the cost of voting, i.e.

$$\frac{P(\text{B-votes} - \text{A-votes} \in \{0,-1\})}{2} = \frac{1}{2}\sum_{x=0}^{\infty}\left[e^{-v_B}\frac{(v_B)^x}{x!}e^{-v_A}\frac{(v_A)^x}{x!} + e^{-v_B}\frac{(v_B)^x}{x!}e^{-v_A}\frac{(v_A)^{(x+1)}}{(x+1)!}\right] = c, \tag{10}$$

is satisfied at the same values of (v_A, v_B).

It remains to show that the other citizen types have no incentive to vote, given (v_A, v_B). In the proof of Proposition 3, we show more generally that the maximal benefit of voting obtains for a type t who votes for a candidate that he ranks highest, while he ranks the other relevant candidates at the bottom of his preference ranking (in any permutation, if there are more than 2 candidates). Thus, if the benefit of voting for types AB and BC is equal to the cost of voting c, it follows that the benefit of voting is smaller than c for all other types.

In summary, this proves that the stipulated profile is the unique AB-equilibrium,[4] and since A and B were arbitrary, it proves the first claim of the proposition.

2. The characterization results follow immediately from the proof above: Because $v_A = v_B$, it follows that both candidates have the same probability of winning. Moreover, the only types who vote in equilibrium with positive probability are types AC and BC, the types who rank one of the two candidates highest and his opponent lowest. □

[4] For completeness, we should also mention that there cannot be an AB-equilibrium in which type AC votes with probability 0, but other types vote for A. Again, this follows immediately from the fact that type AC's benefit of voting for A is higher than any other type's benefit of doing so.

7.2. Proof of Proposition 2

Proof. As in Proposition 1, the assumption that N is sufficiently large implies that no citizen types vote with probability 1, and that therefore each type who votes with positive probability must be indifferent between voting and not voting.

Our proof proceeds in three steps. In Step 1, we prove the existence of an ABC equilibrium. In Step 2, we prove that, in every such equilibrium, every candidate receives the same vote share. Finally, in Step 3, we show that there is no strategic voting.

Step 1. We start by showing the existence of an ABC equilibrium with the stipulated characteristics. Since the votes of AB and AC type voters are independent Poisson distributed random variables, the probability that there are n votes for candidate A is given by

$$P(A \textit{ receives n votes}) = e^{-(\nu_{AB}+\nu_{AC})}\frac{(\nu_{AB}+\nu_{AC})^n}{n!}$$

Similarly, the corresponding probabilities for candidates B and C are given by

$$P(B \textit{ receives n votes}) = e^{-(\nu_{BA}+\nu_{BC})}\frac{(\nu_{BA}+\nu_{BC})^n}{n!}$$

$$P(C \textit{ receives n votes}) = e^{-(\nu_{CA}+\nu_{CB})}\frac{(\nu_{CA}+\nu_{CB})^n}{n!}$$

Note that these probabilities only depend on the sums $\nu_{AB}+\nu_{AC} \equiv \nu_A$, $\nu_{BA}+\nu_{BC} \equiv \nu_B$ and $\nu_{CA}+\nu_{CB} \equiv \nu_C$.

Consider the expected benefit of voting for A for a voter of type AB. There are the following different possibilities of a pivot event:

1. Among the other voters, A receives the same number of votes as both B and C. With an additional vote, A wins outright, replacing either B or C with probability 1/3 each. The expected benefit of voting is therefore $\frac{1}{3}(1-\lambda)+\frac{1}{3}=\frac{2-\lambda}{3}$.
2. Among the other voters, A is one vote behind both B and C. With an additional vote for A, there is a chance that A replaces either B (probability $\frac{1}{3}\frac{1}{2}=\frac{1}{6}$; with a benefit of $1-\lambda$) or C (again probability $\frac{1}{6}$; but now benefit 1). Thus, the benefit is $\frac{1}{6}(1-\lambda)+\frac{1}{6}=\frac{2-\lambda}{6}$.
3. A is one vote behind B, and ahead of or tied with C; the benefit of voting is $\frac{1-\lambda}{2}$.
4. A is one vote behind C, and ahead of or tied with B; the benefit of voting is $\frac{1}{2}$.
5. A and B are tied for the lead, with C strictly behind them; in this case, the benefit of voting is $\frac{1-\lambda}{2}$.
6. A and C are tied for the lead, with B strictly behind them; in this case, the benefit of voting is $\frac{1}{2}$.

Multiplying the probabilities of these events with the corresponding benefits, and adding up yields that the expected benefit of voting for our AB-type voter is

$$\begin{aligned} EB_{AB}(\nu_A,\nu_B,\nu_C) = e^{-(\nu_A+\nu_B+\nu_C)} \Bigg\{ \sum_{x=0}^{\infty}\Bigg[\frac{\nu_A^x\nu_B^x\nu_C^x}{(x!)^3}\left(\frac{2-\lambda}{3}\right) + \frac{\nu_A^x\nu_B^{x+1}\nu_C^{x+1}}{x!((x+1)!)^2}\left(\frac{2-\lambda}{6}\right) \\ + \frac{\nu_A^x\nu_B^{x+1}e^{\nu_C}\Gamma(x+1,\nu_C)}{(x!)^2(x+1)!}\frac{1-\lambda}{2} + \frac{\nu_A^x\nu_C^{x+1}e^{\nu_B}\Gamma(x+1,\nu_B)}{(x!)^2(x+1)!}\frac{1}{2}\Bigg] + \\ \sum_{x=1}^{\infty}\Bigg[\frac{\nu_A^x\nu_B^x e^{\nu_C}\Gamma(x,\nu_C)}{(x-1)!(x!)^2}\frac{1-\lambda}{2} + \frac{\nu_A^x\nu_C^x e^{\nu_B}\Gamma(x,\nu_B)}{(x-1)!(x!)^2}\frac{1}{2}\Bigg]\Bigg\}, \end{aligned} \tag{11}$$

where $\Gamma(x,\nu)=(x-1)!\sum_{n=0}^{x-1}\frac{\nu^n}{n!}e^{-\nu}$ is the upper incomplete gamma function.

The probability that an A-voter is pivotal between A and B is equal to

$$Piv_{AB,A} = \left(\sum_{x=0}^{\infty}\left[\frac{\nu_A^x\nu_B^{x+1}e^{\nu_C}\Gamma(x+1,\nu_C)}{(x!)^2(x+1)!}\right] + \sum_{x=1}^{\infty}\left[\frac{\nu_A^x\nu_B^x e^{\nu_C}\Gamma(x,\nu_C)}{(x-1)!(x!)^2}\right]\right)e^{-(\nu_A+\nu_B+\nu_C)} \tag{12}$$

and the probability that an A-voter is pivotal between A and C is equal to

$$Piv_{AC,A} = \left(\sum_{x=0}^{\infty}\left[\frac{\nu_A^x\nu_C^{x+1}e^{\nu_B}\Gamma(x+1,\nu_B)}{(x!)^2(x+1)!}\right] + \sum_{x=1}^{\infty}\left[\frac{\nu_A^x\nu_C^x e^{\nu_B}\Gamma(x,\nu_B)}{(x-1)!(x!)^2}\right]\right)e^{-(\nu_A+\nu_B+\nu_C)} \tag{13}$$

It is useful to define

$$\sum_{x=0}^{\infty}\frac{\nu_A^x\nu_B^x\nu_C^x}{(x!)^3} = {}_0F_2(;1,1;\nu_A\nu_B\nu_C)$$

and

$$\sum_{x=0}^{\infty}\frac{v_A^x v_B^x v_C^x}{x!((x+1)!)^2} = {}_0F_2(;2,2;v_A v_B v_C),$$

where ${}_0F_2(;1,1;v_A v_B v_C)$ and ${}_0F_2(;2,2;v_A v_B v_C)$ are confluent hypergeometric functions.

Using this notation, the expected benefit formulas can be expressed as follows

$$\begin{aligned} EB_{AB}(v_A,v_B,v_C) = &\left\{{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Bv_C\frac{2-\lambda}{6}\right\}e^{-(v_A+v_B+v_C)} \\ &+ Piv_{AB,A}\frac{1-\lambda}{2} + Piv_{AC,A}\frac{1}{2}, \end{aligned} \tag{14}$$

$$\begin{aligned} EB_{AC}(v_A,v_B,v_C) = &\left\{{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Bv_C\frac{2-\lambda}{6}\right\}e^{-(v_A+v_B+v_C)} \\ &+ Piv_{AB,A}\frac{1}{2} + Piv_{AC,A}\frac{1-\lambda}{2}, \end{aligned} \tag{15}$$

$$\begin{aligned} EB_{BA}(v_A,v_B,v_C) = &\left\{{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Av_C\frac{2-\lambda}{6}\right\}e^{-(v_A+v_B+v_C)} \\ &+ Piv_{BA,B}\frac{1-\lambda}{2} + Piv_{BC,B}\frac{1}{2}, \end{aligned} \tag{16}$$

$$\begin{aligned} EB_{BA}(v_A,v_B,v_C) = &\left\{{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Av_C\frac{2-\lambda}{6}\right\}e^{-(v_A+v_B+v_C)} \\ &+ Piv_{BA,B}\frac{1}{2} + Piv_{BC,B}\frac{1-\lambda}{2}, \end{aligned} \tag{17}$$

$$\begin{aligned} EB_{CA}(v_A,v_B,v_C) = &\left\{{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Av_B\frac{2-\lambda}{6}\right\}e^{-(v_A+v_B+v_C)} \\ &+ Piv_{CA,C}\frac{1-\lambda}{2} + Piv_{CB,C}\frac{1}{2}, \end{aligned} \tag{18}$$

$$\begin{aligned} EB_{CB}(v_A,v_B,v_C) = &\left\{{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Av_B\frac{2-\lambda}{6}\right\}e^{-(v_A+v_B+v_C)} \\ &+ Piv_{CA,C}\frac{1}{2} + Piv_{CB,C}\frac{1-\lambda}{2}. \end{aligned} \tag{19}$$

In a mixed strategy equilibrium, we must have that

$$EB_{ij}(v_A,v_B,v_C) = c \tag{20}$$

for all $i, j \in \{A, B, C\}$.

Step 2. We now show that, in every ABC equilibrium, the expected number of voters for each candidate must be the same: $v_A = v_B = v_C$.

Since the benefit of voting is the same for each type, (20) implies that $EB_{AB}(v_A,v_B,v_C) = EB_{AC}(v_A,v_B,v_C) = c$. As the first two terms in both expressions coincide, we have

$$Piv_{AB,A}(\frac{1}{2}-\frac{\lambda}{2}) + Piv_{AC,A}\frac{1}{2} = Piv_{AB,A}\frac{1}{2} + Piv_{AC,A}(\frac{1}{2}-\frac{\lambda}{2})$$

which implies $Piv_{AB,A} = Piv_{AC,A}$. Analogously, we have that $Piv_{BA,B} = Piv_{BC,B}$ for B voters, and $Piv_{CA,C} = Piv_{CB,C}$ for C voters.

Using $Piv_{AB,A} = Piv_{AC,A}$ in (14), and the corresponding equality in EB_{BA}, we have that

$$\begin{aligned} &\underbrace{\left[{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Bv_C\frac{2-\lambda}{6}\right]e^{-(v_A+v_B+v_C)} + Piv_{AB,A}(1-\frac{\lambda}{2})}_{EB_{AB}(v_A,v_B,v_C)} = \\ &\underbrace{\left[{}_0F_2(;1,1;v_Av_Bv_C)\frac{2-\lambda}{3} + {}_0F_2(;2,2;v_Av_Bv_C)v_Av_C\frac{2-\lambda}{6}\right]e^{-(v_A+v_B+v_C)} + Piv_{BA,B}(1-\frac{\lambda}{2})}_{EB_{BA}(v_A,v_B,v_C),} \end{aligned} \tag{21}$$

where $EB_{AB} = EB_{BA}$ follows from (20).

Suppose, to the contrary of the claim, that $v_A > v_B$. Canceling the common first term in (21) and observing that $v_A > v_B$ implies ${}_0F_2(;2,2;v_Av_Bv_C)v_Bv_C < {}_0F_2(;2,2;v_Av_Bv_C)v_Av_C$, so it follows that $Piv_{AB,A} > Piv_{BA,B}$.

However, if $v_A > v_B$, using (12) and the corresponding definition of $Piv_{BA,B}$ shows that

$$\begin{aligned}
Piv_{AB,A} &= \left(\sum_{x=0}^{\infty}[\frac{v_A^x v_B^{x+1} e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} = \\
&= \left(v_B\sum_{x=0}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} < \\
&< \left(v_A\sum_{x=0}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} = \\
&\left(\sum_{x=0}^{\infty}[\frac{v_A^{x+1} v_B^x e^{v_C}\Gamma(x+1,v_C)}{(x!)^2(x+1)!}] + \sum_{x=1}^{\infty}[\frac{v_A^x v_B^x e^{v_C}\Gamma(x,v_C)}{(x-1)!(x!)^2}]\right)e^{-(v_A+v_B+v_C)} = Piv_{BA,B}
\end{aligned}$$

This contradiction proves that $v_A \le v_B$. Analogously, one can show that $v_A \ge v_B$, which then implies $v_A = v_B$. Likewise, we can show $v_B = v_C$.

Using $v_A = v_B = v_C = v$, the expected benefit of voting for each type is

$$\begin{aligned}
&\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{x!^3}\frac{2-\lambda}{3} + \frac{v^{3x+2}}{x!(x+1)!^2}\frac{2-\lambda}{6} + \frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x}\frac{v^n}{n!}\frac{1}{2} + \frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x}\frac{v^n}{n!}\frac{1-\lambda}{2}\right]e^{-3v} \\
&+\sum_{x=1}^{\infty}\left[\frac{v^{2x}}{x!x!}\sum_{n=0}^{x-1}\frac{v^n}{n!}\frac{1}{2} + \frac{v^{2x}}{x!x!}\sum_{n=0}^{x-1}\frac{v^n}{n!}\frac{1-\lambda}{2}\right]e^{-3v} = \\
&e^{-3v}\left(1-\frac{\lambda}{2}\right)\left\{\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{x!^3}\frac{2}{3} + \frac{v^{3x+2}}{x!(x+1)!^2}\frac{1}{3} + \frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x}\frac{v^n}{n!}\right] + \sum_{x=1}^{\infty}\left[\frac{v^{2x}}{x!x!}\sum_{n=0}^{x-1}\frac{v^n}{n!}\right]\right\}.
\end{aligned} \tag{22}$$

We now show that (22) is decreasing in v.

Lemma 2. *Equation (22) is a decreasing function of v.*

Proof. Let $f(v)$ denote e^{-3v} times the term in curly brackets in (22). Since $\sum_{n=0}^{x}\frac{v^n}{n!} = \frac{e^v\Gamma(x+1,v)}{x!}$, $f(v)$ can be rewritten

$$\begin{aligned}
f(v) &= \left\{\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{(x!)^3}\frac{2}{3} + \frac{v^{3x+2}}{x!((x+1)!)^2}\frac{1}{3} + \frac{v^{2x+1}e^v\Gamma(x+1,v)}{x!x!(x+1)!}\right] + \sum_{x=1}^{\infty}\left[\frac{v^{2x}e^v\Gamma(x,v)}{x!x!(x-1)!}\right]\right\}e^{-3v} \\
&\equiv \left[f_1(v)\frac{2}{3} + f_2(v)\frac{1}{3} + f_3(v) + f_4(v)\right]e^{-3v}.
\end{aligned} \tag{23}$$

Differentiating (23) yields

$$\begin{aligned}
f'(v) &= \sum_{x=0}^{\infty}\left[2\frac{v^{3x+2}}{x!((x+1)!)^2}e^{-3v} + \frac{v^{3x+1}}{x!x!(x+1)!} - \frac{1}{3}\frac{v^{3x+1}}{x!((x+1)!)^2}\right] + \\
&+\sum_{x=1}^{\infty}\left[\frac{2xv^{2x-1}e^v\Gamma(x,v) + v^{2x}e^v\Gamma(x,v) - v^{x-1}v^{2x}}{x!x!(x-1)!}\right]e^{-3v} + \\
&+\sum_{x=0}^{\infty}\left[\frac{(2x+1)v^{2x}e^v\Gamma(x+1,v) + v^{2x+1}e^v\Gamma(x+1,v) - v^x v^{2x+1}}{x!x!(x+1)!}\right]e^{-3v} - \\
&-3\left[\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{(x!)^3}\frac{2}{3} + \frac{v^{3x+2}}{x!((x+1)!)^2}\frac{1}{3} + \frac{v^{2x+1}e^v\Gamma(x+1,v)}{x!x!(x+1)!}\right] + \sum_{x=1}^{\infty}\left[\frac{v^{2x}e^v\Gamma(x,v)}{x!x!(x-1)!}\right]\right]e^{-3v},
\end{aligned} \tag{24}$$

where we use $\frac{\partial\Gamma(x,v)}{\partial v} = -v^{x-1}e^{-v}$. Simplifying the second and third lines of this expression, we get

$$\sum_{x=1}^{\infty}\left[\frac{2xv^{2x-1}e^{v}\Gamma(x,v)+v^{2x}e^{v}\Gamma(x,v)-v^{3x-1}}{x!x!(x-1)!}\right]=$$
$$2\sum_{x=0}^{\infty}\left[\frac{v^{2x+1}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]+\sum_{x=1}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x,v)}{x!x!(x-1)!}\right]-\sum_{x=0}^{\infty}\left[\frac{v^{3x+2}}{x!((x+1)!)^{2}}\right]= \tag{25}$$
$$2f_3(v)+f_4(v)-f_2(v)$$

and

$$\sum_{x=0}^{\infty}\left[\frac{(2x+1)v^{2x}e^{v}\Gamma(x+1,v)+v^{2x+1}e^{v}\Gamma(x+1,v)-v^{x}v^{2x+1}}{x!x!(x+1)!}\right]=$$
$$\sum_{x=0}^{\infty}\left[\frac{(2(x+1)-1)v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]+\sum_{x=0}^{\infty}\left[\frac{v^{2x+1}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]-\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right]=$$
$$2\sum_{x=1}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x,v)}{x!x!(x-1)!}\right]+2\sum_{x=0}^{\infty}\left[\frac{v^{3x}}{x!x!x!}\right]-\sum_{x=0}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]+ \tag{26}$$
$$\sum_{x=0}^{\infty}\left[\frac{v^{2x+1}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]-\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right]=$$
$$2f_4(v)+2f_1(v)-\sum_{x=0}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]+f_3(v)-\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right],$$

where we use $\Gamma(x+1,v)=\frac{x!\Gamma(x,v)}{(x-1)!}+\frac{v^{x}}{e^{v}}$, and $\Gamma(1,v)=e^{-v}$. Therefore,

$$\begin{aligned}
f'(v)=e^{-3v}\Bigg\{&\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}-\frac{1}{3}\frac{v^{3x+1}}{x!((x+1)!)^{2}}\right]+2f_2(v)+2f_3(v)+f_4(v)-\\
&-f_2(v)+2f_4(v)+2f_1(v)-\sum_{x=0}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]+f_3(v)-\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!x!(x+1)!}\right]-\\
&-2f_1(v)-f_2(v)-3f_3(v)-3f_4(v)\Bigg\}=\\
&=-\frac{1}{3}\sum_{x=0}^{\infty}\left[\frac{v^{3x+1}}{x!((x+1)!)^{2}}\right]e^{-3v}-\sum_{x=0}^{\infty}\left[\frac{v^{2x}e^{v}\Gamma(x+1,v)}{x!x!(x+1)!}\right]e^{-3v}<0. \qquad \square
\end{aligned}$$

Lemma 2 implies that the value of v for which the expected benefit of voting is equal to c is a decreasing function of c. To prove that a unique solution exists for all $c<\frac{2}{3}(1-\frac{\lambda}{2})$, it is then sufficient to prove that limit of benefit at 0 is $\frac{2}{3}(1-\frac{\lambda}{2})$ and at infinity is 0. Since the expected benefit in (22) is equal to $(1-\frac{\lambda}{2})f(v)$, it is enough to prove that $\lim_{v\to 0}f(v)=\frac{2}{3}$ and $\lim_{v\to\infty}f(v)=0$.

Since $f(v)$ is continuous function, the limit at zero is just the value at 0, which is $\frac{2}{3}$. For the case of infinity, we use the squeeze theorem.

$$0\le f(v)\le\left(\sum_{x=0}^{\infty}\left[\frac{v^{2x}}{x!x!}\sum_{n=0}^{x}\frac{v^{n}}{n!}\right]+\sum_{x=0}^{\infty}\left[\frac{v^{2x+1}}{x!(x+1)!}\sum_{n=0}^{x+1}\frac{v^{n}}{n!}\right]\right)e^{-3v}\le\left(\sum_{x=0}^{\infty}\frac{v^{2x}}{x!x!}+\sum_{x=0}^{\infty}\frac{v^{2x+1}}{x!(x+1)!}\right)e^{-2v}, \tag{27}$$

where certainly $\sum_{n=0}^{x}\frac{v^{n}}{n!}\le e^{v}$, for each x, because $e^{v}=\sum_{n=0}^{\infty}\frac{v^{n}}{n!}$. The right hand side of (27) can be expressed in terms of modified Bessel functions of order zero and one.

$$\left(\sum_{x=0}^{\infty}\frac{v^{2x}}{x!x!}+\sum_{x=0}^{\infty}\frac{v^{2x+1}}{x!(x+1)!}\right)e^{-2v}=\frac{I_0(2v)+I_1(2v)}{e^{2v}}$$

By formula 9.7.1 of Abramowitz and Stegun (1964), we have that

$$\lim_{v\to\infty}\frac{I_0(2v)+I_1(2v)}{e^{2v}}=0.$$

Therefore,

$$0 \leq \lim_{v\to\infty} f(v) \leq \lim_{v\to\infty} \frac{I_0(2v)+I_1(2v)}{e^{2v}} = 0.$$

Step 3. To prove that no strategic voting (i.e., voting for second preferred candidate) can occur in any ABC equilibrium, we again, like in the proof of Proposition 1, refer the reader to the proof of Proposition 3 below in which we show that the maximal benefit of voting obtains for a type t who votes for his highest-ranked candidate. □

7.3. Proof of Proposition 3

Proof. As in Proposition 1 and in Proposition 2, for every $c \leq c_R$, there exists some $\bar{N}(c) > 0$ such that, for all $N > \bar{N}(c)$, no citizen types vote with probability 1, and citizen types who vote with positive probability must be indifferent between voting and not voting.

Consider a strategy profile in which each candidate in R receives votes in a way that his expected number of votes is v. The benefit from voting of a voter of type t who votes for his i-th preferred candidate C_i (i.e., $u_t(C_i) = \lambda_i$) is

$$\begin{aligned} BOV(v) = &\sum_{k\in C}(e^{-v})^r(\lambda_i-\lambda_k)\frac{1}{\#C}+ \\ &+\sum_{j\in R}\Bigg\{\sum_{x=1}^{\infty}\left[(e^{-v}\frac{v^x}{x!})^2\sum_{s=0}^{r-2}\binom{r-2}{s}F(x-1)^{r-2-s}(e^{-v}\frac{v^x}{x!})^s\frac{1}{(2+s)}\right]+ \\ &+\sum_{x=0}^{\infty}\left[e^{-v}\frac{v^x}{x!}e^{-v}\frac{v^{x+1}}{(x+1)!}\sum_{s=0}^{r-2}\binom{r-2}{s}F(x)^{r-2-s}(e^{-v}\frac{v^{x+1}}{(x+1)!})^s\frac{1}{(2+s)}\frac{1}{(1+s)}\right]\Bigg\}(\lambda_i-\lambda_j). \end{aligned}$$

The first line captures the event that none of the r relevant candidates receives any votes. Note that this is the only case in which the candidates not in R may win, but if our voter votes for C_i, C_i replaces the voter's kth ranked candidate with probability $\frac{1}{\#C}$. The second and third line turn to pivot events among the set of relevant candidates. The second line captures the event that, among the other voters, the candidates who our voter ranks ith and jth, as well as s other candidates, all tie for the first place. In this case, without our voter's vote, the probability that j would win is $1/(2+s)$, so this is the probability that j is replaced by i if our voter votes, with resulting payoff change $(\lambda_i - \lambda_j)$. The third line captures the event that, among other voters, i is one behind j, as well as s other candidates who are tied with j for first place. In this case, the probability that j would win if our voter does not vote is $1/(1+s)$, and the probability that i wins if he votes is $1/(2+s)$, so the probability that j gets replaced by i in this scenario is $\frac{1}{(1+s)(2+s)}$. Summing over all candidates in R gives the benefit of voting.

Let $c_R = BOV(0)$; this is the maximum cost above which nobody votes. Clearly, $BOV(\cdot)$ is a continuous function so to show that a solution for $BOV(v) = c$ exists for $c \leq c_R$, it is sufficient to show that $BOV(0) > 0$ and $\lim_{v\to\infty} BOV(v) = 0$. Substituting $v = 0$ into BOV, we get

$$BOV(0) = \sum_{k\in C}(\lambda_i - \lambda_k)\frac{1}{\#C} = 1 - \frac{\sum_{k\in C}\lambda_k}{\#C} > 0. \tag{28}$$

To prove that $\lim_{v\to\infty} BOV(v) = 0$, note that

$$\begin{aligned} BOV(v) &< \sum_{j\in R}2^{r-2}\left[\sum_{x=0}^{\infty}e^{-v}\frac{v^x}{x!}e^{-v}\frac{v^{x+1}}{(x+1)!}+\sum_{x=0}^{\infty}(e^{-v}\frac{v^x}{x!})^2\right](\lambda_i-\lambda_j) = \\ &= \sum_{j\in R}2^{r-2}\left[\frac{I_0(2v)+I_1(2v)}{e^{2v}}\right](\lambda_i-\lambda_j) \end{aligned} \tag{29}$$

where I_0 and I_1 are Bessel Functions. Using formula 9.7.1 in Abramowitz and Stegun, we have that $\lim_{v\to\infty} \frac{I_0(2v)+I_1(2v)}{e^{2v}} = 0$, so $\lim_{v\to\infty} BOV(v) = 0$ because it is finite sum. This establishes existence of a solution, for any $\lambda_i > 0$.

Furthermore, observe that, for any given v, the first term in BOV is increasing in λ_i, and the second term is increasing in $\sum_{j\in R}(\lambda_i - \lambda_j)$. Thus, the maximal benefit of voting obtains for a type t who votes for a candidate that he ranks highest, while he ranks the other relevant candidates at the bottom of his preference ranking (in any permutation).

For N sufficiently large, no type can vote with probability 1 (by the same argument as in Propositions 1 and 2), so $BOV(v) = c$ for all types who vote in equilibrium with positive probability, and no type can have $BOV(v) > c$. Thus, given R, only types who rank one candidate $i \in R$ highest, and all the other ones in R lowest, can vote in equilibrium with positive probability, and if they do, they have to vote sincerely.

Finally, the equal winning probability claim follows directly from the fact that the vote distribution is the same for every relevant candidate. □

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