THE TOPOLOGY OF REAL ALGEBRAIC VARIETIES: DETERMINISTIC AND RANDOM ASPECTS

SHANKS WORKSHOP

VANDERBILT UNIVERSITY, MARCH 10 - 11, 2017

Titles and Abstracts

Saugata Basu, Purdue University

Title: On the number of lines on projective hypersurfaces

Abstract: I will describe new results comparing three quantities. The number of lines, C_n, on a generic complex projective hypersurface in $\mathbb{P}^n$ of degree $2n-3$, the expected number, E_n, of lines on a random real hypersurface of the same degree, and the number $R_n = (2n-3)!!$ of the number of lines counted with intrinsic signs on a generic real hypersurface of the same degree. Clearly, $R_n \leq E_n \leq C_n$. We prove that $E_3 = 6\sqrt{2} - 3$, and that $\lim_{n\to\infty} \frac{\log(E_n)}{\log(C_n)} = \frac{1}{2}$. I will also discuss new proofs of the well known fact that $C_3 = 27$, and also that $R_n = (2n-3)!!$ (a result of Kharlamov and Finashin, and Okonek and Teleman) which follow from our methods. Joint work with A. Lerario, E. Lundberg and C. Peterson.

Andrei Gabrielov, Purdue University

Title: Spherical quadrilaterals and non-algebraic dessins d'enfants

Abstract: A spherical polygon (membrane) is a bordered surface homeomorphic to a closed disc, with$n$ distinguished boundary points called corners, equipped with a Riemannian metric of constant curvature 1, except at the corners, and such that the boundary arcs between the corners are geodesic. Spherical polygons with $n = 3$ and $n = 4$ are called spherical triangles and quadrilaterals, respectively. Classification of spherical quadrilaterals is a very old problem, related to the properties of solutions of the Heun equation (second order linear ordinary differential equation with four regular singular points). The corresponding problem for generic spherical triangles, related to the hypergeometric equation, was solved by Felix Klein

more than 100 years ago, while non-generic cases were completely classified as late as 2011.

When all angles at the corners are integer multiples ofπ, classification of spherical quadrilaterals is equivalent to classification of rational functions with four real critical points, a special case of the B. and M. Shapiro conjecture, related to real Schubert Calculus and control theory. Rational functions with real critical points can be characterized by their nets, combinatorial objects similar to Grothendieck's dessins d'enfants. Similarly, spherical polygons can be characterized by their multi-colored nets, which can be understood as non-algebraic dessins d'enfants. This reduces the problem to a combinatorial problem of classification of nets.

If time permits, I'll tell about recent progress in classification of circular quadrilaterals (with the sides mapped to some circles on the sphere, not necessarily geodesic), and about connection between isomonodromic deformations of the Heun equation and solutions of the Painlevé VI equation. This connection, discovered by Richard Fuchs in 1905, allows one to study real solutions of the Painlevé VI equation by using sequences of the nets of special (with one angle equal to 2π) circular pentagons.

Penka Georgieva, Institut de Mathématiques de Jussieu, Paris

Title: Real Gromov-Witten theory in all genera

Abstract: We construct positive-genus analogues of Welschinger's invariants for many real symplectic manifolds, including the odd-dimensional projective spaces and the quintic threefold. Our approach to the orientability problem is based entirely on the topology of real bundle pairs over symmetric surfaces. This allows us to endow the uncompactified moduli spaces of real maps from symmetric surfaces of all topological types with natural orientations and to verify that they extend across the codimension-one boundaries of these spaces. In reasonably regular cases, these invariants can be used to obtain lower bounds for counts of real curves of arbitrary genus. Joint work with A. Zinger.

Erik Lundberg, Florida Atlantic University

Title: The topology of random lemniscates

Abstract: A lemniscate is the level set of the modulus of a polynomial or rational function. While sampling from an ensemble of random rational lemniscates that is invariant under rotations of the Riemann sphere, we study basic geometric and topological properties. For instance, what is the average (spherical) length of a random lemniscate? How many connected components are there and how are they arranged in the plane? We will make these questions precise and address each of them. This is joint work with Antonio Lerario.

Jincheng Niu, University of Arizona

Title: Complex and real Gromov-Witten invariants and curve counts

Abstract: I will present an overview of the (complex) Gromov-Witten invariants and their relation to curve counts provided by Gopakumar-Vafa formula for Fano classes. I will then present a similar formula that transforms the real postive-genus GW-invariants of many real-orientable symplectic threefolds into signed counts of curves. These integer invariants provide lower bounds for counts of real curves of a given genus that pass through conjugate pairs of constraints. Joint work with A. Zinger.

Frank Sottile, Texas A&M University

Title: Irrational Toric Varieties

Abstract: Classical toric varieties come in two flavours: Normal toric varieties are given by rational fans in $\mathbb{R}^n$. A (not necessarily normal) affine toric variety is given by finite subset A of $\mathbb{Z}^n$. When A is homogeneous, it is projective. Applications of mathematics have long studied the positive real part of a toric variety as the main object, where the points A may be arbitrary points in $\mathbb{R}^n$. For example, in 1963 Birch showed that such an irrational toric variety is homeomorphic to the convex hull of the set A.

Recent work showing that all Hausdorff limits of translates of irrational toric varieties are toric degenerations suggested the need for a theory of irrational toric varieties associated to arbitrary fans in $\mathbb{R}^n$. These are $\mathbb{R}^n_>$-equivariant cell complexes dual to the fan. Among the pleasing parallels with the classical theory is that the space of Hausdorff limits of the irrational projective toric variety of a finite set A in $\mathbb{R}^n$ is homeomorphic to the secondary polytope of A.

This talk will sketch this story of irrational toric varieties. It represents work with Garcia-Puente, Zhu, Postinghel, Villamizar, and Pir.

Sara Tukachinsky, Université de Montréal

Title: Bounding chains in defining genus zero invariants

Abstract: Welschinger's invariants for real symplectic manifolds were originally defined in dimensions 2 and 3, counting real curves with point constraints. Equivalently, they count disks with Lagrangian boundary constraints and point constraints at interior and boundary marked points. Georgieva extended the definition to disks with various interior constraints in higher odd dimensions. In a joint work with Jake Solomon, we extend further the definition in odd dimensions to allow for boundary constraints. The general construction does not rely on real structure, instead using A_∞ algebras, and assigning bounding chains for boundary constraints. In the presence of a real structure, bounding chains take simpler form, and in particular amount to a point in the 3-dimensional case. Our invariants satisfy a version of the WDVV equations. For $\mathbb{C}P^n$, these equations allow the computation of all invariants by means of recursion formulae.