# Chapter One: Introduction

Mario Crucini and Robert Driskill

April 20, 2020

## List of Figures


1 Inverse demand function . . . 12
2 Inverse supply function . . . 13
3 Graphical solution: $\overline{Y} = 1$. . . . 17
4 Comparative statics . . . 18
5 Cartesian Plane . . . 21
6 $y = \ln x$ . . . 24
7 Slope of a function . . . 26
8 $\text{Lim}_{\Delta \to 0} m$ . . . 27
9 Profit function . . . 29
10 Depicting the maximum . . . 30
11 $\rho = -1,\ -\frac{1}{2},\ 0, 1, \infty$ . . . 33


## 1 What is international economics?

Economics is often defined as the study of the allocation of scarce resources. Thus we can define international economics as the study of how the interactions among economic entities–individuals, firms, governments, and organizations–located in different sovereign nations affect the allocation of scarce resources.

Note that it is not necessarily *Nations* that "decide" to trade with other nations, but rather the decision-making entities within these nations, e.g., people, or firms, that decide to trade.

Examples might help make this definition more concrete. Genesco, a company headquartered in Nashville Tennessee, often buys shoes from production sources in Vietnam. It then transports these to the United States, where it plans to sell them to U.S. residents. This is an interaction between firms in two different countries.

Sometimes interactions across sovereign borders are between the same economic entity. The Nissan automaker–with its headquarters and corporate charter in Japan–buys plastic from a firm in the United States, ships it to one of its plants in Mexico where it molds it into bumpers, then ships those bumpers back to a firm in the United States to apply chrome plating. It then ships the chrome-plated plastic bumpers back to an assembly plant in Mexico, where they are attached to cars in their final assembly. The cars are sold around the world.

Sometimes the entities involved include governments. The United States federal government decides to buy oil from a company headquartered in Saudi Arabia in order to replenish its Strategic Petroleum Reserve.

The United States imposes a tax on products produced in other countries– steel and aluminum from all countries, wood from Canada, for example– but transported across the United States border. This is known as imposition of a tariff.

Professor Driskill buys a book from a company in the United Kingdom. His credit card company charges him $20, and uses those $20 to buy Sixteen Pounds Stirling (the currency of the United Kingdom), which it uses to pay the British bookseller.

A Brazilian volleyball player emigrates to the Republic of Georgia to play for their Olympic volleyball team.

Mexican residents obtain visas–which allow them to travel to the United States–and work picking produce on a California farm.

As long as there is an interaction across sovereign borders, such an interaction falls under the purview of International Economics.

# 2 Why is it interesting

To have a discussion about what makes International Economics interesting, it helps to introduce a very few concepts (that are probably well-known to most people).

First, when a good crosses a border into a sovereign nation from abroad, this is an **import** for the country into which the good comes, and an **export** for the country from whence the good comes. The *value* of imports for a particular good is simply the price of the import times the quantity imported. The *value* of exports for a particular good is simply the price of the export times the quantity exported.

A **tariff** is a tax levied by a government on imports into its nation. For example, if some firm in the United States–West End Liquors in Nashville Tennessee, for example–imported a bottle of wine from Italy that costs $20, and the tariff on that wine was 10%, then West End Liquors would have to pay the U.S. government $2. A tariff is quite similar to a sales tax, except it is only levied on imports.

We usually assume–as economists–that when the price of a good increases, people buy fewer units of that good.

## 2.1 Winners and Losers

Without fail, changes in economic interactions across sovereign nations leads to some people being made better off and some worse off.

We can see this illustrated in an earlier episode of global integration that dates back to pre-World War One. The pre-eminent economist of the twentieth century, John Maynard Keynes, describes that earlier world of **globalization**– the interaction among economic entities located in different locations around the globe, especially when these interactions cross the borders of sovereign nations– as follows:

> "What an extraordinary episode in the economic progress of man that age was which came to an end in August, 1914! ... The inhabitant of London could order by telephone, sipping his morning tea in bed, the various products of the whole earth, in such quantity as he might see fit, and reasonably expect their early delivery upon his doorstep; he could at the same moment and by the same means adventure his wealth in the natural resources and new enterprises of any quarter of the world, and share, without exertion or even trouble, in their prospective fruits and advantages; or he could decide to couple the security of his fortunes with the good faith of the townspeople of any substantial municipality in any continent that fancy or information might recommend."
>
> John Maynard Keynes, 1920, 11-12, *The Economic Consequences of the Peace,* New York, Harcourt, Brace, and Howe

Keynes described a world somewhat similar to that in which we find ourselves today. Instead of ordering by phone, we now order via the internet, and our packages arrive in days, not weeks. We venture our wealth in new enterprises far from our own country via thick financial markets interconnected via high-speed cables and the internet. Nonetheless, the commercial world today is in many ways as much intertwined as it was about one hundred years ago.

That "extraordinary episode" was built upon technological advances in transport and communications, namely steam power and the telegraph and telephone. In Keynes' view, these developments led to increased grain exports from America in exchange for increased manufactured goods from Europe. This globalization then, as it does now, created frictions: the people for whom globalization created opportunities were often not the same people for whom globalization created challenges. Lower grain prices in Europe brought about by the growth of American imports led to a condition whereby, as Keynes put it, in Europe "a unit of labor applied to industry yielded year by year a purchasing power over an increasing quantity of food."[1] That is, for example, an hour of labor in industry in 1900 earned a wage which allowed the purchase of, say, one potato. In 1880, though, that same hour of work in industry earned a wage which allowed the purchase of, say, one-half of a potato.

This benefit to industrial workers, though, meant worse times for farmers: if a unit of labor applied to industry yielded a wage for these factory workers that bought more food than it had in the past, then this meant that a unit of labor

[1] John Maynard Keynes. The Economic Consequences of the Peace, Chapter Two. (Kindle Locations 100-101)

applied to food production yielded food that exchanged for fewer industrial goods than it had in the past. Hence, farmers mounted political efforts to impose tariffs as they suffered from the cheap imports.[2]

Consider now some more recent examples. In 2002, The Bush Administration imposed (temporary) tariffs on imported steel.[3] The domestic price of steel rose, and employment and profits of the domestic producers of steel also rose. Domestic steel producers–and their workers–were winners.

As was pointed out at the time, U.S. users of steel–makers of products that use steel such as automobiles, or ceiling fans, or a myriad of other products–paid that higher price, and reduced their purchases of steel. This meant that these steel-using companies made fewer of their automobiles, or of their fans, or of any of the other myriad products made from steel. This meant these fabricators also hired fewer workers, made less profit, and had to sell their products at a higher price which reflected their higher input prices. Some consumers of their products also were made worse off as they now paid more for less steel.

Also around 2002, China began to expand exports to both Europe and the United States of low-cost textiles and apparel. This led to a significant contraction of the U.S. and European textile and apparel industry while lowering the price of clothing for U.S. and European consumers. Again, the Chinese export expansion led to winners–some of the consumers of clothing–and losers–the domestic producers of clothing.[4]

And more recently, the Trump administration imposed new tariffs on steel in March 2018. As in the 2002 episode, domestic producers of steel–and their workers–have been helped, while steel users–automobile manufacturers, ceiling fan manufacturers, and final consumers of these products– have been hurt.

These examples highlight that in actual economies, trade policy changes invariably lead to both winners and losers. This means that trade policy is more interesting than one might expect: how does a government–and its citizens–think about what policies it should pursue when these policies create both winners and losers?

## 2.2 Unanticipated consequences: General Equilibrium

The study of International Economics provides an exceptional opportunity to learn about how less salient features of a change in economic circumstances are also important. For example, imposition of a tariff affects exports as well as imports, in some non-obvious but inescapable ways. And as alluded to

[2] See *Reluctant Partners* by Andrew Brown (2003), "Cooperation before 1914."

[3] Steel is not a homogenous product. The tariffs were by and large imposed on steel products produced at the time by so-called "Big Steel" producers whose U.S. manufacturing plants were located in the important electoral states of Pennsylvania and Ohio, and whose workers were unionized. We will ignore these important distinctions and just talk about steel as if it were a homogenous product.

[4] To be sure, not all clothing consumers are net winners: if you lose your job in a textile mill, the lower cost of your clothing does not come close to compensating your loss of income. In a more subtle way that we will explore later, higher relative prices for clothing means lower relative prices for some other good or goods, which might make some consumers better off.

in an earlier footnote, a decrease in U.S. and European clothing prices from the entry of China into the WTO inexorably leads–in not-so-obvious ways–to higher relative prices of some other good or service. The study of International Economics will train you to be alert to these important but indirect effects in other economic policy issues.

# 3 How we approach the subject

## 3.1 The Three C's: Causes, Consequences, and Conduct

We organize thought about international economics by considering the *causes* of international economic interactions, the *consequences* of these interactions, and the way in which these interactions are *conducted.*

This can best be introduced by way of example. First we consider a report of an actual economy that was so simple that the basic ideas are easily described. Then we consider a more recent and complex example: U.S. tariffs on steel imposed by the Bush "43" administration in 2002.

**The economic organization of a prisoner of war camp** The basic rationale for existence of trade among residents of different locations is perhaps best understood by way of a description of the workings of a World War II Prisoner of War (POW) camp. In an article titled "The Economic Organization of a Prisoner of War Camp," R.A. Radford described the economic workings of a prison camp. As he noted, a POW camp "provides a living example of a simple economy ... and its simplicity renders the demonstration of certain economic hypotheses ... instructive." The POW society, he noted, was "small and simple enough to prevent detail from obscuring the basic pattern and ... from obscuring the working of the system." The most interesting part of his description for our purposes concerns the development of trade between compounds composed of different nationalities that were housed in separate parts of the camp.

**Causes** In particular, Radford noted there was a British and a French compound. He recounted (from his position as a British POW):

> "The people who first visited the highly organized French trading centre, with its stalls and known prices found coffee extract—relatively cheap among the tea-drinking English—commanding a fancy price in biscuits or cigarettes, and some enterprising people made small fortunes that way."

This description of trade between the British and the French captures the fundamental reason for trade among different countries: prices *in the absence* of trade differ across distinct locations. These differences in prices create *incentives* for individuals to buy a good in the location where it is relatively cheap–in this case, coffee in the English (British) compound–and sell it in the location where it is relatively expensive–in this case the French compound. That is, the different prices that prevail in the absence of trade create incentives for people to *export* products from the low-price location and *import* products to the high-priced location.

A number of questions arise from this brief description of the generic reason for trade. First, what caused prices to differ across locations in the first place? In Radford's example, the fundamental reason was the differences in tastes between the "tea-drinking English" and the "coffee-drinking French." But a reasonable question is: What are the other major reasons that prices might differ across location in the absence of trade? Much of the analysis of the great theme of providing an understanding of the pattern of trade is devoted to answering this question. As we will see, a short list of these reasons would include, along with differences in tastes: differences across locations in *resources*, such as differing ratios of capital to labor or differing amounts of natural resources; differences across locations in *technology*; differences across locations in *policies*; and differences across locations in *institutions*.

A second question that arises is: what happens to the prices in the two locations *after* trade? Before trade, they differed: this gave people the incentive to export from low-price locations and import to high-priced locations. But this process means demand increases in the low-price location and supply increases in the high-price location. Prices in the two locations can thus be expected to change, increasing in the low-price location and decreasing in the high-price location. In other words, prices of identical goods are becoming more equalized as a consequence of trade flows.

**Consequences** This tendency towards equalization of prices is important because changes in prices affect different individuals differently. For example, imagine you were a member of Radford's POW compound, and furthermore imagine that you were the odd Englishman who liked coffee better than tea. According to the logic just developed, trade between your compound and the French compound would increase the price of coffee, making you worse off than you were before.

Of course, the more typical British POW, who preferred tea to coffee, would most likely benefit because he would be selling coffee in exchange for tea. The higher price of coffee means he gets more tea in exchange for his coffee.

These distributional consequences of trade lead us to the second great theme of international trade: how should a community think about the effects of changing economic circumstances when some members of that community are made better off but some worse off?

**Conduct** Trade between the British and French compounds initially was conducted by "some enterprising individuals" who bribed the German guards to let them move across the barriers between the camps. These barriers gave people the opportunity to make "small fortunes" by buying goods in the cheaper location and selling them in the more expensive location. In economic jargon, we would say that these enterprising individuals incurred "transport costs"–the bribes to the guards, and the time and effort invested in moving between the camps–in their pursuit of gain. There was initially no "trade policy" enacted by the hierarchy in the English camp, i.e., "the representatives of the Senior British Officer."[5]

But Radford notes:

> " Eventually public opinion grew hostile to these monopoly profits–not everyone could make contact with the French–and trading with them was put on a regulated basis. Each group of beds was given a quota of articles to offer and, the transaction was carried out by accredited representatives from the British compound ... ."

Put another way, the "government" of the British camp put rules and regulations on the trade between the two camps. As we will see, the conduct of trade in more complex actual economies can be quite subject to quite elaborate rules and regulations.

**Steel tariffs** In 2002, the Bush administration imposed temporary tariffs on a variety of types of steel–especially the types produced in large mills that were owned by so-called "Big Steel" companies–imported from various countries in Europe, Asia, and South America. What was the cause of importation of steel into the U.S., and what were the consequences of the tariffs? Finally, why and how did the Bush administration engage in this conduct?

**Causes** One immediate question is: why did firms in the United States–at that time–produce only about three-fourths of the amount of steel used in U.S. fabrication, i.e., in making things like automobiles and fans and pumps and lawn furniture and so on? That is, what was the *cause* of the US economy as a whole being a net importer of steel?

As with the Prisoner of War Camp example, at one level the answer is straightforward: in the absence of trade, foreign steel would have been cheaper. At a deeper level, though, we would want to know why foreign steel would have been cheaper. Did foreign firms have lower labor costs, and if so, why? Did they have fewer regulations, and hence could price their steel lower? Or did they receive subsidies from their governments?

**Consequences** A second question arises: what were the *consequences* of these tariffs? Steel fabricators, i.e., firms that used steel to produce end products, and the end users of steel, i.e., the buyers of these fabricated products, were both hurt by these tariffs: they had to pay more for the steel they bought. On the other hand, domestic steel producers were helped: they could charge higher prices for the steel they produced.

[5] Radford does not go into much detail about camp governance, but does mention the "the representatives of the Senior British Officer." We can assume that military rank and associated hierarchical organization was maintained even in POW camps. That is, there was a "government" of sorts.

**Conduct** Finally, what governed the imposition of these tariffs? How do we characterize the *conduct* of the Bush administration, and the conduct of the countries that exported steel to the U.S.? What was the legal authority that allowed the Administration to impose these tariffs? What were the political considerations that affected how the Administration conducted its trade policy with respect to steel? What other constraints exist that affect their conduct of trade policy?

It turns out the Administration used their authority under Section 201 of the Trade Act of 1974 to impose the tariffs. This allowed the Administration to argue they did not break the treaty-like obligations the U.S. had made when it joined the World Trade organization. This perceived legality seemed important to the administration: they wanted to maintain their reputation as proponents of "free trade."

The administration's decision to target "Big Steel" rather than other types–types produced all over the country by so-called mini-mills–was governed by the anticipated importance of Pennsylvania and Ohio in the anticipated 2004 presidential election. This political targeting, though, involved a trade-off. In retaliation for the steel tariffs, both the European Union and Japan threatened to impose tariffs on oranges, an industry important to another electorally-important state, namely Florida.

Finally, the conduct was motivated in part by *another* political motivation: the Administration desired so-called **fast track** authority from Congress for other trade negotiations. The administration believed they could get votes from Midwestern congresspeople for fast track authority if they protected the steel industry in their states.

The conduct of trade frequently involves numerous Byzantine-like political twists and turns like seen in the Bush tariff example.

## 3.2 The economist's approach to international economics

Economics is the study of the allocation of scarce resources. At Vanderbilt University in 2020, someone interested in economics could take the class titled "Religion, Culture and Commerce: The World Economy in Historical perspective." Surely this is the purview of International Economics. The class description is: The role of religion in economic exchange and the movement of commodities.

Or, at Vanderbilt, you could take "Economic Anthropology." Or you could study economics with the help of the Catholic Church, by reading "A Catholic Framework for Economic Life," which offers ten key principles to help Catholics reflect on the values that should shape our participation in economic life, and was written by the bishops of the United States based on the Catechism of the Catholic Church, papal encyclicals, the pastoral letter Economic Justice for All, and other statements of the U.S. Catholic bishops.[6]

The point here is that there are many ways, most of which are also rewarding and useful, to study the allocation of scarce resources. Economists do it differ-

[6] http://www.usccb.org/issues-and-action/human-life-and-dignity/economic -justice-economy/catholic-framework-for-economic-life.cfm.

ently, though. For our purposes, we emphasize (1) our focus on the individual as a rational entity, and (2) our use of models.

### 3.2.1 The idea of a rational individual

For economists, we model a rational individual as someone who knows what they want and who thinks logically about how to get what they want. Thinking logically means a person considers (1) the constraints she faces, and (2) the feasible actions that will obtain for her what she wants. The constraints an individual faces could be the knowledge that other equally rational individuals are competing for the same thing.

### 3.2.2 Models

**Overview** We all use models in everyday life, such as the iconic London Underground map. For economists, though, models are such an important component of how they study the allocation of scarce resources that we take some time to clarify the concept. To help keep thoughts organized, consider the following definition of an economic model:

**Definition 1** *An economic model is a logical (usually mathematical) representation of whatever a priori or theoretical knowledge economic analysis suggests is most relevant for treating a particular problem. ( Economic Statistics and Econometrics, by Edward Kane, Harper and Row, NY, NY, 1968, p. 12)*

While *any* model can be expressed in terms of sentences written in plain language, economic models are frequently amenable to representation as lists of equations (which can frequently be represented as graphs). Equations express logical relationships among variables, e.g., "The price of a good is equal to the number two (2) divided by the quantity per unit of time of good $Q$". In symbolic notation, we would write:

$$p = \frac{2}{Q}.$$

The variables in this equation are price ($p$) and the quantity per unit of time of good $Q$.

**Definition 2** *A* ***variable*** *is a quantity free to take on any of a number of permissible values.*

The reason most international economic models are expressed in terms of graphs and equations is that the complexity of the models makes them difficult to understand without use of a more symbolic system. It is nearly impossible to think through a complex problem without using some symbolic notation system. Furthermore, the language of graphs and equations is well suited for constructing the long chains of deductive reasoning that lead to the conclusions of interest to an economist.

The first step in making a model is a decision about what is essential to understanding the question at hand, and what can be ignored. The London Underground model, for example, does not include any topographical features such as hills. These features exist, but are assumed not important to the question at hand: how to get from one station to another via London's underground railway system.

The second step is the logical structuring of the relationships among the assumed–to–be essential economic concepts.

The models we use in International Economics are indeed sets of equations that symbolize relationships between variables, e.g., "as price increases, the quantity demanded goes down." Part of constructing a model is making a decision about which variables are **exogenous**–a variable whose value is given to us from "outside the model"– and which are **endogenous**–a variable whose value is jointly determined by the particular values taken by the exogenous variables and by the logical relationships among variables within the model.

**Definition 3** *An exogenous variable is a variable whose value is given from "outside the model."*

**Definition 4** *An endogenous variable is a variable whose value is jointly determined by the values of the exogenous variables and by the logical relationships among the variables within the model.*

We will develop a sequence of models to help us understand the international economy. What helps in understanding these different models is that all are used to answer the following canonical question:

> What is the relationship between the values of the endogenous variables and the values of the exogenous variables?

Answering this question is referred to as "solving the model."[7] It is the reason we model, as it lets us make falsifiable predictions that provide a way of evaluating the usefulness of these models.

**Example** We illustrate a model via an example from Principles of Economics classes: the partial-equilibrium competitive model of supply and demand.

**Elements of models: variables** The first step in construction of a model is making a decision about what variables are important for the model. *For our purposes* we assume that, for some commodity $C$ (coffee, for example), quantity demanded per unit of time, symbolically denoted as $C^d$, quantity supplied per unit of time, symbolically denoted as $C^s$, the price of the commodity, denoted as $P_C$, and income per unit of time, denoted as $Y$, are the variables of interest.

For the model as a whole, the endogenous variables are $C^d$, $C^s$, $C$, and $P_C$, and the exogenous variable is $Y$.

[7] We also refer to having solved a model as having reached "Miller time!" in homage to a classic beer commercial. Google it if you are interested.

**Elements of models: equations** The logical structuring and representation of basic interrelationship for this model is expressed by three equations.

- The demand equation: *The quantity of coffee demanded per unit of time is a decreasing function of the dollar price of coffee and an increasing function of income.* This can be expressed succinctly in symbols as:

$$C^d = f(\overset{-}{P_C}; \overset{+}{\overline{Y}}). \tag{M.1}$$

The minus sign over the argument $P_C$ tells us that an increase in $P_C$ decreases $C^d$, while the plus sign over the argument $Y$ tells us that an increase in $Y$ increases $C^d$. We separate the arguments in equation M.1 by a semicolon to indicate that $Y$ is exogenous. We also place a bar over $Y$ to indicate that it is an exogenous variable.

Equation M.1 is known as a demand function, or equivalently, a demand curve. For a variety of reasons, we might want to express this relationship in terms of a specific functional form. For example, we might specify the demand function as:

$$C^d = \frac{\gamma \cdot \overline{Y}}{P_C},\ \gamma > 0. \tag{M.2}$$

The symbol $\gamma$ (gamma) is a **parameter**. This is a stand-in for the value of member of a set of permissible numbers–all the real numbers greater than zero, in this case. It could be four (4), for example.

A parameter looks a lot like a variable: it too is a quantity free to take on any number from a set of permissible values. We assume, though, that parameters are always exogenous, and that they remain a constant number over the time span of interest to us. We can think of them as constant numbers that tie together the variables in the equations that make up a model.

**Definition 5** *A parameter is a constant, which, while not always known, is assumed to have fixed numerical value for the time period in question.*

An equation is a collection of ordered pairs, and thus can be depicted in the Catesian plane as a curve. Because of a quirk in the history of economics, graphs of demand and supply functions are always depicted with price on the vertical axis and quantity on the horizontal axis. Thus to depict a demand function, we first find the **inverse demand function**, in which price is on the left-hand-side of the equation and quantity on the right:

$$P_C = \frac{\gamma \cdot \overline{Y}}{C^d}. \tag{M.3}$$

The graphical depiction of a member of this parametric equation in the $P_C - C^d$ plane is seen in Figure 1, where we have set $\gamma = 4$ and $\overline{Y} = 1$.

Equation M.1 (and its specific functional form counterparts) is known as a **behavioral relationship** because it describes an entity's behavior: how the entity (presumeably consumers in aggregate) responds to changes in circumstances.

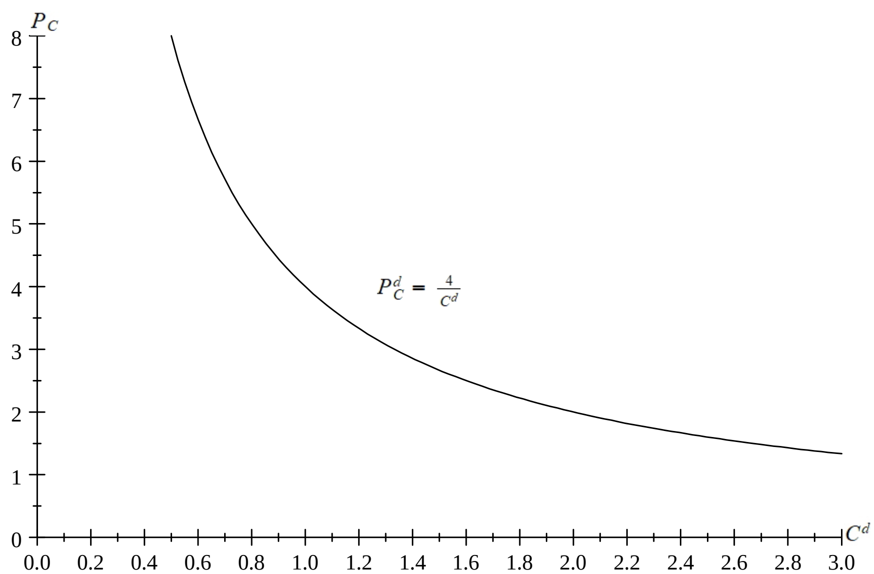


Figure 1: Inverse demand function

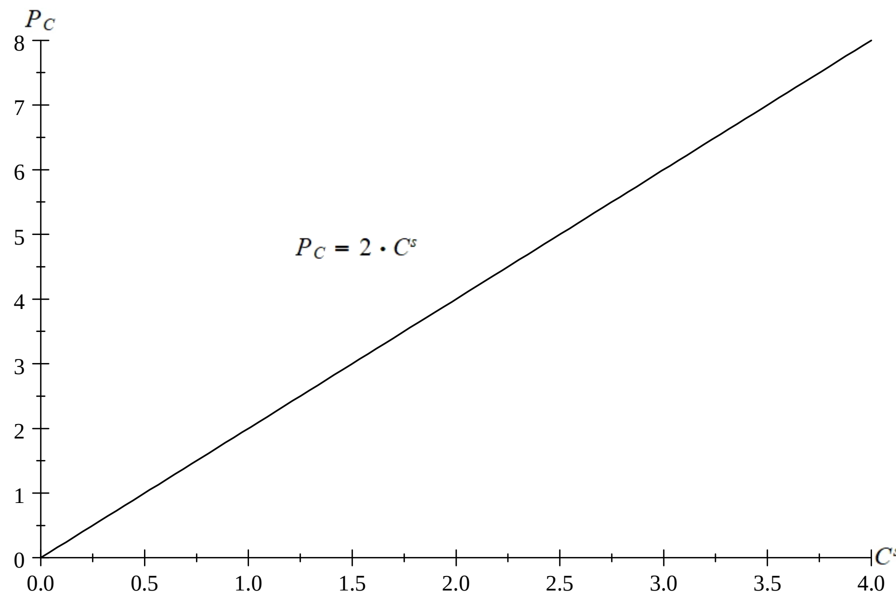


Figure 2: Inverse supply function

- The supply equation: *The quantity of coffee supplied by the EU per unit of time is an increasing function of the dollar price of coffee.* This can be expressed succinctly in symbols as:

$$C^s = g(\overset{+}{P_C}). \tag{M.4}$$

A specific functional form to express this functional relationship could be the following:

$$C^s = \beta \cdot (P_C)\,; \beta > 0 \tag{M.5}$$

where $\beta$ is a parameter.

Again because of the quirks of economic history, in order to depict this function as a graph we find the inverse supply function as:

$$P_C = \frac{1}{\beta} \cdot C^s; \tag{M.6}$$

The pictures would look like figure 2, for $\beta = \frac{1}{2}$.

The supply relationship is also a **behavioral relationship**: it describes how some entity (presumeably firms in aggregate) responds to changes in circumstances, namely price.

- The equilibrium condition: *the quantity demanded equals the quantity supplied*.

This can be succinctly expressed symbolically as:

$$C^d = C^s. \tag{M.7}$$

The three equations (M.1), (M.4), and (M.7) are the **structural equations** of the model and together constitute the **structural form** of the model. These equations are a simple listing of all the equations of the model. They depict the detailed economic behavior of the model embodied in the assumptions of logical interrelationships among variables without taking advantage of any opportunities for algebraic simplification and consolidation. Each equation in the structural form might include only endogenous variables or both endogenous and exogenous variables.

**Definition 6** *The structural form of a model is all of the equations of the model that depict the assumed logical interrelationships among variables. Each equation may have either only endogenous variables or both endogneous and exogenous variables.*

**Solving the model: analytic solution** A solution to a model answers the canonical question: what is the relationship between exogenous components and endogenous components of a model? For our purposes there are two (2) ways of solving a model: graphically and analytically. We do both. In particular, we use graphs to illustrate the analytic approach. For expository ease, we solve using the specific functional forms we introduced.

First let us reproduce in one place the three structural equations:

$$C^d = \frac{\gamma \cdot \overline{Y}}{P_C}, \ \gamma > 0. \tag{M.2}$$

$$C^s = \beta \cdot (P_C)\,; \beta > 0 \tag{M.5}$$

$$C^d = C^s. \tag{M.7}$$

These three equations have three (3) endogenous variables: $C^d$, $C^s$, and $P_C$. Our task is to express each of these three variables as a function solely of the exogenous components of the model, namely the exogenous variable $\overline{Y}$ and the parameters $\gamma$ and $\beta$. The protocol we usually follow in solving this type of model is first to substitute the behavioral relationships into the equilibrium condition:

$$\overbrace{\frac{\gamma \cdot \overline{Y}}{P_C}}^{C^d} = \overbrace{\beta \cdot (P_C)}^{C^s}. \tag{M.8}$$

With some algebra, this yields our first **solution equation:**

$$\beta \cdot (P_C)^2 = \gamma \cdot \overline{Y};$$
$$(P_C)^2 = \frac{\gamma \cdot \overline{Y}}{\beta};$$
$$\widehat{P}_C = \left(\frac{\gamma \cdot \overline{Y}}{\beta}\right)^{\frac{1}{2}}. \tag{M.9}$$

We describe this as a solution equation because it answers the question: what is the value of the endogenous variable $P_C$ given values of the exogenous variable $\overline{Y}$ and the values of the exogenous parameters $\gamma$ and $\beta$? We put the "hat" over $P_C$ to emphasize that this is a *particular* value of the price: the *equilibrium* price.

We can then solve for the other endogenous variables by substitution of the equilibrium price equation into the two original behavioral relationships:

$$\widehat{C}^d = \frac{\gamma \cdot \overline{Y}}{\left(\frac{\gamma \cdot \overline{Y}}{\beta}\right)^{\frac{1}{2}}};$$
$$\widehat{C}^s = \beta \cdot \left(\frac{\gamma \cdot \overline{Y}}{\beta}\right)^{\frac{1}{2}}.$$

Some algebraic simplification yields the simpler solution equations:

$$\widehat{C}^d = \left(\beta \cdot \gamma \cdot \overline{Y}\right)^{\frac{1}{2}}; \tag{M.10}$$
$$\widehat{C}^s = \left(\beta \cdot \gamma \cdot \overline{Y}\right)^{\frac{1}{2}}. \tag{M.11}$$

Not surprisingly, $\widehat{C}^d = \widehat{C}^s$: it was required by the equilibrium condition.

Equations (M.9),(M.10), and (M.11) constitute the **reduced form** for our model. It consists of one equation for each endogenous variable. Each equation has one and only one endogenous variable expressed as an explicit function of exogenous variables and exogenous parameters.

**Definition 7** *A reduced form for a model consists of a set of equations, one for each endogenous variable, for which each equation has one and only one endogenous variable expressed as an explicit function of exogenous variables and parameters.*

A reduced form allows us to "read off" the values of the endogenous variables from the values of the exogenous variables and parameters. A reduced form equation is also called a **solution equation.**

To illustrate, imagine our exogenous parameters and variables were:

$$\gamma = 2,\ \beta = \frac{1}{2},\ \overline{Y} = 1.$$

Our solution of the model would then be:

$$\widehat{P}_C = \left(\frac{\gamma \cdot \overline{Y}}{\beta}\right)^{\frac{1}{2}} = \left(\frac{2 \cdot 1}{\frac{1}{2}}\right)^{\frac{1}{2}} = 2; \qquad \text{(M.12.1)}$$

$$\widehat{C}^d = \left(\beta \cdot \gamma \cdot \overline{Y}\right)^{\frac{1}{2}} = \left(\frac{1}{2} \cdot 2 \cdot 1\right)^{\frac{1}{2}} = 1; \qquad \text{(M.12.2)}$$

$$\widehat{C}^s = \left(\beta \cdot \gamma \cdot \overline{Y}\right)^{\frac{1}{2}} = \left(\frac{1}{2} \cdot 2 \cdot 1\right)^{\frac{1}{2}} = 1. \qquad \text{(M.12.3)}$$

A reduced form lets us easily carry out **comparative statics** analysis:

**Definition 8** *Comparative statics analysis is determining how endogenous variables change when an exogenous variable changes, ceterus paribus.*

Suppose, for example, $\overline{Y}$ were to change from one (1) to two and one-quarter ($2\frac{1}{4}$). Our solutions would now be:

$$\widehat{P}_C = \left(\frac{\gamma \cdot \overline{Y}}{\beta}\right)^{\frac{1}{2}} = \left(\frac{2 \cdot 2.25}{\frac{1}{2}}\right)^{\frac{1}{2}} = 3;$$

$$\widehat{C}^d = \left(\beta \cdot \gamma \cdot \overline{Y}\right)^{\frac{1}{2}} = \left(\frac{1}{2} \cdot 2 \cdot 2.25\right)^{\frac{1}{2}} = 1.5;$$

$$\widehat{C}^s = \left(\beta \cdot \gamma \cdot \overline{Y}\right)^{\frac{1}{2}} = \left(\frac{1}{2} \cdot 2 \cdot 2.25\right)^{\frac{1}{2}} = 1.5.$$

**Solving the model: graphical analysis** We depict in figure 3 the superposition of the inverse demand and inverse supply functions. The equilibrium condition requires that the quantity demanded equal the quantity supplied, which means that the intersection of the two curves in the $C - P_C$ plane is the unique pair $(C, P_C)$ that solves the model.

We can carry out comparative statics analysis by shifting the demand curve as $\overline{Y}$ increases. This is depicted in figure 4.

Often, we only know some qualitative information about our model. For example, if all we knew was that the demand curve was downward-sloping and shifted to the right as income went up, and that the demand curve sloped up, our graphical analysis would still give us the right "direction" of the movement of the endogenous variables as, ceterus paribus, income changed.

# 4 The way forward

We first tackle the "micro" part of international trade. We do this by moving through a sequence of models, starting with the most simple–an **endowment economy**–and ending with more complex ones that include strategic behavior.

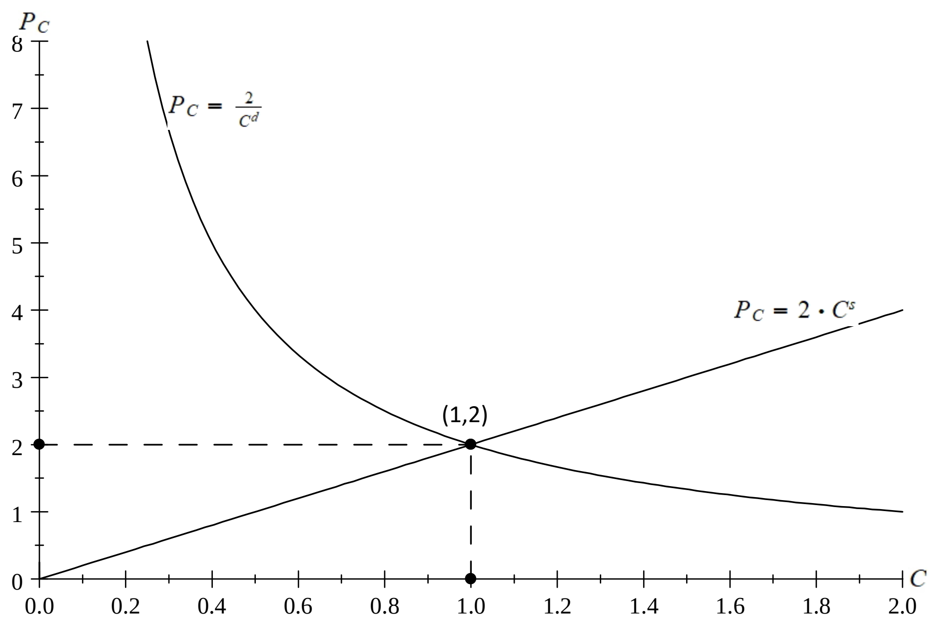


Figure 3: Graphical solution: $\overline{Y} = 1$.

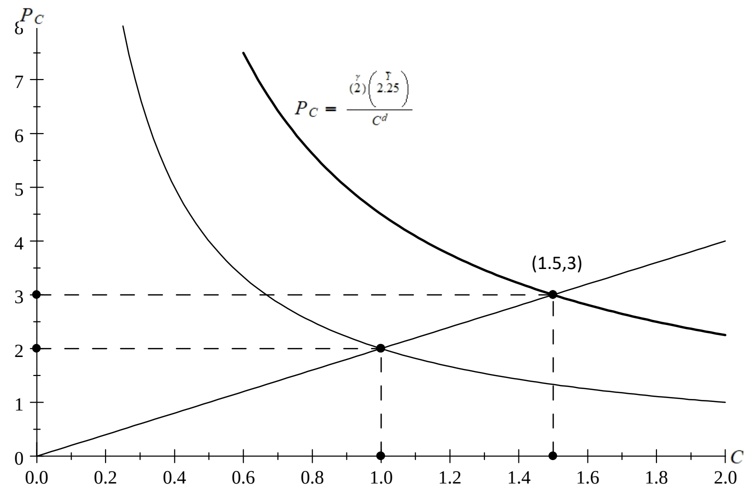


Figure 4: Comparative statics

Our strategy with all these models is to break the complete model at hand into more manageable sub-models. We use equations and graphs to illustrate the interrelations between variables, and to help us solve the model.

We now review the basic mathematical tools we use.

# 5 Mathematics and calculus review

This is a brief review designed to re-acquaint the reader with the tools we use in this class.

## 5.1 Real numbers

Perhaps a brief description of the number system will help clarify the distinction and usefulness of real numbers. Most of us are most familiar (from counting) with numbers such as 1, 2, 3, ... , the *positive integers.* The numbers $-1$, $-2$, $-3$, ... are known as the *negative integers*. When we collect all the positive and negative integers and add the number 0 (zero), we have *the set of all integers*.

Almost as familiar as integers are fractions, such as $\frac{1}{2}$, $\frac{5}{3}$, $\frac{13}{16}$, $-\frac{1}{2}$, and so on. If we envision a ruler, every such fraction would fall between the integers. Membership in the set of fractions is defined by an ability to be expressed as a *ratio* of two integers, from whence comes their designation as the **rational**

**numbers**. Integers are also rational numbers because any integer $n$ can be expressed as the fraction $\frac{n}{1}$.

Finally, we have the concept of **irrational numbers***:* numbers that cannot be expressed as the ratio of two integers. A familiar example is the number $\pi = 3.1415...$, which measures the ratio of the circumference of a circle to its diameter. The "dot-dot-dot" symbol after the digit "5" in our expression for $\pi$ means that an infinite number of other numbers follow. We sometimes use the symbol " $\approx$ " to indicate that a number such as $\pi$ is *approximately* equal to $3.14: \quad \pi \approx 3.14$. What is interesting and useful for us is that each irrational number falls between two rational numbers and can be thought to "fill in" those spaces on a ruler. Thus, the collection of all rational and irrational numbers creates a continuum of numbers known as the *real numbers.* The set of real numbers can be represented as a line along a ruler (of infinite length) in which there are no empty spaces.

Using real numbers in a theory is more convenient for most purposes that using just rational numbers. For example, for purposes of development of a theory we may allow the "permissible values" of things like consumption per year and income per year to be the set of non-negative real numbers. This is an approximation to reality because we can't really sub-divide the measurement of things like consumption per year and income per year more finely than small discrete units such as $\frac{1}{100}$ of a dollar per year. That is, we can't have a measurement of these variables that is, for example, the number $\pi \approx 3.1415...$ . This lack of realism is offset by the convenience of thinking about the permissible values of such variables as coming from the *continuum* of values represented by the real line. The point here is not that we think most economic variables are measured continuously. In fact, we think most are measured in discrete units. For most models, though, these discrete units are quite fine, and consequently a listing of all possible measurements would be quite large. By assuming that economic measurements are members of the set of real numbers, we avail ourselves of the convenience of concise expressions and depictions of members of collections of variables. These collections of variables are important components of our models.

## 5.2 Analytic geometry

Analytic geometry allows us to picture ordered pairs of numbers and algebraic equations in terms of points and geometric curves. For many people, this depiction is the key tool for understanding logical interactions among variables. The key idea, the discovery of which is credited to the French mathematician Descartes (1596-1650) involves locating a point in a plane by means of its distance from two perpendicular axis. Such a plane is known as a "Cartesian plane" and points in this plane are located by pairs of numbers known as "Cartesian coordinates." This terminology is in commemoration of Descartes. We briefly review these concepts before developing graphical representations of the logical relationships that make up our sub-model of demand.

### 5.2.1 Coordinates

The fundamental idea in analytic geometry is the establishment of a one-to-one correspondence between numbers or groups of numbers and points in a geometric space. "One-to-one" means that for every unique point there corresponds a unique pair of numbers. Of most use to undergraduate-level economics is the correspondence between points in a plane and pairs of numbers. For our partial-equilibrium demand-supply model, the pairs of numbers of interest are $(C, P_C)$. Generic notation familiar to some from mathematics classes denotes pairs as $(x, y)$. Most people are familiar with this basic concept from knowledge of map coordinates. A point on a map is described by its *coordinates* : a pair of numbers, one of which specifies latitude and the other longitude.

To establish the one-to-one correspondence between points in a plane and pairs of numbers, start with a horizontal line in a plane, extending indefinitely to the left and the right. In generic notation, this line is known as the $x-axis$. For the model of the previous section, we might want to think of this axis as the $C-axis$ . A reference point $O$ on this axis and a unit of length (e.g., one kilo of coffee per unit of time) are then chosen. The axis is scaled by this unit of length so that the number zero is attached to point $O$, the number $+a$ is attached to the point $a$ units of length to the right of $O$, the number $+2a$ is attached to the point $2a$ units to the right of $O$, the number $-a$ is attached to the point $a$ units to the left of $O$, and so on. In this way, every point on the $x-axis$ corresponds to a unique real number. The perhaps non-intuitive feature of this real number line is that between any two real numbers, no matter how close to each other, there can always to interspersed another real number. This implies that a point on the line takes up no space. For the purposes of economic models, one can think of the real number line as a convenient approximation to numbers that represent small but discrete units.

Now place another straight line vertical in the plane, i.e., at a right angle to the x-axis, through point $O$. In generic notation, this is the $y - axis$. For the model of the preceding section, we would think of this as the $P_C - axis$. It also extends indefinitely up and down. Choose a unit of length, such as price per kilo of coffee, and scale the y-axis with this unit, much as with the x-axis. That is, the number $b$ is attached to the point on the y-axis $b$ units above point $O$, the number $2b$ is attached to the point on the y-axis $2b$ units above point $O$, and so on.

Now draw a line parallel to the $y-axis$ through point $a$ on the $x-axis$ and a line parallel to the $x - axis$ through point $b$ on the $y - axis$. These two lines intersect at point $R$, which corresponds to the pair of numbers $(a, b)$. Clearly, for any two real numbers $a$ and $b$, there corresponds a unique point $R$, which we denote as $R(a, b)$. Conversely, we say that the coordinates of $R$ are $(a, b)$. Figure 5 illustrates the point associated with the pair $(2, 1)$.

The two axis divide the plane into four quadrants labeled I, II, III, and IV, where standard terminology denotes quadrant I as that section for which all points have two positive coordinates. This quadrant is of most use in economics because most economic variables, such as prices and quantities, are inherently

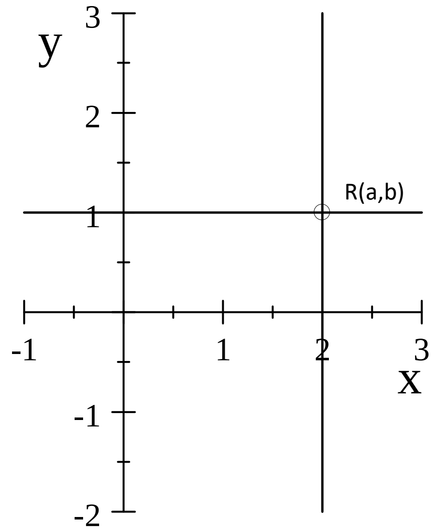


Figure 5: Cartesian Plane

non-negative numbers. Quadrant II has points with signs $(-,+)$, quadrant III has points with signs $(-,-)$ and quadrant IV has points with signs $(+,-)$.

## 5.3 Relations and Functions

### 5.3.1 Ordered pairs

Most economic models consider relationships among a number of variables. Consider two variables, $y$ and $x$. An **ordered pair** is denoted by two numbers, the first associated with the variable $x$ and the second associated with the variable $y$, written within parenthetical brackets, separated by a comma, with the number associated with the variable $x$ listed first and the number associated with the variable $y$ listed second, as in:

$$(x, y)$$

### 5.3.2 Relations and functions: definitions and notation

A collection of ordered pairs $(x, y)$ constitutes a **relation** between $x$ and $y$.

Such a collection for which for every value of $x$ there exists only one, i.e., a **unique**, value of $y$ is known as a **function**. In this case we say that $y$ is a function of $x$, and write it as:

$$y = f(x)$$

Note: this does not mean multiply $f$ times $x$. This functional expression is a general statement expressing that such a collection of ordered pairs exists, but does not make explicit the actual rule that relates $y$ to $x$.

We might also express this relationship as

$$y = y(x).$$

We do this usually because we have run out of symbols.

We also sometimes express functions and relations in **implicit form**:

$$F(x, y) = 0.$$

### 5.3.3 Types of functions

**Polynomial functions** A polynomial function is a linear collection of terms, where each term has a *coefficient*–which is a real number–times the variable $x$ raised to a non-negative integer power:

$$y = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n.$$

The $a_i's$ are the coefficients. One of our favorites in economics is the quadratic:

$$y = a_0 + a_1 x + a_2 x^2.$$

**Rational functions**

$$\frac{a_0 + a_1 x + a_2 x^2 + ... + a_n x^n}{b_0 + b_1 x + b_2 x^2 + ... + b_m x^m}$$

A rational ( from "ratio") function is just a ratio of polynomial functions. One of our favorite in economics is

$$y = \frac{a}{x}$$

which when written in implicit form is

$$xy = a.$$

**Algebraic and Nonalgebraic functions** Any function that is expressed in terms of polynomials and/or roots (such as square roots) of polynomials is an **algebraic function**. An algebraic (but not rational) function is, for example,

$$y = \sqrt{x^3 + 1}.$$

There exist, though, non-algebraic functions known as **transcendental functions**. These include **exponential**, ***logarithmic***, and **trigonometric** functions. Exponential functions have the independent variable appearing as an exponent:

$$y = b^x.$$

Our most useful exponential function is the *natural* exponential function:

$$y = e^x$$

where

$$e = \lim_{m \to \infty} \left(1 + \frac{1}{m}\right)^m = 2.71828... \,.$$

Our favorite logarithmic function (in economics) is the natural logarithmic function (closely related to the exponential function):

$$y = \ln x$$

It looks like this (Figure 6):

Key properties for our purposes are:

$$\begin{aligned} \ln(1) &= 0; \\ \ln e &= 1; \end{aligned}$$

$$\ln e^n = n$$

where $n$ is any real number, and, for $x$ "small,"

$$\ln(1 + x) \approx x.$$

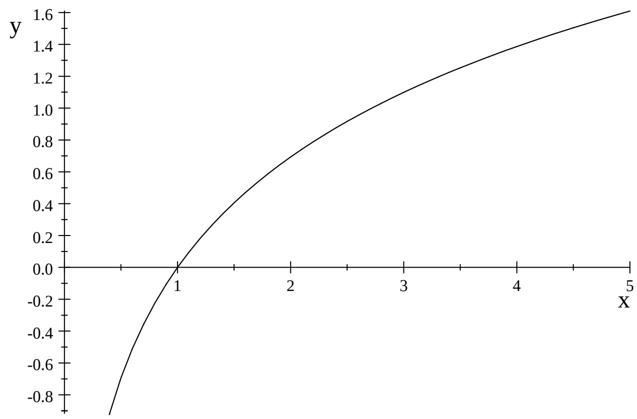


Figure 6: $y = \ln x$

# 6 Exponents

The expression $6^2$ means that six (6) is to be raised to the second power, which means multiply six (6) time six (6). In general, we have

$$x^n \equiv \underbrace{x \times x \times x \times ... \times x}_{n\ terms}$$

## 6.1 Rules of exponents

1. $x^n \times x^m = x^{n+m}$ (example: $x^2 \times x^5 = x^7$)
2. $\frac{x^n}{x^m} = x^{n-m}$
3. $x^{-n} = \frac{1}{x^n}$
4. $x^0 = 1$
5. $x^{\frac{1}{n}} = \sqrt[n]{x}$
6. $(x^n)^m = x^{nm}$
7. $x^n y^n = (xy)^n$

# 7 Solving coupled systems of two linear equations

Consider two linear equations:

$$a_{11}x + a_{12}y = b$$
$$a_{21}x + a_{22}y = c$$

where the $a_{ij}$ and $b$ and $c$ are constants. Simple substitution can be used to solve these two equations for the solution values of $x$ and $y$. First, solve $y$ as a function of $x$ for the first equation:

$$y = \frac{b}{a_{12}} - \frac{a_{11}}{a_{12}}x.$$

Substitute this expression for $y$ into the second equation:

$$a_{21}x + a_{22}\left(\frac{b}{a_{12}} - \frac{a_{11}}{a_{12}}x\right) = c.$$

Simplify and collect terms:

$$a_{21}a_{12}x + a_{22}b - a_{11}a_{22}x = a_{12}c$$
$$x = \frac{a_{22}b - a_{12}c}{a_{11}a_{22} - a_{12}a_{21}}$$

Substitute this expression for $x$ into either the first or second equation:

$$y = \frac{a_{11}c - a_{12}b}{a_{11}a_{22} - a_{12}a_{21}}$$

# 8 Solving coupled systems of two nonlinear equations

Checkhov famously wrote: "All happy families are alike; each unhappy family is unhappy in its own way." So it is with linear and nonlinear equations: linear means one and only one thing, while there are many ways equations can be nonlinear. Hence, there is no canonical method for solving systems of nonlinear equations. All we can say is that cleverness is required!

We usually supply the cleverness, as we assume our readers are not necessarily preparing for graduate school.

# 9 Derivatives and differentiation

## 9.1 Slope of a function

Consider two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on the curve $y = f(x)$, as displayed in Figure 7. The slope of the ray connecting the two points is

$$m \equiv \frac{y_1 - y_2}{x_1 - x_2} = \frac{\Delta y}{\Delta x}.$$

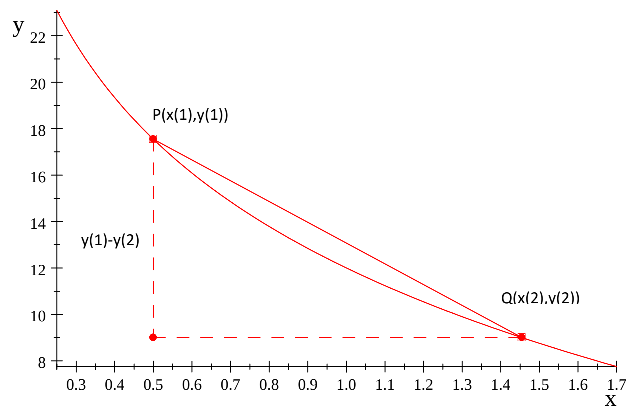


Figure 7: Slope of a function

Imagine we now hold $P$ fixed and move $Q$ along the curve towards $P$. As we do, the slope of the connecting ray $PQ$ (for the functions we are interested in) will vary by smaller and smaller amounts, eventually approaching a constant limiting value, which we call the slope of the tangent to the curve at $P$ or the slope of the curve at $P$.

The depiction of this limiting value of the slope $m$ of $y = f(x)$ at $P(x_1, y_1)$ is the slope of the straight line that "just touches" or, equivalently, is tangent to, the curve at point $P$. This is depicted in figure 8:

## 9.2 Rules of differentiation

1. Constant-function rule: The derivative of a constant function $y = f(x) = c$ is zero:

$$\frac{dy}{dx} = 0$$

Example:

$$\frac{d(3)}{dx} = 0.$$

2. Power-function rule:

$$\frac{d}{dx}cx^n = ncx^{n-1}$$

Example:

$$\frac{d(x^{(-3)})}{dx} = -3x^{-4}.$$

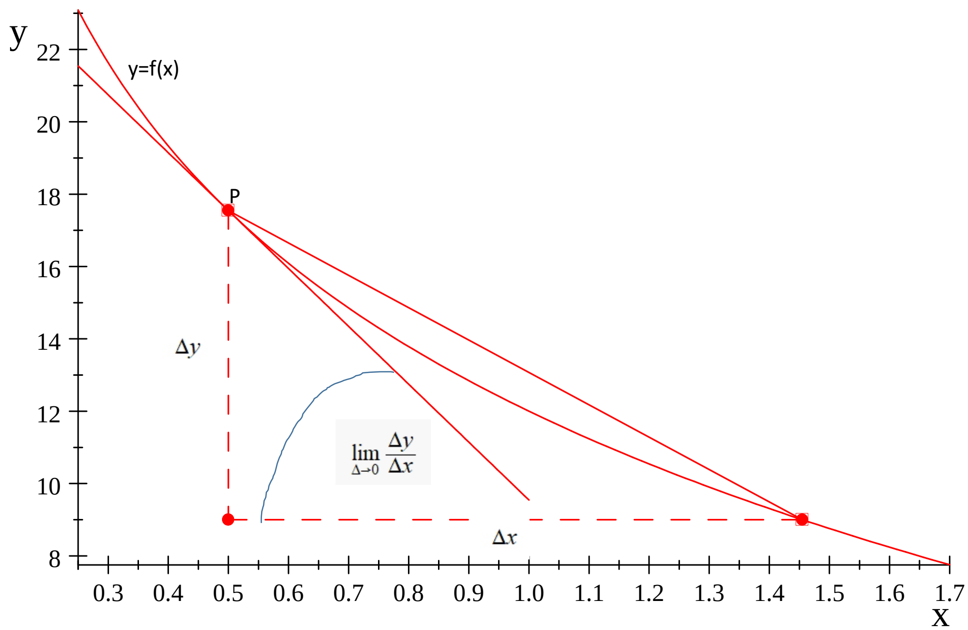


Figure 8: $\text{Lim}_{\Delta\to 0} m$

3. Sum-difference rule:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$$

4. Product rule:

$$\frac{d}{dx}[f(x)g(x)] = f(x)g\prime(x) + g(x)f\prime(x)$$

5. Quotient rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f\prime(x)g(x) - f(x)g\prime(x)}{[g(x)]^2}$$

6. Differentiation of natural log function:

$$\frac{d \ln x}{dx} = \frac{1}{x}.$$

7. Differentiation of the natural exponential function:

$$\frac{de^x}{dx} = e^x$$

8. Chain rule: If we have a function $z = f(y)$, where $y$ in turn is a function of $x$, $y = g(x)$, then:

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = f\prime(y)g\prime(x)$$

# 10 Finding a maximum

Imagine you are the owner of a firm that produces cloth, the quantity of which is signified by $C$. You take as exogenous the price of your output, $P_C$, and the price of labor, the only variable whose value you get to decide. Let $L_C$ denote the amount of labor you use in your production process, and $w$ the price of labor.

Production of cloth is given by the production function

$$C = \alpha (L_C)^\theta, \ \alpha > 0, \theta \in (0, 1).$$

Your profits are thus the following function of $L_C$:

$$\Pi_C = P_C C(L_C) - wL_C.$$

Your job is to choose $L_C$ so as to maximize $\Pi_C$. The value of $L_C$ that does the trick is found by taking the derivative of the function $\Pi_C$ with respect to $L_C$, setting that derivative equal to zero, and then solving for $L_C$. Lets put some

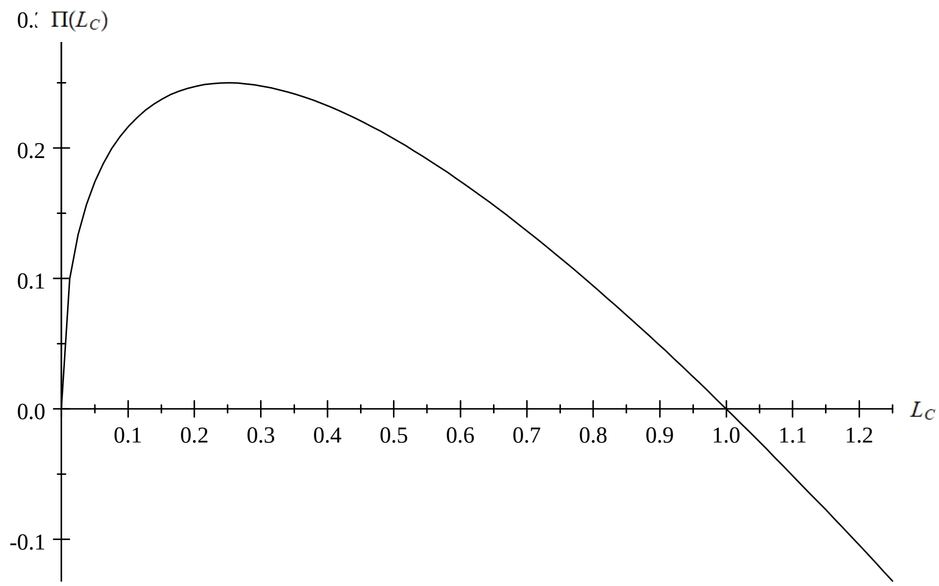


Figure 9: Profit function

numbers in for the parameters $\alpha$ and $\theta$, and numbers in for the values of the exogenous variables $P_C$ and $w$ :

$$\alpha = 1,\ \theta = \frac{1}{2},\ P_C = 1,\ w = 1.$$

The function $\Pi_C$ is thus:

$$\Pi_C = (L_C)^{\frac{1}{2}} - L_C.$$

It looks like this (Figure 9):

Its maximum is where $\frac{d\Pi_C}{dL_C} = 0$. The derivative function is

$$\frac{d\Pi_C}{dL_C} = \frac{1}{2}(L_C)^{-\frac{1}{2}} - 1.$$

Setting this to zero yields:

$$\begin{aligned} \frac{1}{2}(L_C)^{-\frac{1}{2}} &= 1; \\ (L_C)^{-\frac{1}{2}} &= 2. \end{aligned}$$

Hence, raising both sides of the equation to the power minus two ($-2$), we have that the value of $L_C$ that maximizes profits is:

$$L_C = \frac{1}{2^2} = \frac{1}{4}$$

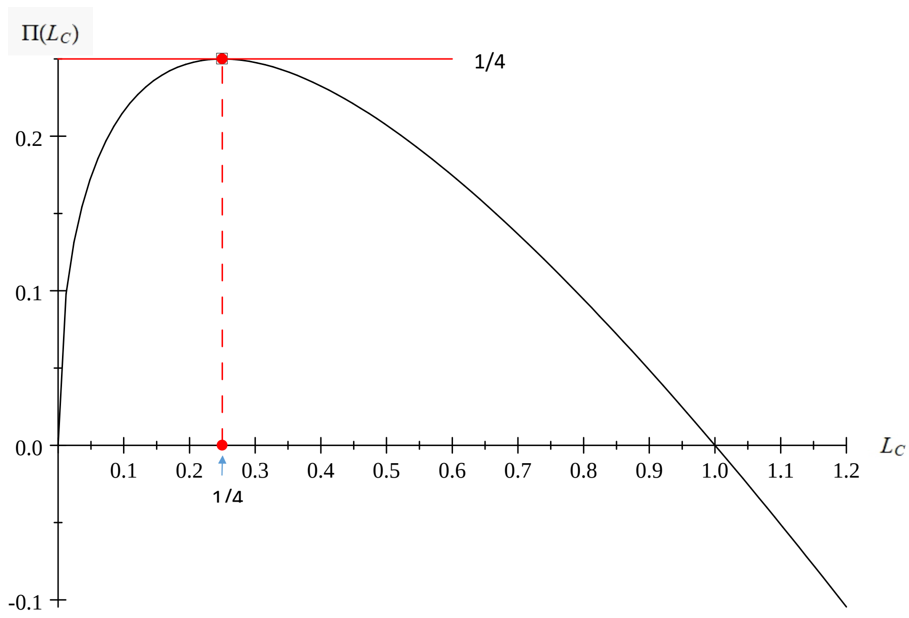


Figure 10: Depicting the maximum

What are profits at this maximum?

$$\Pi_C = \left(\frac{1}{4}\right)^{\frac{1}{2}} - \frac{1}{4} = \frac{1}{4}.$$

The picture: (Figure 10)

# 11 Partial differentiation

Consider a function

$$y = f(x_1, x_2).$$

where there is no independent functional relationship between $x_1$ and $x_2$. The partial derivative $\frac{\partial f}{\partial x_!}$ is found by treating $x_2$ as a constant and taking the derivative of $f$ with respect to $x_1$. For example:

$$U = \frac{xy}{x+y};$$

$$\frac{\partial U}{\partial x} = \frac{y(x+y) - xy}{(x+y)^2}.$$

Analogous steps give $\frac{\partial f}{\partial x_2}$.

## 12 Differentials

We can think of $\frac{dy}{dx}$ as the ratio of two quantities, $dy$ and $dx$. Thus, for example, if $y = 2x^{\frac{1}{2}}$, then

$$dy = [x^{-\frac{1}{2}}]dx.$$

The interpretation of this is that $dy$ is the change in the variable $y$ arising from the change in the value of the variable $x$.

## 13 Total differentials

We can extend the notion of a differential to a multivariate function. Let $U$ be a function of $x$ and $y$ :

$$U = f(x, y).$$

To find the total change in $U$ from a small change in $x$ and a small change in $y$, we have:

$$dU = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Example:

$$U = [\alpha x^{-\rho} + (1-\alpha)y^{-\rho}]^{-(\frac{1}{\rho})}$$

where $\rho$ and $\alpha$ are parameters that obey the following restrictions:

$$\begin{aligned} 0 &\leq \alpha \leq 1; \\ \rho &\neq 0; \\ -1 &< \rho < +\infty. \end{aligned}$$

$$dU = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ : First, define $g(x, y) = \alpha x^{-\rho} + (1-\alpha)y^{-\rho}$ (remember $y$ is treated as a constant when we find $\frac{\partial g}{\partial x}$, and $x$ is treated as a constant when we find $\frac{\partial g}{\partial y}$). So, $f(g(x, y)) = g(x, y)^{-\left(\frac{1}{\rho}\right)}$. Hence,

$$\frac{\partial f}{\partial g} = -\left(\frac{1}{\rho}\right) g(x, y)^{-(\frac{1}{\rho})-1}$$

By the chain rule:

$$\frac{\partial f}{\partial g}\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x}$$

Now,

$$\frac{\partial g}{\partial x} = -\alpha\rho x^{-\rho-1}$$

so

$$\frac{\partial f}{\partial x} = \overbrace{\left[-\left(\frac{1}{\rho}\right) g(x,y)^{-\left(\frac{1}{\rho}\right)-1}\right]}^{\frac{\partial f}{\partial g}} \overbrace{\left[-\alpha\rho x^{-\rho-1}\right]}^{\frac{\partial g}{\partial x}}$$

Now for $\frac{\partial f}{\partial y}$. By the chain rule

$$\frac{\partial f}{\partial g}\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y}.$$

Hence,

$$\frac{\partial f}{\partial y} = \left[-\left(\frac{1}{\rho}\right) g(x,y)^{-\left(\frac{1}{\rho}\right)-1}\right]\left[-(1-\alpha)\rho y^{-\rho-1}\right]$$

# 14 Finding the slope of an indifference curve for preferences represented by above function

Set $dU = 0$ :

$$\frac{\partial f}{\partial x}dx = -\frac{\partial f}{\partial y}dy$$

Or,

$$\frac{dy}{dx} = \frac{\frac{\partial f}{\partial x}}{-\frac{\partial f}{\partial y}}$$

Substituting from above:

$$\begin{aligned}\frac{dy}{dx} &= -\frac{\left[-\left(\frac{1}{\rho}\right) g(x,y)^{-\left(\frac{1}{\rho}\right)-1}\right]\left[-\alpha\rho x^{-\rho-1}\right]}{\left[-\left(\frac{1}{\rho}\right) g(x,y)^{-\left(\frac{1}{\rho}\right)-1}\right]\left[-(1-\alpha)\rho y^{-\rho-1}\right]} \\ &= -\frac{[\alpha]\,[x]^{-\rho-1}}{[1-\alpha]\,[y]^{-\rho-1}} \\ &= -\frac{[\alpha]}{[1-\alpha]}\left(\frac{y}{x}\right)^{\rho+1}\end{aligned}$$

For example, if $\alpha = \frac{1}{2}$ and $\rho = 1$, then $\frac{\alpha}{1-\alpha} = 1$ and $-\rho - 1 = -2$, so

$$\frac{dy}{dx} = -\left(\frac{y}{x}\right)^2$$

## 14.1 The power of the parametric solution

The indifference-curve slope depends on the parameter $\rho$. We can easily see how the features of this slope vary for different values of $\rho$. For some examples

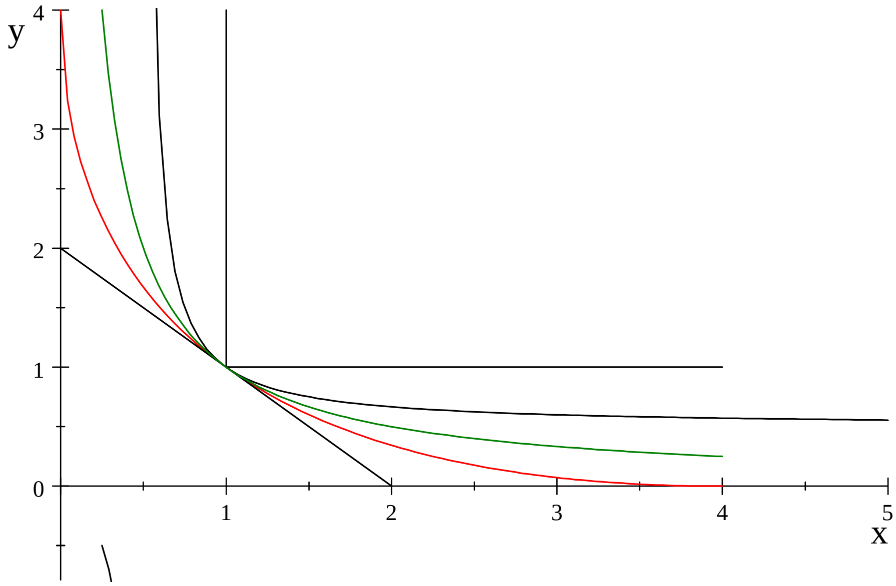


Figure 11: $\rho = -1,\ -\frac{1}{2},\ 0, 1, \infty$

consider the following permissible values of $\rho$ :

$$
\begin{aligned}
\lim_{\rho \to 0} &: \quad \frac{dy}{dx} = -\frac{[\alpha]}{[1-\alpha]}\left(\frac{y}{x}\right); \\
\rho &= -1 : \ \frac{dy}{dx} = -\frac{[\alpha]}{[1-\alpha]}. \\
\rho &= \infty : \\
\frac{dy}{dx} &= -\infty \ if \ y > x; \ \frac{dy}{dx} = 0 \ if \ y < x.
\end{aligned}
$$

The values $\rho = -1$ and $\rho = \infty$ illustrate the two extreme types of indifference curves: those depicting perfect substitutability between $x$ and $y$, and those depicting perfect complementarity between $x$ and $y$. The value $\rho = 0$ (to be precise, the limit as $\rho \to 0$) yields indifference curves that are associated with linear income expansion paths.

Figure 11 presents pictures of these indifference curves for $\rho = -1,\ -\frac{1}{2}, 0, 1, \infty$.