Diego's theorem for nuclear implicative semilattices

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joint work with
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The study of implicative semilattices was pioneered by Nemitz in 1960s. Since then they have been studied rather extensively (also under the name of Brouwerian semilattices).

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Diego worked in the $\to$-signature, but local finiteness is preserved if we add $\wedge$ (and even 0) to the signature.

Of course, we cannot further add $\vee$ to the signature as the variety of Heyting algebras is not locally finite.

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Diego's theorem also plays an important role in the algebraic proof of Fine's 1984 result that all subframe logics above **K4** have the fmp.

It is also critical for an algebraic account of Zakharyaschev's canonical formulas which provide a uniform axiomatization of all intermediate logics and logics above **K4**.

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A nuclear implicative semilattice is a pair $\mathfrak{A} = (A, j)$ where A is an implicative semilattice and j is a nucleus on A.

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3. In logic, nuclei model the so-called lax modality. The corresponding Lax Logic is an intuitionistic modal logic with interesting links to computer science since lax modality is used to reason about formal verification of hardware.
4. Nuclei also provide a unifying tool for different semantics of intuitionistic logic.

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Our technique is based on duality theory and the construction of universal models from modal logic.

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A map $f : P \to Q$ between posets is called a bounded morphism (or p-morphism) if

1. $p \leqslant p' \Rightarrow f(p) \leqslant f(p')$;
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Since there are more implicative semilattice homomorphisms, we need to generalize the above notion.

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Köhler duality (1981): The category of finite implicative semilattices and implicative semilattice homomorphisms is dually equivalent to the category of finite posets and Köhler maps.

Generalizations

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Priestley-like duality and the connections with the previous approaches was developed jointly with Ramon Jansana (2008,2013).

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Up to isomorphism, each nucleus on a finite implicative semilattice arises this way.

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What does it take for $f^*(j_T U) = j_S f^*(U)$?

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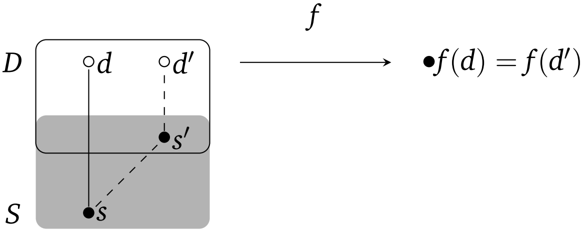
1. $f^{-1}(T) = D \cap S$,
2. if $s \in S$, $d \in D$, and $s \leqslant d$, then there are $s' \in S \cap D$ and $d' \in D$ such that $s \leqslant s' \leqslant d'$ and $f(d) = f(d')$.



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Given two S-posets (P, S) and (Q, T), call a Köhler morphism $f : P \to Q$ an S-morphism if it satisfies the above two conditions.

Theorem: The category of finite nuclear implicative semilattices and nuclear homomorphisms is dually equivalent to the category of finite S-posets and S-morphisms.

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Conversely, each coloring gives rise to the n-tuple $U_1, \ldots, U_n \in \mathsf{Up}(X)$ given by

$$U_i = \{x \in X \mid i \in c(x)\}$$

for each $i = 1, \ldots, n$.

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There is a one-to-one correspondence between finite irreducible models and finite n-generated nuclear implicative semilattices. It is obtained by associating with each finite irreducible model $\mathfrak{M} = (X, S, c)$ the finite nuclear implicative semilattice $(\mathsf{Up}(X), j_S)$ generated by $U_1, \ldots, U_n$ where $U_i = \{x \in X \mid i \in c(x)\}$ for each i.

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Coloring Theorem: A finite model $\mathfrak{M} = (X, S, c)$ is irreducible iff the following two conditions are satisfied:

1. $c(x) = c(\nabla x) \Rightarrow x \in S \ \& \ \nabla x \nsubseteq S$,
2. $\nabla x = \nabla y \ \& \ c(x) = c(y) \ \& \ (x \in S \Leftrightarrow y \in S) \Rightarrow x = y$.

Universal models

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A model $\mathfrak{L} = (X, S, c)$ is n-universal provided for every finite irreducible model $\mathfrak{M} = (Y, T, c)$ there is a unique embedding of posets $e : Y \to X$ such that $e(Y)$ is an upset of X, $e^{-1}(S) = T$, and $c(e(y)) = c(y)$ for all $y \in Y$.

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The n-universal model $\mathfrak{L}$ is then the union of the models $\mathfrak{L}_k$.

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1. X_{k+1} is obtained by adding for each antichain $\alpha \subseteq X_k$ with $\alpha \nsubseteq X_{k-1}$ the following new elements to X_k:
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    The partial order on X_{k+1} extends the partial order on X_k so that the covers of the elements of $X_{k+1} \setminus X_k$ are defined as $\nabla r_{\alpha,\sigma} = \nabla s_{\alpha,\sigma} = \alpha$.

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Finally, we define $\mathfrak{L} = (X, S, c)$ by setting

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Key results

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Theorem 2: The n-universal model $\mathfrak{L}$ is finite.

1-universal model

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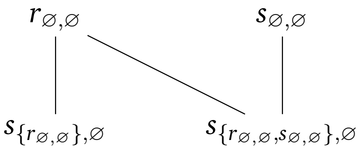

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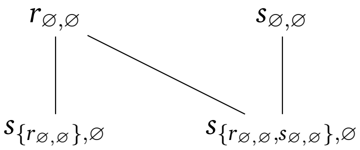

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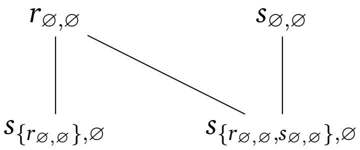

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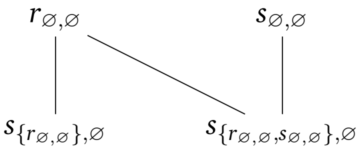

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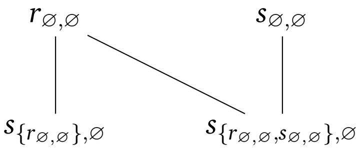

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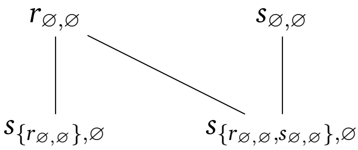

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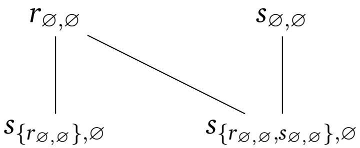

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Rule (3) gives an element in S with empty color for each antichain not contained in S. There are two such antichains: $\{r_{\varnothing,\varnothing}\}$ and $\{r_{\varnothing,\varnothing}, s_{\varnothing,\varnothing}\}$. Therefore, the second layer is made of the two elements $s_{\{r_{\varnothing,\varnothing}\},\varnothing}$ and $s_{\{r_{\varnothing,\varnothing},s_{\varnothing,\varnothing}\},\varnothing}$.

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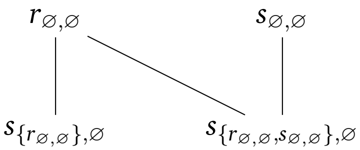

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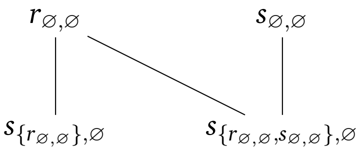

The third layer is empty because Rules (1) and (2) do not apply since every element has empty color, and Rule (3) does not apply as every antichain that is not contained in S is contained entirely in the first layer.

Free cyclic nuclear implicative semilattice

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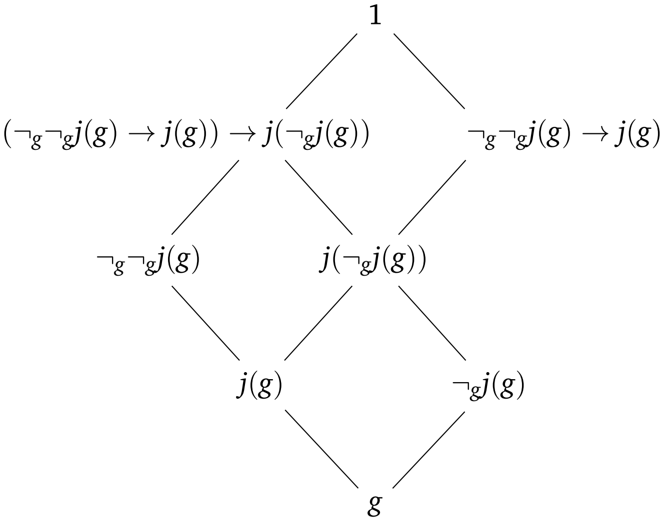

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Thank You!