Priestley duality for MV-algebras: A new perspective

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6. Splitting of the operation $\ominus$ on duals to obtain a more expressive environment.
7. Specializing this to MV-algebras.

Part I:
Priestley duality and canonical extensions

Priestley spaces

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$$\hat{a} = \{x \in X(L) : a \in x\}.$$

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$$R(x, y, z) \iff x \subseteq \downarrow\{a \cdot b : a \in y, b \in z\}.$$

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Canonical extensions are a view of Priestley duality that exploits this.

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Every doubly algebraic distributive lattice C is determined by its poset $M^{\infty}(C)$ of completely meet-irreducible elements.

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Definition:

The *canonical extension* of a bounded distributive lattice L is a doubly algebraic lattice L^δ that contains L as a compact, separating sublattice.

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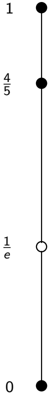

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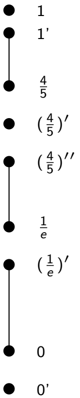

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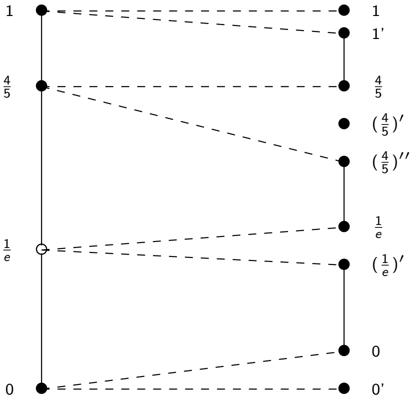

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And here the fact that the points in the dual space are idealized meet-irreducibles is much more explicit.

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$$I_{(-)}\colon X \to \mathrm{PrIdl}(L),$$

$$F_{(-)}\colon X \to \mathrm{PrFil}(L),$$

$$\mu\colon X \to M^\infty(L^\delta),$$

$$\nu\colon X \to J^\infty(L^\delta),$$

Connecting the different presentations

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These isomorphisms are connected via

$$I_x = L \cap {\downarrow}\mu(x),$$

$$\mu(x) = \bigvee I_x,$$

$$F_x = L \cap {\uparrow}\nu(x),$$

$$\nu(x) = \bigwedge F_x,$$

$$\kappa(\nu(x)) = \mu(x),$$

$$F_x^{\mathsf{c}} = I_x.$$

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- the $\bigvee$-closure of L in L^δ the *open* elements and denote them by $O(L)$.

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Operations of higher arity can be extended by using the fact that $(A \times B)^\delta \cong A^\delta \times B^\delta$.

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The fact that the non-lattice operations of MV-algebras aren't smooth is fundamental to the difficulty of MV.

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We call the restriction of $f^{\delta\sharp}$ to $M^\infty(B^\delta)$ the *dual* of f.

Part II:

MV-algebras, $\ominus$-algebras, and the duality

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Instead of the residuated lattice signature, it's convenient for us to work with the MV-algebra addition/subtraction. In this language...

Definition:

An *MV-algebra* is an algebraic structure $(A, \oplus, 0, \neg)$ such that:

1. $(A, \oplus, 0)$ is a commutative monoid,
2. $\neg\neg x = x$ for all $x \in A$,
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The last condition is often called (MV6).

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define the join operation $\vee$ of a lattice, so MV-algebras are lattice-ordered.

If we set $x \ominus y = \neg(\neg x \oplus y)$, we also obtain an operation that satisfies the (co-)residuation law

$$x \leq y \oplus z \iff x \ominus z \leq y.$$

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This fact is behind a lot of the complexity. For example, Cabrer and Cignoli gave a duality for many algebras in the vicinity but could only find complicated second-order conditions that dualize (MV6).

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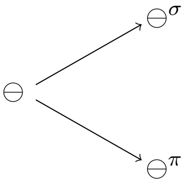

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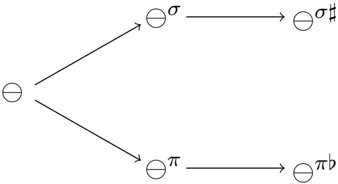

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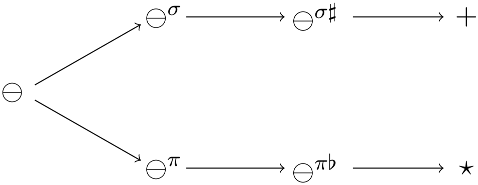

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For each completely join-irreducible $j \in A^\delta$, the map

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And we may show that this map has an adjoint $\ominus^{\pi\flat}$ determined by the property that for all $j \in J^\infty(A^\delta)$, $v \in A^\delta$, $u \in [0, \neg^\delta j]$,

$$u \le v \ominus^\pi j \iff u \ominus^{\pi\flat} j \le v.$$

The σ-extension of $\ominus$

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Similar remarks show that $\ominus^\sigma$ has an adjoint determined for all $u, v \in A^\delta$ and $m \in M^\infty(A^\delta)$ by

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$\ominus^{\sigma\sharp}$ manifests as a partial operation that we call $+$, and $\ominus^{\pi\flat}$ as an operation that we call $\star$.

Abstract $\ominus$-spaces

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6. for any $x \in X$, the image of the left translation $y \mapsto x + y$ is a totally-ordered subset of $\uparrow x$, and moreover this function has an upper adjoint.

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3. for all $x \in X_1$ and $z \in X_2$, if $(f(x), z) \in \mathrm{dom}(+_2)$, then there exists $w' \in X_1$ such that $(x, w') \in \mathrm{dom}(+_1)$, $z \leq f(w')$, and $f(x +_1 w') = f(x) +_2 z$.

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(v) If there exists $w \not\leq y$ such that $z + w \leq x \star y$, then $z \leq x$.

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$$w \not\leq y \text{ and } z + w \leq x \star y \implies z \leq x.$$

This axiom did not even have a first-order equivalent in previously-known dualities, but in the language of these two partial operations it is transparent.

An example: The Chang algebra

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Consider the Chang MV-algebra as the rotation of the cancellative hoop $\{0, -1, -2, \dots\}$.

The following describes $+$ and $\star$ when defined:

$$\downarrow(0, a) + \downarrow(0, b) = \downarrow(0, b) + \downarrow(0, a) = \downarrow(0, a + b)$$

$$\mathcal{C} + \downarrow(0, a) = \downarrow(0, a) + \mathcal{C} = \mathcal{C}, \text{ and}$$

$$\downarrow(0, a) + \downarrow(1, b) = \downarrow(1, b) + \downarrow(0, a) = \downarrow(1, a - b) \text{ for } a < b,$$

$$\downarrow(0, a) \star \downarrow(0, b) = \downarrow(0, b) \star \downarrow(0, a) = \downarrow(0, a + b - 1),$$

$$\downarrow(0, a) \star \mathcal{C} = \mathcal{C} \star \downarrow(0, a) = \mathcal{C},$$

$$\downarrow(0, a) \star \downarrow(1, b) = \downarrow(1, a - b - 1) \text{ if } b < a, \text{ and}$$

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One could similarly compute the duals of quotients at other prime MV-ideals.

These quotients are the stalks in the sheaf representation of $F_{MV}(1)$ (Gehrke–van Gool–Marra 2014) over its space of prime MV-ideals.

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So examples like the above give you the dual of $F_{MV}(1)$ (and even $F_{MV}(n)$).

Thank you!

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