On commutative (pseudo-) BCK-algebras

Jan Kühr

Joint work with Petr Ševčík

- BCK-algebras are the $\{\to, 1\}$-subreducts of commutative integral residuated lattices.
- Pseudo-BCK-algebras or biresiduation algebras are the $\{\backslash, /, 1\}$-subreducts of integral residuated lattices.
- Pseudo-ŁBCK-algebras or cone algebras are the $\{\backslash, /, 1\}$-subreducts of integral GMV-algebras, i.e., integral residuated lattices satisfying $(x/y)\backslash x = y/(x\backslash y)$.
- Commutative pseudo-BCK-algebras are pseudo-BCK-algebras satisfying $(x/y)\backslash x = y/(x\backslash y)$. Not BCK-algebras.

A pseudo-BCK-algebra or a biresiduation algebra is an algebra $\mathbf{A} = (A, \backslash, /, 1)$ of type $(2,0,0)$ satisfying the equations

$$\big((x\backslash z)/(y\backslash z)\big)/(x\backslash y) = 1, \quad (y/x)\backslash\big((z/y)\backslash(z/x)\big) = 1,$$
$$1\backslash x = x, \quad x/1 = x,$$
$$x\backslash 1 = 1, \quad 1/x = 1,$$

and the quasi-equation

$$x\backslash y = 1 \quad \& \quad y\backslash x = 1 \quad \Rightarrow \quad x = y.$$

A BCK-algebra is a pseudo-BCK-algebra satisfying $x\backslash y = y/x$.
The underlying poset is defined by

$$x \leqslant y \quad \text{iff} \quad x\backslash y = 1 \quad \text{iff} \quad y/x = 1.$$

A commutative pseudo-BCK-algebra is a pseudo-BCK-algebra satisfying the equation

$$(x/y)\backslash x = y/(x\backslash y).$$

In this case the underlying poset is a join-semilattice where

$$x \vee y = (x/y)\backslash x = y/(x\backslash y).$$

A pseudo-ŁBCK-algebra (or a cone algebra) is a commutative pseudo-BCK-algebra satisfying the equation

$$(x\backslash y) \vee (y\backslash x) = 1.$$

The class of commutative pseudo-BCK-algebras is a variety – congruence distributive and 1-regular.

Interval algebras: In any commutative pseudo-BCK-algebra $\mathbf{A} = (A, \backslash, /, 1)$, all intervals $[a, 1] \subseteq A$ are subuniverses of $\mathbf{A}$. In fact, the pseudo-BCK-algebra $[\mathbf{a}, \mathbf{1}] = ([a, 1], \backslash, /, 1)$ is the $\{\backslash, /, 1\}$-reduct of the bounded GMV-algebra $[\mathbf{a}, \mathbf{1}]^+ = ([a, 1], \vee, \wedge_a, \cdot_a, \backslash, /, a, 1)$, where

$$x \cdot_a y = \big((a/y)/x\big)\backslash a = a/\big(y\backslash(x\backslash a)\big),$$
$$x \wedge_a y = \big((a/x) \vee (a/y)\big)\backslash a = a/\big((x\backslash a) \vee (y\backslash a)\big),$$

for all $x, y \in [a, 1]$.

The algebras $\mathbf{C}_n$: For any integer $n \geqslant 2$, let

$$C_n = \left\{0, \tfrac{1}{n-1}, \ldots, \tfrac{n-2}{n-1}, 1\right\}.$$

The algebra $\mathbf{C}_n = (C_n, \to, 1)$ with

$$x \to y = \min\{1, 1 - x + y\}$$

is a linearly ordered ŁBCK-algebra. Up to isomorphism, $\mathbf{C}_n$ is the only n-element linearly ordered ŁBCK-algebra.

Komori (1978): The varieties of ŁBCK-algebras are

$$\mathscr{T} \subset \mathscr{C}_2 \subset \cdots \subset \mathscr{C}_n \subset \cdots \subset \bigvee_{n \geqslant 2} \mathscr{C}_n,$$

where $\mathscr{C}_n = \mathrm{V}(\mathbf{C}_n)$.

Kowalski (1995): The covers of $\mathscr{C}_2$ is the lattice of varieties of BCK-algebras are $\mathscr{C}_3$ and $\mathrm{V}(\mathbf{H}_3)$, where $\mathbf{H}_3$ is $(\{0, 1/2, 1\}, \to, 1)$ with $1/2 \to 0 = 0$.

The algebras $\mathbf{C}_{n,\kappa}$: For any integer $n \geqslant 3$ and any cardinal $\kappa \geqslant 1$, the bottom element in $\mathbf{C}_n$ is replaced with κ minimal elements. In particular, we let $\mathbf{D}_n = \mathbf{C}_{n,2}$.

There are $2^{\aleph_0}$ varieties of commutative BCK-algebras.

For any $\emptyset \neq N \subseteq \{3, 4, 5, \dots\}$, let $\mathscr{D}_N = \mathrm{V}(\{\mathbf{D}_n \colon n \in N\})$.
We know that $\mathbf{D}_m$ satisfies

- $x^n \to y = x^{n-1} \to y$ iff $m \leqslant n$,
- $((x \to y) \vee (y \to x))^{n-2} \to y \leqslant (x \to y) \vee (y \to x)$ iff $m \geqslant n$.

Here $u^k \to v$ means $u \to (\cdots \to (u \to v) \dots)$.
Then ..., whence $\mathbf{D}_m \in \mathscr{D}_N$ iff $m \in N$.

A commutative pseudo-BCK-algebra $\mathbf{A}$ is a pseudo-ŁBCK-algebra iff it satisfies the following condition, for all $a, b, c \in A$:

$$\text{if } a \vee b \leqslant c \text{ and } c \backslash a = c \backslash b, \text{ then } a = b.$$

We say that (a, b, c) is a forbidden triple in $\mathbf{A}$ if

$$\text{if } a \vee b \leqslant c,\ c \backslash a = c \backslash b \text{ and } a \neq b.$$

In this case, a, b don't have a common lower bound.

We say that a commutative pseudo-BCK-algebra $\mathbf{A}$ is sectionally of finite length if every interval $[a, 1]$ is of finite length (as a lattice).

If a commutative pseudo-BCK-algebra $\mathbf{A}$ is sectionally of finite length, then $\mathbf{A}$ is a BCK-algebra.

Let $\mathbf{A}$ be a commutative BCK-algebra that is not an ŁBCK-algebra and let (a, b, c) be a forbidden triple in $\mathbf{A}$. Then

- $a \to b = b \to a$;
- (a, b, z) is a forbidden triple iff $z \in [a \vee b, a \to b]$;
- for every $x \in [a, a \vee b]$ there is a unique $y \in [b, a \vee b]$ such that (x, y, z) is a forbidden triple for every $z \in [a \vee b, a \to b]$.

Let $\mathbf{A}$ be a commutative BCK-algebra sectionally of finite length. Then $\mathbf{A}$ is *not* an ŁBCK-algebra if and only if $\mathbf{A}$ contains a subalgebra isomorphic to $\mathbf{D}_n$ for some integer $n \geqslant 3$.

Suppose that $\mathbf{A}$ is not an ŁBCK-algebra; then it has a forbidden triple, say (a, b, c). We may assume that the element a is maximal in the sense that whenever (x, y, z) is a forbidden triple such that $x \geqslant a$, then $x = a$. Then:

- a, b are covered by $a \vee b$ and $a \to b = b \to a$ is a coatom;
- $B = \{a, b\} \cup [a \vee b, 1]$ is a subuniverse of $\mathbf{A}$;
- $a \to b = b \to a$ is the only coatom in $[a \vee b, 1]$;
- $[a \vee b, 1]$ is a finite chain;
- $\mathbf{B} \cong \mathbf{D}_n$ for some $n \geqslant 3$.

The covers of the variety $\mathscr{C}_{n,p}$ (for $n \geqslant 3$, $p \geqslant 1$) in the lattice of varieties of commutative BCK-algebras are the varieties:

- $\mathscr{C}_{n,p} \vee \mathscr{C}_{n+1}$,
- $\mathscr{C}_{n,p+1}$,
- if $n \geqslant 4$, then $\mathscr{C}_{n,p} \vee \mathscr{D}_k$ for every $k \in \{3, \ldots, n-1\}$.

Every variety of commutative BCK-algebras that properly contains $\mathscr{C}_{n,p}$ contains at least one of these covers.

Let $\mathscr{K}$ be a variety of commutative BCK-algebras such that $\mathscr{C}_{n,p} \subsetneq \mathscr{K}$.

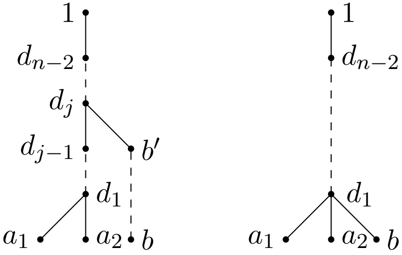
Case 1 – There is an ŁBCK-algebra in $\mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_{n,p})$:
If $\mathbf{A} \in \mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_{n,p})$, then $\mathbf{A}$ is linearly ordered and $|A| \geqslant n+1$, whence $\mathbf{C}_{n+1}$ is isomorphic to a subalgebra of $\mathbf{A}$, and so $\mathscr{C}_{n,p} \vee \mathscr{C}_{n+1} \subseteq \mathscr{K}$.

Case 2 – There is no ŁBCK-algebra in $\mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_{n,p})$:
If $\mathbf{A} \in \mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_{n,p})$, then for every $e \in A$, the subalgebra $[\mathbf{e}, \mathbf{1}]$ is a linearly ordered ŁBCK-algebra and $|[e, 1]| \leqslant n$. Hence $\mathbf{A}$ is sectionally of finite length and contains a subalgebra isomorphic to $\mathbf{D}_m$ for some $m \leqslant n$.

Case 2a – Some algebra in $\mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_{n,p})$ has a subalgebra isomorphic to $\mathbf{D}_m$ for some $m < n$: Then $\mathscr{C}_{n,p} \vee \mathscr{D}_m \subseteq \mathscr{K}$.

Case 2b – No algebra in $\mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_{n,p})$ contains a subalgebra isomorphic to $\mathbf{D}_m$ for $m < n$: If $\mathbf{A} \in \mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_{n,p})$, then $\mathbf{A}$ has a subalgebra $\mathbf{B} \cong \mathbf{D}_n$ with universe $B = \{a_1, a_2, d_1, \dots, d_{n-2}, 1\}$.

For $p = 1$, $\mathbf{D}_n \in \mathrm{Si}(\mathscr{K} \setminus \mathscr{C}_n)$ and so $\mathscr{C}_{n,2} = \mathscr{D}_n \subseteq \mathscr{K}$.
For $p \geqslant 2$, $\mathbf{D}_n \in \mathscr{C}_{n,p}$ and so $\mathbf{B}$ is a proper subalgebra of $\mathbf{A}$:



It follows that $\mathbf{A} \cong \mathbf{C}_{n,\kappa}$ for some $\kappa \geqslant p+1$, whence $\mathscr{C}_{n,p+1} \subseteq \mathscr{K}$.

The variety $\mathscr{C}_{n,p}$ (for $n \geqslant 3$, $p \geqslant 2$) is axiomatized, relative to commutative BCK-algebras, by the equations

$$x^n \to y = x^{n-1} \to y,$$
$$\big((x \to y) \vee (y \to x)\big)^{n-2} \to y \leqslant (x \to y) \vee (y \to x),$$
$$\bigvee_{0 \leqslant i \neq j \leqslant p} (x_i \to x_j) = 1.$$

Here $u^k \to v$ means $u \to (\cdots \to (u \to v) \dots)$.

A (normal) filter in a commutative pseudo-BCK-algebra $\mathbf{A}$ is $F \subseteq A$ such that:

- $1 \in F$;
- if $x, x\backslash y \in F$, then $y \in F$;
- if $x \in F$, then $\lambda_y(x) = (x\backslash y)\backslash y, \rho_y(x) = y/(y/x) \in F$.

The map $\theta \mapsto [1]_\theta$ is an isomorphism $\mathbf{Con}(\mathbf{A}) \cong \mathbf{Fi}(\mathbf{A})$.

Let $\mathscr{U}, \mathscr{V}$ be varieties of commutative pseudo-BCK-algebras. The Maltsev product $\mathscr{U} \circ \mathscr{V}$ is the class of those commutative pseudo-BCK-algebras $\mathbf{A}$ which have a filter $F \in \mathrm{Fi}(\mathbf{A})$ such that $\mathbf{F} \in \mathscr{U}$ and $\mathbf{A}/F \in \mathscr{V}$.

$\mathscr{C}_n \circ \mathscr{C}_n = \mathscr{C}_n$ for every $n \geqslant 1$, and $\mathscr{C}_{n,p} \circ \mathscr{C}_{n,p} = \mathscr{C}_{n,p}$ for every $n \geqslant 3, p \geqslant 2$.

The varieties of commutative (pseudo-) BCK-algebras form a non-commutative po-monoid.

Let $\mathbf{A}$ be a commutative pseudo-BCK-algebra. We say that an element $a \in A$ is idempotent if

$$a\backslash(a\backslash x) = a\backslash x \text{ for all } x \in A,$$

or equivalently,

$$a \vee (a\backslash x) = 1 \text{ for all } x \in A.$$

The idempotent elements of $\mathbf{A}$ form a subalgebra of $\mathbf{A}$, $\mathbf{I}(\mathbf{A})$. Moreover, $\mathbf{I}(\mathbf{A}) \in \mathscr{C}_2$.

We say that a commutative pseudo-BCK-algebra $\mathbf{A}$ has enough idempotents if $A = \bigcup_{a \in I(\mathbf{A})} [a, 1]$.

For any $a \in A$, both $[a, 1]$ and $a^{\perp} = \{x \in A : a \vee x = 1\}$ are subuniverses of $\mathbf{A}$.

If $a \in I(\mathbf{A})$, then

- $a \backslash x = x / a$ for all $x \in A$,
- the map $h_a : x \mapsto (a \backslash x, a \vee x)$ is an embedding of $\mathbf{A}$ into $\mathbf{a}^{\perp} \times [\mathbf{a}, \mathbf{1}]$.

We call $a \in I(\mathbf{A})$ central if h_a is an isomorphism.

The central elements of $\mathbf{A}$ form a subalgebra of $\mathbf{I}(\mathbf{A})$, $\mathbf{C}(\mathbf{A})$.

Suppose that a commutative pseudo-BCK-algebra $\mathbf{A}$ has enough idempotents and let

$\mathbf{M} = \prod_{e \in I(\mathbf{A})} [\mathbf{e}, \mathbf{1}]$ . . . pseudo-ŁBCK-algebra,

$\mathbf{M}^+ = \prod_{e \in I(\mathbf{A})} [\mathbf{e}, \mathbf{1}]^+$ . . . bounded GMV-algebra.

- The map $f \colon x \mapsto (x \vee e)_{e \in I(\mathbf{A})}$ is an embedding of $\mathbf{A}$ into $\mathbf{M}$.
- $\mathbf{A}$ is pseudo-ŁBCK-algebra.
- For every $x \in A$, $x \in I(\mathbf{A})$ iff $f(x) \in I(\mathbf{M})$.

Let's identify $\mathbf{A}$ with the subalgebra $f[\mathbf{A}]$ of $\mathbf{M}$, which is the reduct of $\mathbf{M}^+$. Let

$$L = \{a_1 \wedge \cdots \wedge a_n \colon a_i \in A\}.$$

Then $(L, \cdot, \backslash, /, 1)$ is a subalgebra of the GMV-algebra $(M, \cdot, \backslash, /, 1)$. In addition, the pseudo-ŁBCK-algebra $(L, \backslash, /, 1)$ has enough idempotents and A is an up-set in L.

1. If $0 \in L$, then $\mathbf{L} = (L, \cdot, \backslash, /, 0, 1)$ is a subalgebra of the bounded GMV-algebra $\mathbf{M}^+$. Let

$$\mathbf{K}_{\mathbf{A}} = \mathbf{L} \times \mathbf{C}_2^+.$$

Clearly, $f_{\mathbf{A}} \colon x \mapsto (f(x), 1)$ is an embedding of $\mathbf{A}$ into $\mathbf{K}_{\mathbf{A}}^- = \mathbf{L}^- \times \mathbf{C}_2$.

2. If $0 \notin L$, let

$$L^{\sim} = \{x^{\sim} \colon x \in L\} \quad \text{and} \quad L^{-} = \{x^{-} \colon x \in L\}$$

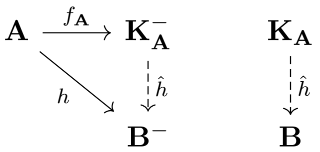
where $x^{\sim} = x \backslash 0$ and $x^{-} = 0/x$ are the negations in $\mathbf{M}^{+}$.
Then $L^{\sim} = L^{-}$, $L \cap L^{\sim} = \emptyset$ and

$$\mathbf{K}_{\mathbf{A}} = (L \cup L^{\sim}, \cdot, \backslash, /, 0, 1)$$

is a subalgebra of the bounded GMV-algebra $\mathbf{M}^{+}$.
Moreover, A and L are up-sets in $L \cup L^{\sim}$.
The natural embedding $f_{\mathbf{A}}$ of $\mathbf{A}$ into $\mathbf{K}_{\mathbf{A}}^{-} = (L \cup L^{\sim}, \backslash, /, 1)$ is f.

Let $\mathbf{A}$ be a pseudo-ŁBCK-algebra with enough idempotents. Let $\mathbf{K_A}$ and $f_{\mathbf{A}}$ be as before. Then:

- $\mathbf{A}$ is a subalgebra of $\mathbf{K_A^-}$ and it is a union of filters of $\mathbf{K_A}$;
- $I(\mathbf{A}) \subseteq I(\mathbf{K_A})$;
- for any bounded GMV-algebra $\mathbf{B}$ and any $\{\backslash, /, 1\}$-homomorphism $h\colon A \to B$ with the property that $h[I(\mathbf{A})] \subseteq I(\mathbf{B})$ there exists a unique $\{\cdot, \backslash, /, 0, 1\}$-homomorphism $\hat{h}\colon K_{\mathbf{A}} \to B$ such that $\hat{h} \circ f_{\mathbf{A}} = h$.



- Let $\mathbb{bGMV}$ be the category of bounded GMV-algebras with homomorphisms.
- Let $\mathbb{pLBCK}_{\mathbb{ei}}$ be the category of pseudo-ŁBCK-algebras with enough idempotents with homomorphisms that preserve idempotents.

The forgetful functor $U: \mathbb{bGMV} \to \mathbb{pLBCK}_{\mathbb{ei}}$ is adjoint; its co-adjoint $F: \mathbb{pLBCK}_{\mathbb{ei}} \to \mathbb{bGMV}$ is given as follows:

- For any $\mathbf{A} \in \mathbb{pLBCK_{ei}}$, $F(\mathbf{A}) \in \mathbb{bGMV}$ is the bounded GMV-algebra $\mathbf{K_A}$ constructed above.
- For any morphism $\mathbf{A} \xrightarrow{h} \mathbf{A}'$ in $\mathbb{pLBCK_{ei}}$, the morphism $F(\mathbf{A} \xrightarrow{h} \mathbf{A}')$ in $\mathbb{bGMV}$ is the morphism $\mathbf{K_A} \xrightarrow{F(h)} \mathbf{K_{A'}}$ which is given by

$$
\begin{array}{ccc}
\mathbf{A} & \xrightarrow{f_{\mathbf{A}}} & \mathbf{K}^{-}_{\mathbf{A}} \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \hat{g}} \\
\mathbf{A}' & \xrightarrow[f_{\mathbf{A}'}]{} & \mathbf{K}^{-}_{\mathbf{A}'}
\end{array}
\qquad\qquad
\begin{array}{c}
\mathbf{K_A} \\
\downarrow{\scriptstyle F(h) = \hat{g}} \\
\mathbf{K_{A'}}
\end{array}
$$

where g is $f_{\mathbf{A}'} \circ h$.

Let $\mathbf{A}$ and $\mathbf{B}$ be commutative pseudo-BCK-algebras satisfying condition $\mathscr{P}$. If $\mathbf{A} \cong \mathbf{a}^{\perp}$ for some $a \in C(\mathbf{B})$ and $\mathbf{B} \cong \mathbf{b}^{\perp}$ for some $b \in C(\mathbf{A})$, then $\mathbf{A} \cong \mathbf{B}$. Equivalently, if $\mathbf{A} \cong \mathbf{B} \times \mathbf{C}$ and $\mathbf{B} \cong \mathbf{A} \times \mathbf{D}$ where $\mathbf{C}, \mathbf{D}$ are bounded, then $\mathbf{A} \cong \mathbf{B}$.

The condition $\mathscr{P}$ can be:

1. the algebra is orthogonally σ-complete, i.e., if $\{x_i : i \in I\}$ is a countable subset s.t. $x_i \vee x_j = 1$ for all $i \neq j$, then $\bigwedge\{x_i : i \in I\}$ exists;
2. if $\{a_i : i \in I\}$ is a countable set of central elements s.t. $a_i \vee a_j = 1$ for all $i \neq j$, then $\bigwedge\{x_i : i \in I \cup \{0\}\}$ exists for every subset $\{x_i : i \in I \cup \{0\}\}$ s.t. (i) $x_i \geqslant a_i$ for all $i \in I$ and (ii) $x_0 \vee a_i = 1$ for all $i \in I$.

The latter condition is weaker and entails that whenever $\{a_i : i \in I\}$ is a countable set of central elements s.t. $a_i \vee a_j = 1$ for all $i \neq j$, then $a = \bigwedge\{a_i : i \in I\}$ exists and is central, and the algebra is isomorphic to $\mathbf{a}^{\perp} \times \prod_{i \in I} [\mathbf{a}_i, \mathbf{1}]$.

Let $\mathbf{A}$ be a commutative pseudo-BCK-algebra satisfying $\mathscr{P}$. Let $a_1, a_2 \in C(\mathbf{A})$, $a_1 \geqslant a_2$. If $\mathbf{A} \cong \mathbf{a}_2^\perp$, then $\mathbf{A} \cong \mathbf{a}_1^\perp$.

Let $X_i = a_i^\perp$ for $i = 1, 2$; then $X_1 \supseteq X_2$. Let f be an isomorphism $\mathbf{A} \cong \mathbf{X}_2 = \mathbf{a}_2^\perp$. Let $X_n = f[X_{n-2}]$ for each $n \geqslant 3$. We get

- $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$;
- $\mathbf{A} \cong \mathbf{X}_2 \cong \mathbf{X}_4 \cong \dots$ and $\mathbf{X}_1 \cong \mathbf{X}_3 \cong \dots$;
- $\mathbf{X}_{k-1} \cong \mathbf{X}_k \times [\mathbf{b}_k, \mathbf{1}]$, where the elements $b_1 = a_1, b_2, b_3, \dots$ form an orthogonal sequence of central elements;
- $[\mathbf{b}_1, \mathbf{1}] \cong [\mathbf{b}_3, \mathbf{1}] \cong \dots$ and $[\mathbf{b}_2, \mathbf{1}] \cong [\mathbf{b}_4, \mathbf{1}] \cong \dots$;
- $b = \bigwedge_{k \geqslant 1} b_k \in C(\mathbf{A})$ and $c = \bigwedge_{k \geqslant 2} b_k \in C(\mathbf{X}_1)$;
- $\mathbf{A} \cong \mathbf{b}^\perp \times \prod_{k \geqslant 1} [\mathbf{b}_k, \mathbf{1}] \cong \mathbf{b}^\perp \times [\mathbf{b}_1, \mathbf{1}] \times [\mathbf{b}_2, \mathbf{1}] \times [\mathbf{b}_1, \mathbf{1}] \times \dots$;
- $\mathbf{X}_1 \cong (\mathbf{X}_1 \cap \mathbf{c}^\perp) \times \prod_{k \geqslant 2} [\mathbf{b}_k, \mathbf{1}] \cong$ $(\mathbf{X}_1 \cap \mathbf{c}^\perp) \times [\mathbf{b}_2, \mathbf{1}] \times [\mathbf{b}_1, \mathbf{1}] \times [\mathbf{b}_2, \mathbf{1}] \times \dots$;
- $\mathbf{b}^\perp = \mathbf{X}_1 \cap \mathbf{c}^\perp$, hence $\mathbf{A} \cong \mathbf{X}_1$.

Thank you!