Semisimplicity, the excluded middle, and Glivenko theorems

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Semisimplicity: algebraic definition

An algebra $\mathbf{A}$ is **simple** if it has exactly two congruences ($\Delta_{\mathbf{A}} < \nabla_{\mathbf{A}}$).

An algebra $\mathbf{A}$ is **semisimple** if it is a subdirect product of simple algebras, or equivalently the intersection of maximal non-trivial congruences is $\Delta_{\mathbf{A}}$.

A variety K is **semisimple** if each algebra $\mathbf{A} \in \mathsf{K}$ is semisimple.

More generally, consider a **generalized quasivariety** K (axiomatized by a set of possibly infinite implications over a given set of variables).

A K**-congruence** on $\mathbf{A}$ is a congruence θ such that $\mathbf{A}/\theta \in \mathsf{K}$. Replacing congruences by K-congruences defines the **relative semisimplicity** of K.

Two problems: algebraic formulation

Given a variety K, describe its semisimple subvarieties.

Theorem (Kowalski). A variety K of $\mathrm{FL}_{\mathrm{ew}}$-algebras is semisimple if and only if K validates the equation $x \vee \neg(x^n) = 1$ for some n.

Theorem (Kowalski and Kracht). A variety K of BAOs is semisimple if and only if K validates $x \leq \Box\Diamond_n x$ and $\Box_{n+1} x = \Box_n x$ for some n.

(**Reminder.** $\mathrm{FL}_{\mathrm{ew}}$-algebras are bounded integral commutative residuated lattices. BAOs are Boolean algebras with a normal box operator ($\Box_n x := x \wedge \Box x \wedge \cdots \wedge \Box^n x$).)

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(**Reminder.** FL_{ew}-algebras are bounded integral commutative residuated lattices. BAOs are Boolean algebras with a normal box operator ($\Box_n x := x \wedge \Box x \wedge \cdots \wedge \Box^n x$).)

Given a variety K, describe its semisimple algebras.

Theorem. A Heyting algebra is semisimple iff it is Boolean ($x \vee \neg x = 1$).

Theorem. An S4-algebra is semisimple iff it is an S5-algebra ($x \leq \Box\Diamond x$).

(**Reminder.** S4-algebras are Boolean algebras with an interior operator.)

In contrast, semisimple algebras are not characterized equationally in FL_{ew}.

Semisimplicity and the LEM

The paper of Kowalski on the semisimple subvarieties of $\mathrm{FL}_{\mathrm{ew}}$ ends on the following note:

> *"[...] the argument used in the proof seems to have a certain generality to it, especially in view of its being a modification of [an analogous proof for Boolean algebras with operators]. It would be interesting to see what exactly that generality amounts to."*

We try to pinpoint what this generality consists in:

Theorem. A well-behaved logic or generalized quasivariety is semisimple if and only if it enjoys some form of the law of the excluded middle (LEM).

This general theorem can then be used to deduce the theorems of Kowalski and Kracht. In fact we can extend them to FL_{e}-algebras and Heyting algebras with operators. Our strategy will be (almost) purely syntactic throughout.

Semisimplicity: logical definition

The two problems can be rephrased in logical terms.

Recall that the **models** of a logic L are structures $\langle \mathbf{A}, F \rangle$ with $F \subseteq \mathbf{A}$ where each rule $\Gamma \vdash \varphi$ of L holds. We call F an L**-filter** on $\mathbf{A}$ if $\langle \mathbf{A}, F \rangle$ is a model of L.

(**Reminder.** $\Gamma \vdash \varphi$ holds in $\langle \mathbf{A}, F \rangle$ if $v[\Gamma] \subseteq F$ implies $v(\varphi) \in F$ for each $v \colon \mathbf{Fm} \to \mathbf{A}$.)

The lattice of L-filters on $\mathbf{A}$ corresponds to the lattice of K-congruences. We can therefore phrase the definition of semisimplicity in terms of L-filters.

A logic L is **semantically semisimple** if each model of L is semisimple, i.e. each L-filter on each algebra $\mathbf{A}$ is an intersection of simple L-filters on $\mathbf{A}$.

Restricting to $\mathbf{A} := \mathbf{Fm}$ yields **syntactic semisimplicity**. The L-filters on $\mathbf{Fm}$ are the **theories** of L, a theory T of L is **simple** if it is a maximal non-trivial theory of L, and it is **semisimple** if it is an intersection of simple theories.

Two problems: logical formulation

If L is **(weakly) algebraizable**, then L is semantically semisimple if and only if its algebraic counterpart K is semisimple as a generalized quasivariety.

This correspondence extends to the level of individual models: the simple (semisimple) models of L correspond to simple (semisimple) algebras in K.

This is in principle just window dressing, but it is a very convenient one. We could talk directly about equational consequence in K instead.

Two problems: logical formulation

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This is in principle just window dressing, but it is a very convenient one. We could talk directly about equational consequence in K instead.

Given a logic L, describe its semisimple axiomatic extensions.

For example, an axiomatic extensions of $\mathrm{FL}_{\mathrm{ew}}$ is semisimple if and only if $x \vee \neg(x^n)$ is a theorem for some n.

Given a logic L, describe its semisimple models.

For example, a model of IL is semisimple if and only if it validates $x \vee \neg x$.

Syntactic principles at play

We attack these problems armed with the following syntactic principles:

- deduction–detachment theorems (DDTs),
- inconsistency lemmas (ILs),
- dual inconsistency lemmas (dual ILs),
- the law of the excluded middle (LEM).

Each of these has a certain semantic import as well, of course.

These principles come in (parametrized) local and global forms. For the sake of simplicity, we will mostly ignore their parametrized forms.

Global ILs and their duals were introduced by Raftery (2013). He proved the equivalence between semisimplicity and the LEM in the global case.

Running examples: $\mathrm{FL}_{\mathrm{ew}}$ and K

Our running examples will be substructural and global modal logics.

K is the global (classical) modal logic of all Kripke frames $\langle W, R \rangle$.

S4 (S5) is the global modal logic of reflexive transitive (and symmetric) Kripke frames. It extends K by $\Box x \to x$ and $\Box x \to \Box\Box x$ (and $x \to \Box\Diamond x$).

(**Reminder.** Global means: if Γ holds at all worlds, then φ holds at worlds.)

$\mathrm{FL}_{\mathrm{ew}}$ is the logic of all bounded commutative integral residuated lattices:

$$\Gamma \vdash_{\mathrm{FL}_{\mathrm{ew}}} \varphi \iff \{1 \leq \gamma \mid \gamma \in \Gamma\} \vDash_{\mathrm{FL}_{\mathrm{ew}}} 1 \leq \varphi.$$

Commutativity and boundedness will be essential, integrality will not.

Running examples: Ł and $Ł_\infty$

Finitary Łukasiewicz logic Ł is the logic **all** MV-algebras:

$$\Gamma \vdash_{Ł} \varphi \iff \text{for each finite } \Delta \subseteq \Gamma \text{ the rule } \Delta \vdash \varphi \text{ holds in } [0,1]_{Ł}.$$

Infinitary Łukasiewicz logic $Ł_\infty$ is the logic of **semisimple** MV-algebras:

$$\Gamma \vdash_{Ł} \varphi \iff \text{the rule } \Gamma \vdash \varphi \text{ holds in } [0,1]_{Ł}.$$

Ł is an axiomatic extension of FL_{ew}. $Ł_\infty$ is the extension of Ł by the rule

$$\{x \to y^n \mid x \in \omega\} \vdash \neg x \vee y.$$

The same distinction can be made between the finitary and infinitary version of product fuzzy logic (Π and Π_∞) defined by $[0,1]_\Pi$.

Global DDTs

Intuitionistic logic satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{IL}} \psi \iff \Gamma \vdash_{\mathrm{IL}} \varphi \to \psi.$$

Global modal logic S4 satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{S4}} \psi \iff \Gamma \vdash_{\mathrm{S4}} \Box\varphi \to \psi.$$

The $(k+1)$-valued Łukasiewicz logic $Ł_{k+1}$ satisfies the equivalence

$$\Gamma, \varphi \vdash_{Ł_{k+1}} \psi \iff \Gamma \vdash_{Ł_{k+1}} \varphi^k \to \psi.$$

These are examples of **global deduction–detachment theorems (DDTs)**: there is some set of formulas $\mathrm{I}(x,y)$ such that

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}(\varphi, \psi).$$

Local DDTs

Global modal logic K satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{K}} \psi \iff \Gamma \vdash_{\mathrm{K}} \Box_n \varphi \to \psi \text{ for some } n \in \omega,$$

where we use the abbrevation

$$\Box_n \varphi := \varphi \wedge \Box \varphi \wedge \cdots \wedge \Box^n \varphi.$$

The logic $\mathrm{FL_{ew}}$ satisfies the equivalence

$$\Gamma, \varphi \vdash_{\mathrm{FL_{ew}}} \psi \iff \Gamma \vdash_{\mathrm{FL_{ew}}} \varphi^n \to \psi \text{ for some } n \in \omega.$$

These are examples of **local deduction–detachment theorems (DDTs)**: there is a family $\Phi(x,y)$ of sets of formulas $\mathrm{I}(x,y)$ such that

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}(\varphi, \psi) \text{ for some } \mathrm{I}(x,y) \in \Phi(x,y).$$

Global inconsistency lemmas (ILs)

Intuitionistic logic satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{IL}} \emptyset \iff \Gamma \vdash_{\mathrm{IL}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n).$$

Global modal logic S4 satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{IL}} \emptyset \iff \Gamma \vdash_{\mathrm{IL}} \neg\Box(\varphi_1 \wedge \cdots \wedge \varphi_n).$$

The $(k+1)$-valued Łukasiewicz logic $Ł_{k+1}$ satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{Ł_{k+1}} \emptyset \iff \Gamma \vdash_{Ł_{k+1}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)^n.$$

These are examples of **global inconsistency lemmas (ILs)**: for each $n \geq 1$ there is some set of formulas $\mathrm{I}_n(x_1, \ldots, x_n)$ such that

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L}} \emptyset \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}_n(\varphi_1, \ldots, \varphi_n).$$

Of course, for logics with a conjunction we can restrict to $n := 1$.

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Of course, for logics with a conjunction we can restrict to $n := 1$.

Local inconsistency lemmas (ILs)

Global modal logic K satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{K}} \emptyset \iff \Gamma \vdash_{\mathrm{K}} \neg\Box_n(\varphi_1 \wedge \cdots \wedge \varphi_n) \text{ for some } n \in \omega,$$

The logic $\mathrm{FL_{ew}}$ satisfies the equivalence

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{FL_{ew}}} \emptyset \iff \Gamma \vdash_{\mathrm{FL_{ew}}} \neg(\varphi_1 \wedge \cdots \wedge \varphi_n)^n \text{ for some } n \in \omega.$$

The infinitary Łukasiewicz logic Ł_∞ satisfies the same equivalence because Ł_∞ is compact and the finitary Łukasiewicz logic Ł inherits it from $\mathrm{FL_{ew}}$.

These are examples of **local inconsistency lemmas (ILs)**: for each $n \geq 1$ there is a family $\Phi_n(x_1, \ldots, x_n)$ of sets of formulas $\mathrm{I}_n(x_1, \ldots, x_n)$ such that

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_{\mathrm{L}} \emptyset \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}_n(\varphi_1, \ldots, \varphi_n) \text{ for some } \mathrm{I}_n \in \Phi_n.$$

Substitution swapping

The local DDT and IL can also be formulated in terms of substitutions.

Theorem. A logic L enjoys a (parametrized) local DDT if and only if for each (surjective) substitution σ and each theory T of L

$$\sigma(\varphi), T \vdash_{\mathrm{L}} \sigma(\psi) \implies \varphi, \sigma^{-1}[T] \vdash_{\mathrm{L}} \psi.$$

Theorem. A logic L enjoys a (parametrized) local IL if and only if for each (surjective) substitution σ and each theory T of L

$$\sigma(\varphi), T \vdash_{\mathrm{L}} \emptyset \implies \varphi, \sigma^{-1}[T] \vdash_{\mathrm{L}} \emptyset.$$

The right-to-left implication always holds in both cases.

The DDT and the IL

Logics which have a DDT and a *falsum* constant $\bot$ automatically have the IL. However, there is still some benefit to thinking of these two separately:

- a logic may enjoy a stronger IL than DDT,
- an infinitary logic may still be compact.

The finitary product fuzzy logic Π only has the local DDT (which it inherits from FL_{ew}) but it has a global IL:

$$\Gamma, \varphi_1, \ldots, \varphi_n \vdash_\Pi \emptyset \iff \Gamma \vdash_\Pi \neg(\varphi_1 \wedge \cdots \wedge \varphi_n).$$

The infinitary Łukasiewicz logic Ł is not finitary, but it is **compact**:

$$\Gamma \vdash_{Ł} \emptyset \implies \Delta \vdash_{Ł} \emptyset \text{ for some finite } \Delta \subseteq \Gamma.$$

The same holds for the infinitary product fuzzy logic Π_∞.

The infinitary DDT and IL

The IL looks more complicated than the DDT due to the parameter $n \geq 1$. This difference disappears if we allow for infinitary DDTs and ILs.

The **κ-ary local DDT** allows us to transport each tuple of $\alpha < \kappa$ formulas $\Phi := \{\varphi_i \mid i \in \alpha\}$ (α ordinal) from the left of the turnstile to the right:

$$\Gamma, \Phi \vdash_{\mathrm{L}} \psi \iff \Gamma \vdash_{\mathrm{L}} \mathrm{I}_\alpha(\varphi_0, \dots) \text{ for some } \mathrm{I}_\alpha(x_0, \dots) \in \Phi_\alpha(x_0, \dots).$$

These are usually not studied for at least two reasons:

- it does not make sense for the global DDT,
- this generalization is trivial if L is finitary.

We shall see that $Ł_\infty$ is an example of a logic which enjoys an infinitary local DDT of this kind despite not being finitary.

Global dual inconsistency lemmas (dual ILs)

Classical logic satisfies the equivalence

$$\Gamma \vdash_{\mathrm{CL}} \varphi \iff \Gamma, \neg\varphi \vdash_{\mathrm{CL}} \emptyset.$$

The $(k+1)$-ary Łukasiewicz logic satisfies the equivalence

$$\Gamma \vdash_{\text{Ł}_{k+1}} \varphi \iff \Gamma, \neg\varphi^{k} \vdash_{\text{Ł}_{k+1}} \emptyset.$$

The global modal logic S5 satisfies the equivalence

$$\Gamma \vdash_{\mathrm{S5}} \varphi \iff \Gamma, \neg\Box\varphi \vdash_{\mathrm{S5}} \emptyset.$$

These are **global dual ILs**: there is some set of formulas $\mathrm{J}(x)$ such that

$$\Gamma \vdash_{\mathrm{L}} \varphi \iff \Gamma, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \emptyset.$$

Local dual inconsistency lemmas (dual ILs)

The infinitary Łukasiewicz logic $Ł_\infty$ satisfies the equivalence

$$\Gamma \vdash_{Ł_\infty} \varphi \iff \Gamma, \neg\varphi^n \vdash_{Ł_\infty} \emptyset \text{ for each } n \in \omega.$$

This is a **local dual IL**: there is a family of sets of formulas $\Psi(x)$ such that

$$\Gamma \vdash_{\mathrm{L}} \varphi \iff \Gamma, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \emptyset \text{ for each } \mathrm{J}(x) \in \Psi(x).$$

Note the universal quantification (due to the set occuring on the left).

Fact. If L enjoys a local (global) IL w.r.t. a system of families Φ_n and some dual parametrized local IL, then it enjoys the dual local (global) IL w.r.t. Φ_1.

The dual IL and the classical DDT

Fact. Each logic enjoys both a local (global) IL and a dual IL of any form in fact enjoys the local (global) DDT. [Raftery calls this the **classical** DDT.]

Example. We illustrate this for $Ł_\infty$:

$$\begin{aligned}
\Gamma, \varphi \vdash_{Ł_\infty} \psi &\iff \Gamma, \varphi, \neg(\psi^n) \vdash_{Ł_\infty} \emptyset \text{ for each } n \in \omega \\
&\iff \text{for each } n \text{ there is } k \text{ such that } \Gamma \vdash_{Ł_\infty} (\varphi \wedge \neg\psi^n)^k \\
&\iff \text{there is } f\colon \omega \to \omega \text{ s.t. } \Gamma \vdash_{Ł_\infty} (\varphi \wedge \neg\psi^n)^{f(n)} \text{ for each } n \\
&\iff \Gamma \vdash_{Ł_\infty} \{(\varphi \wedge \neg\psi^n)^{f(n)} \mid n \in \omega\} \text{ for some } f\colon \omega \to \omega.
\end{aligned}$$

Example (continued). Because $Ł_\infty$ is compact we get an infinitary DDT:

$$\begin{aligned}
\Gamma, \Phi \vdash_{Ł_\infty} \psi \iff &\text{ there is } f\colon \omega \to \omega \text{ and a finite } \Phi_n \subseteq \Phi \text{ for each } n \text{ s.t.} \\
&\Gamma \vdash_{Ł_\infty} \Big(\bigwedge \Phi_n \wedge \neg\psi^n\Big)^{f(n)} \text{ for each } n \in \omega.
\end{aligned}$$

The law of the excluded middle (LEM)

The dual IL can be rephrased as a law of the excluded middle (LEM).

Classical logic satisfies the global LEM in the form

$$\Gamma, \varphi \vdash_{\mathrm{CL}} \psi \text{ and } \Gamma, \neg\varphi \vdash_{\mathrm{CL}} \psi \implies \Gamma \vdash_{\mathrm{CL}} \psi.$$

Global modal logic S5 satisfies the global LEM in the form

$$\Gamma, \varphi \vdash_{\mathrm{S5}} \psi \text{ and } \Gamma, \neg\Box\varphi \vdash_{\mathrm{S5}} \psi \implies \Gamma \vdash_{\mathrm{CL}} \psi.$$

The infinitary Łukasiewicz logic $Ł_\infty$ satisfies the local LEM in the form

$$\Gamma, \varphi \vdash_{Ł_\infty} \psi \text{ and } \Gamma, \neg\varphi^n \vdash_{Ł_\infty} \psi \text{ for each } n \implies \Gamma \vdash_{Ł_\infty} \psi.$$

A logic L has the **local LEM** w.r.t. a family $\Psi(x)$ if $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ and $\Gamma, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \psi$ for each $\mathrm{J}(x) \in \Psi(x)$ imply $\Gamma \vdash_{\mathrm{L}} \psi$, and moreover $\varphi, \mathrm{J}(\varphi) \vdash_{\mathrm{L}} \emptyset$ for all $\mathrm{J} \in \Psi$.

The dual IL and the LEM

Fact. The local dual IL and the LEM w.r.t. the same family are equivalent.

Proof. Assume for the sake of simplicity that the family is $\neg_n x$ for $n \in \omega$.

(LEM $\implies$ dual IL) If $\Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \emptyset$ for each n, then in particular $\Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \varphi$ for each n. Moreover, $\Gamma, \varphi \vdash_{\mathrm{L}} \varphi$, so $\Gamma \vdash_{\mathrm{L}} \varphi$ by the LEM.

(Dual IL $\implies$ LEM) If $\Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \psi$ for each n, then $\Gamma, \neg_n \varphi, \neg_m \psi \vdash_{\mathrm{L}} \emptyset$ for each m, n by the dual IL, so $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \varphi$ for each m again by the dual IL.

If $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$, then $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \psi$ for each m by Cut. But $\psi, \neg_m \psi \vdash_{\mathrm{L}} \emptyset$, so $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \emptyset$ for each $m \in \omega$ by Cut and $\Gamma \vdash_{\mathrm{L}} \psi$ by the dual IL.

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If $\Gamma, \varphi \vdash_{\mathrm{L}} \psi$, then $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \psi$ for each m by Cut. But $\psi, \neg_m \psi \vdash_{\mathrm{L}} \emptyset$, so $\Gamma, \neg_m \psi \vdash_{\mathrm{L}} \emptyset$ for each $m \in \omega$ by Cut and $\Gamma \vdash_{\mathrm{L}} \psi$ by the dual IL.

Semisimplicity and the LEM

Fact. Each compact logic with the dual local IL is semisimple.

Proof. If $\Gamma \nvdash_{\mathrm{L}} \varphi$, then $\Gamma, \neg_n \varphi \nvdash_{\mathrm{L}} \emptyset$ for some $n \in \omega$. By compactness $\Gamma, \neg_n \varphi$ extends to a simple L-theory Δ. Then $\Delta \vdash_{\mathrm{L}} \neg_n \varphi$, therefore $\Delta \nvdash_{\mathrm{L}} \varphi$.

Fact. Each semisimple logic with the local IL has the dual local IL.

Proof. If $\Gamma \nvdash_{\mathrm{L}} \varphi$, then $\Gamma \subseteq \Delta \nvdash_{\mathrm{L}} \varphi$ for some simple L-theory Δ. Then $\Delta, \varphi \vdash_{\mathrm{L}} \emptyset$ by the simplicity of Δ and $\Delta \vdash_{\mathrm{L}} \neg_n \varphi$ for some $n \in \omega$ by the local IL. But $\Gamma \subseteq \Delta$, so $\Gamma, \neg_n \varphi \nvdash_{\mathrm{L}} \emptyset$ for some $n \in \omega$.

Note. These proofs work equally well for the parametrized local IL.

Semisimplicity and the LEM

Theorem. The following are equivalent for each compact logic with the parametrized local IL w.r.t. some family Φ:

- syntactic semisimplicity,
- the dual parametrized local IL,
- the parametrized local LEM,
- the dual parametrized local IL w.r.t. Φ_1,
- the parametrized local LEM w.r.t. Φ.

The same holds for the local (global) forms of these principles.

This extends to semantic semisimplicity if there are no parameters.

Semantic semisimplicity and the LEM

Theorem. The following are equivalent* for each compact logic with the local IL w.r.t. some family Φ:

- syntactic semisimplicity,
- the dual local IL,
- the local LEM,
- semantic semisimplicity,
- the semantic dual local IL w.r.t. Φ,
- the semantic local LEM w.r.t. Φ.

The same holds for the global forms of these principles.

* Terms and conditions apply: each set in Φ_1 is finite and $|\Phi_n| \leq |\mathrm{Var}\,\mathrm{L}|$ for each n.

Here the semantic dual local IL or LEM is the dual local IL or LEM for filter generation on arbitrary algebras instead of theory generation on **Fm**. That is, $\Gamma \vdash_{\mathrm{L}} \varphi$ is replaced by $a \in \mathrm{Fg}^{\mathbf{A}}(F)$ (the L-filter generated by F on **A**).

Application: semisimple extensions of $\mathrm{FL_{ew}}$

Theorem. An axiomatic extension of $\mathrm{FL_{ew}}$ is semisimple if and only if it validates $x \vee \neg_n x$ for some $n \in \omega$, where $\neg_n := \neg(x^n)$.

Proof. Each axiomatic extension L of $\mathrm{FL_{ew}}$ inherits the local IL of $\mathrm{FL_{ew}}$. The semisimplicity of L is thus equivalent to the following local LEM:

$$\frac{\Gamma, \varphi \vdash_{\mathrm{L}} \psi \qquad (\forall n \in \omega)\, \Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \psi}{\Gamma \vdash_{\mathrm{L}} \psi}$$

The LEM holds if $\varphi \vee \neg_n \varphi$ is a theorem for some $n \in \omega$ by the proof by cases property (disjunction introduction on the left):

$$\Gamma, \varphi \vdash_{\mathrm{L}} \psi \text{ and } \Gamma, \neg_n \varphi \vdash_{\mathrm{L}} \psi \implies \Gamma, \varphi \vee \neg_n \varphi \vdash_{\mathrm{L}} \psi \implies \Gamma \vdash_{\mathrm{L}} \psi.$$

Conversely, if the LEM holds, we choose suitable Γ, φ, ψ:

$$\varphi := x, \qquad \psi := y, \qquad \Gamma := \bigcup_{n \in \omega} \{\neg_n x \to y\} \cup \{x \to y\}.$$

The LEM yields $\Gamma \vdash_{\mathrm{L}} y$. By finitarity there is a finite $\Delta \subseteq \Gamma$ such that $\Delta \vdash_{\mathrm{L}} y$. That is, $x \to y, \neg_n x \to y \vdash_{\mathrm{L}} y$. Substituting $x \vee \neg_n x$ for y yields $\emptyset \vdash_{\mathrm{L}} x \vee \neg_n x$.

Other results in the vicinity

We can also use this method to describe the semisimple varieties of BAOs (and HAOs). The proof is more complicated, since we only have a local PCP:

$$\Gamma, \varphi \vdash_{\mathrm{K}} \chi \text{ and } \Gamma, \psi \vdash_{\mathrm{K}} \chi \iff \Gamma, \Box_n \varphi \vee \Box_n \psi \vdash_{\mathrm{K}} \chi \text{ for some } n \in \omega.$$

Two similar results lie beyond the immediate reach of our method. We hope to be able to extend it to cover these.

Theorem (Kowalski & Ferreirim, unpublished). A variety K of integral commutative residuated lattices is semisimple if and only if K satisfies the equation $x \vee (x^n \to y) = 1$ for some $n \in \omega$.

Obstacle. No antitheorems. The total theory Fm is not compact.

Theorem (Werner & Wille, 1970). A variety K of commutative rings is semisimple if and only if K satisfies $x = x^n$ for some $n \geq 2$.

Obstacle. A parametrized global IL rather than a local one.

Semisimple companions

Given a logic L, describe the semisimple models of L.

We define the **(syntactic) semisimple companion** $\alpha(\mathrm{L})$ as the logic of semisimple theories of L. That is, $\alpha(\mathrm{L})$ is determined by structures of the form $\langle \mathbf{Fm}, T \rangle$ where T ranges over the (semi)simple theories of L.

The **semantic semisimple companion** is defined similarly as the logic of (semi)simple models of L. It need not coincide with $\alpha(\mathrm{L})$ (e.g. for $\mathrm{L} = \mathrm{FDE}$).

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Theorem. The semisimple models of L are precisely the models of the semisimple companion of L, provided that L is compact and has the local IL.* Moreover, the semantic and syntactic semisimple companion then coincide.

* Same terms and conditions apply: each set in Φ_1 is finite and $|\Phi_n| \leq |\mathrm{Var}\,\mathrm{L}|$ for each n.

For example, $\alpha(\mathrm{IL}) = \mathrm{CL}$ and $\alpha(\mathrm{S4}) = \mathrm{S5}$ and $\alpha(\mathrm{BL}) = \text{Ł}_\infty$.

Admisible rules

The problem of describing the semisimple models of L thus reduces to the problem of axiomatizing its semisimple companion.

Here we borrow some ideas from the theory of structural completeness. The **structural completion** $\sigma(\mathrm{L})$ of L is the largest logic with the same theorems as L, or equivalently the logic determined by the structure $\langle \mathbf{Fm}, \mathrm{Thm}\,\mathrm{L} \rangle$.

A rule $\Gamma \vdash \varphi$ holds in $\sigma(\mathrm{L})$ if and only if it is **admissible** in L, i.e. if each instance of $\Gamma \vdash \varphi$ preserves theoremhood from premises to the conclusion:

$$\emptyset \vdash_{\mathrm{L}} \sigma[\Gamma] \implies \emptyset \vdash_{\mathrm{L}} \sigma(\varphi) \text{ for each substitution } \sigma.$$

The structural completion is often infinitary (e.g. for intuitionistic logic).

Antiadmissible rules

The same idea can be applied to antitheorems, provided that L is compact.

The **antistructural completion** of L is the largest logic with the same antitheorems as L, or equivalently the logic determined by the structures $\langle \mathbf{Fm}, T \rangle$ where T ranges over the simple theories of L. This is just $\alpha(\mathrm{L})$!

(**Proof.** Same antitheorems means same consistent sets Γ. But consistent sets are precisely those included in some simple theory T.)

A rule $\Gamma \vdash \varphi$ holds in $\alpha(\mathrm{L})$ if and only if it is **antiadmissible** in L, i.e. if each instance preserves antitheoremhood from the conclusion to the premises:

$$\sigma(\varphi), \Delta \vdash_{\mathrm{L}} \emptyset \implies \sigma[\Gamma], \Delta \vdash_{\mathrm{L}} \emptyset \text{ for each substitution } \sigma \text{ and each } \Delta \subseteq \mathrm{Fm}.$$

The antistructural completion is often infinitary (e.g. $\alpha(\mathrm{BL}) = \text{Ł}_\infty$).

Antiadmissible rules

The quantification over substitutions σ is in fact redundant if L has the IL! This fact has no counterpart in the theory of admissible rules.

Fact. If L enjoys the local IL, then $\Gamma \vdash \varphi$ is antiadmissible if and only if

$$\varphi, \Delta \vdash_{\mathrm{L}} \emptyset \implies \Gamma, \Delta \vdash_{\mathrm{L}} \emptyset \text{ for each } \Delta \subseteq \mathrm{Fm}.$$

Proof. We can restrict the quantification to theories T of L instead of $\Delta \subseteq \mathrm{Fm}$. But recall that the local IL is equivalent to a substitution swapping property:

$$\sigma(\varphi), T \vdash_{\mathrm{L}} \emptyset \implies \varphi, \sigma^{-1}[T] \vdash_{\mathrm{L}} \emptyset \implies \Gamma, \sigma^{-1}[T] \vdash_{\mathrm{L}} \emptyset \implies \sigma[\Gamma], T \vdash_{\mathrm{L}} \emptyset.$$

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But the above implication is just a Glivenko theorem in disguise!

Glivenko theorems and the semisimple companion

Local Glivenko Theorem. Let L be a compact logic with a local IL. Then $\Gamma \vdash_{\alpha(\mathrm{L})} \varphi$ if and only if $\Gamma \vdash_{\mathrm{L}} \{\neg_{f(n)}\neg_n\varphi \mid n \in \omega\}$ for some $f\colon \omega \to \omega$.

Proof. $\Gamma \vdash_{\alpha(\mathrm{L})} \varphi$ is equivalent to the antiadmissibility of $\Gamma \vdash \varphi$, i.e. to

$$\varphi, \Delta \vdash_{\mathrm{L}} \emptyset \implies \Gamma, \Delta \vdash_{\mathrm{L}} \emptyset \text{ for each } \Delta \subseteq \mathrm{Fm}.$$

By the local IL this is equivalent to

$$\Delta \vdash_{\mathrm{L}} \neg_n\varphi \text{ for some } n \implies \Gamma, \Delta \vdash_{\mathrm{L}} \emptyset \text{ for each } \Delta \subseteq \mathrm{Fm},$$

That is, $\Gamma, \neg_n\varphi \vdash_{\mathrm{L}} \emptyset$ for each n. Applying the local IL again yields that

$$\text{for each } n \text{ there is } k \text{ such that } \Gamma \vdash_{\mathrm{L}} \neg_k\neg_n\varphi,$$

or equivalently

$$\Gamma \vdash_{\mathrm{L}} \{\neg_{f(n)}\neg_n\varphi \mid n \in \omega\} \text{ for some } f\colon \omega \to \omega.$$

Glivenko theorems: examples

The Glivenko theorems for intuitionistic logic and S4 are special cases:

$$\Gamma \vdash_{CL} \varphi \iff \Gamma \vdash_{IL} \neg\neg\varphi,$$
$$\Gamma \vdash_{S5} \varphi \iff \Gamma \vdash_{S4} \neg\Box\neg\Box\varphi.$$

The semisimple companion of Hájek's basic fuzzy logic BL is the infinitary Łukasiewicz logic $Ł_\infty$ (the logic of semisimple MV-algebras), so

$$\Gamma \vdash_{Ł_\infty} \varphi \iff \Gamma \vdash_{BL} \neg_{f(n)}\neg_n\varphi \text{ for some } f\colon \omega \to \omega.$$

If Γ is finite, there is a global Glivenko theorem due to Cignoli & Torrens:

$$\Gamma \vdash_{Ł_\infty} \varphi \iff \Gamma \vdash_{Ł} \varphi \iff \Gamma \vdash_{BL} \neg\neg\varphi.$$

This is beyond the scope of our method. Conversely, the Glivenko theorem for S4 lies beyond the immediate scope of the method of Cignoli & Torrens.

Glivenko theorems: other approaches

The existing approaches to Glivenko theorems due to Cignoli & Torrens and Galatos & Ono are quite different and incomparable in strength:

- both consider Glivenko theorems relative to a certain term,
- in particular, Cignoli & Torrens assume that the double negation is a homomorphism onto an algebra of regular elements,
- neither requires a connection between negation and inconsistency,
- both only consider global Glivenko theorems.

Fundamentally, our notion of negation is related to antitheorems (although there are some tricks we can use to get around this).

Axiomatizing the semisimple companion

Even though we can reduce validity in $\alpha(\mathrm{L})$ to antitheoremhood in L, we know no general method of axiomatizing $\alpha(\mathrm{L})$ unless L has a global IL.

If L has the global IL, then:

- If L has the global PCP, we axiomatize $\alpha(\mathrm{L})$ by the axiom $x \vee \neg x$.
- If L has the global DDT, we use the axiom $(x \to y) \to ((\neg x \to y) \to y)$.

For example, the disjunction of S4 (from the PCP) is $\Box x \vee \Box y$ and the negation of S4 (from the IL) is $\neg\Box x$, so the axiom is $\Box x \vee \Box\neg\Box x$, i.e. $\Diamond\Box x \to \Box x$.

(This is more or less obvious, but it a useful to observe that one can just plug in the appropriate negation and disjunction into the axiomatic LEM.)

Conclusion

For compact logics L with the local IL:

- Semisimplicity is equivalent to the LEM (in the form of a metarule).
- This can be used to describe the semisimple subvarieties of a variety.
- The semisimple models of L are the models of its ss. companion α(L).
- A Glivenko-like double negation translation connects L and α(L).

Moreover, it makes sense to consider:

- infinitary DDTs in addition to finitary ones,
- local Glivenko theorems in addition to global ones,
- antistructural completeness in addition to structural completness.

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Thank you for your attention!