Langlands Eisenstein series for $SL(n, \mathbb{Z})$

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Definition:

For n = 2, 3, 4, ... we define the generalized upper half plane

$$\mathfrak{h}^n = \mathrm{GL}(n,\mathbb{R})/\left(\mathrm{O}(n,\mathbb{R})\cdot\mathbb{R}^{\times}\right).$$

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Example 1

$$\mathfrak{h}^2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \; \middle| \; x \in \mathbb{R}, \; y > 0 \right\}.$$

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$$\mathfrak{h}^{n} = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ 1 & x_{2,3} & \cdots & x_{2,n} \\ & \ddots & & \vdots \\ & & 1 & x_{n-1,n} \end{pmatrix} \begin{pmatrix} y_{1}y_{2}\cdots y_{n-1} \\ y_{1}y_{2}\cdots y_{n-2} \\ & \ddots & \vdots \\ & & y_{1} \\ & & & 1 \end{pmatrix} \mid x_{i,j} \in \mathbb{R}, \ y_{i} > 0 \right\}$$

Definition: (Automorphic function)

An automorphic function for $SL(n,\mathbb{Z})$ is a smooth function $\phi:\mathfrak{h}^n\to\mathbb{C}$ satisfying:

$$\phi(\gamma g) = \phi(g)$$

for all $\gamma \in \mathsf{SL}(n,\mathbb{Z})$ and all $g \in \mathfrak{h}^n$.

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Remark: Note that ϕ defined on \mathfrak{h}^n means that $\phi(dgk) = \phi(g)$ for all $g \in GL(n,\mathbb{R})$, $k \in O(n,\mathbb{R})$ and all matrices d in the center of $GL(n,\mathbb{R})$.

Let $n \ge 2$. A Langlands Eisenstein series for $SL(n, \mathbb{Z})$ depends on:

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Notation for Langlands Eisenstein series

For $g \in \mathfrak{h}^n$, the Eisenstein series is denoted:

$$E_{\mathcal{P},\Phi}(g,s)$$
.

Motivating idea of a power function for h²

Let
$$g = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \in \mathfrak{h}^2$$
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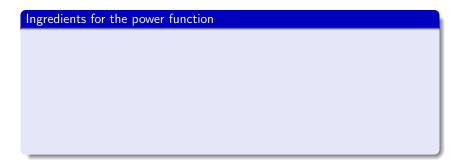
• The classical Eisenstein series for $SL(2,\mathbb{Z})$ is obtained by summing $\frac{y^s}{|c(x+iy)+d|^{2s}}$ over all coprime integers c,d.

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- This can be realized by letting matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$ act on the power function via matrix multiplication.



Ingredients for the power function

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The condition $\sum_{i=1}^{r} n_i s_i = 0$ assures that the above power function is invariant under multiplication by elements of the center of $GL(n,\mathbb{R})$.



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Examples of Langlands Eisenstein series of small rank

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$$E_{\mathcal{P}_{1,1}}(g,s) = \sum_{\gamma \in (\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}) \backslash \mathsf{SL}(2,\mathbb{Z})} \left| \gamma g \right|_{\mathcal{P}_{1,1}}^{s+(1/2,-1/2)} = \sum_{\gamma \in (\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}) \backslash \mathsf{SL}(2,\mathbb{Z})} (\mathrm{Im}\, \gamma z)^{s_1+1/2},$$

where $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}$ and z = x + iy.

The Fourier expansion

$$\begin{split} \mathcal{E}_{\mathcal{P}_{1,1}}(g,s) &= y^{s_1 + \frac{1}{2}} + \phi(s_1 + \frac{1}{2})y^{\frac{1}{2} - s_1} \\ &+ \frac{1}{\zeta^*(2s_1 + 1)} \sum_{m \neq 0} \sigma_{2s_1}(m) |m|^{-s_1} \sqrt{y} \, \mathcal{K}_{s_1}(2\pi |m| y) e^{2\pi i m x}, \end{split}$$

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Functional equation of $E_{\mathcal{P}_{1,1}}(g,s)$

Shifting s by $\frac{1}{2}$ simplifies the functional equation which is given by

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Note that when multiplying by $\zeta^*(2s_1+1)$ we are clearing the denominator in the Fourier expansion.

$$\mathfrak{h}^3 = \left\{ xy = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid x_1, x_2, x_3 \in \mathbb{R}, \ y_1, y_2 > 0 \right\}.$$

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Let $s=(s_1,s_2,s_3)\in\mathbb{C}^3$ with $s_1+s_2+s_3=0$. Then the power function is given by

$$\left| \textit{dxyk} \right|_{\mathcal{P}_{1,1,1}}^{s} := \left| \left(\begin{smallmatrix} y_{1}y_{2} & 0 & 0 \\ 0 & y_{1} & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right) \right|_{\mathcal{P}_{1,1,1}}^{s} = (y_{1}y_{2})^{s_{1}} y_{1}^{s_{2}},$$

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$$\boxed{ E_{\mathcal{P}_{1,1,1}}(g,s) = \sum_{\gamma \in \left(\begin{array}{cc} 1 & * & * \\ & 1 & * \\ & & 1 \end{array} \right) \backslash \operatorname{SL}(3,\mathbb{Z})} \left| \gamma g \right|_{\mathcal{P}_{1,1,1}}^{s + (1,0,-1)} }$$

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The shift by (1,0,-1) makes the form of the functional equations as simple as possible.

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Then $E^*_{\mathcal{P}_{1,1,1}}(g,s)$ satisfies the functional equation

$$E_{\mathcal{P}_{1,1,1}}^*(g,s_1,s_2,s_3) = E_{\mathcal{P}_{1,1,1}}^*(g,s_{\sigma(1)},s_{\sigma(2)},s_{\sigma(3)})$$

for any $\sigma \in S_3$.

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for any $\sigma \in S_3$.

Theorem (G, Stade, Woodbury, 2023)

Functional Equation (Selberg, Langlands, Bump)

Let $g \in \mathfrak{h}^3$ and $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $s_1 + s_2 + s_3 = 0$. Define

$$E_{\mathcal{P}_{1,1,1}}^*(g,s) = \Big(\prod_{1 \leq j < \ell \leq 3} \zeta^*(1+s_j-s_\ell)\Big) \cdot E_{\mathcal{P}_{1,1,1}}(g,s).$$

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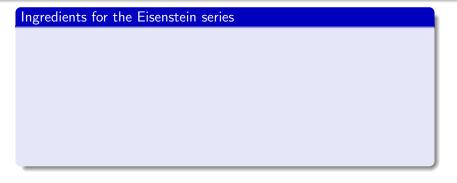
for any $\sigma \in S_3$.

Theorem (G, Stade, Woodbury, 2023)

This functional equation is unique in that, if μ is any real affine transformation of s such that

$$E_{\mathcal{P}_{1,1,1}}^*(g, s_1, s_2, s_3) = E_{\mathcal{P}_{1,1,1}}^*(g, \mu(s)),$$

then $\mu(s)$ is a permutation of s_1, s_2, s_3 .



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$$\mathcal{E}_{\mathcal{P}_{1,2},\; 1\otimes \phi}(g,s) \; = \sum_{\substack{\gamma \in \binom{*\; *\; *\\ *\; *}} \Big\backslash \operatorname{SL}(3,\mathbb{Z})} \phi \left(\mathfrak{m}_2(\gamma g)\right) \, \left| \; \det \mathfrak{m}_1(\gamma g) \right|^{\mathfrak{s}_1+1} \, \left| \; \det \mathfrak{m}_2(\gamma g) \right|^{\mathfrak{s}_2-1/2}$$

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then $\mu(s) = (s_2, s_1)$.

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A "Langlands parameter" for GL(n) is an n-tuple

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Let $F: \mathfrak{h}^n \to \mathbb{C}$ be a smooth $\mathsf{SL}(n,\mathbb{Z})$ invariant function. Suppose F is an eigenfunction of all $\mathsf{GL}(n,\mathbb{R})$ -invariant differential operators on \mathfrak{h}^n .

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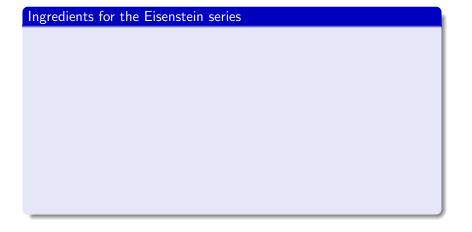
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where \mathcal{B} denotes the partition $n = 1 + 1 + \cdots + 1$, and

$$\rho_{\mathcal{B}} = \left(\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2}\right).$$

The Eisenstein series $E_{\mathcal{P}_{2,2},\ \phi_1\otimes\phi_2}(g,s)$ for $\mathsf{SL}(4,\mathbb{Z})$



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$$\left|g\right|_{\mathcal{P}_{2,2}}^{s} := \left|\begin{pmatrix} y_{1}y_{2}y_{3} & 0 & 0 & 0 \\ 0 & y_{1}y_{2} & 0 & 0 \\ 0 & 0 & y_{1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right|_{\mathcal{P}_{2,2}}^{s} = \left|\det\begin{pmatrix} y_{1}y_{2}y_{3} & 0 \\ 1 & y_{1}y_{2} \end{pmatrix}\right|_{s_{1}}^{s_{1}} \cdot \left|\det\begin{pmatrix} y_{1} & 0 \\ 0 & 1 \end{pmatrix}\right|_{s_{2}}^{s_{2}}$$

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Every $g \in \mathfrak{h}^4$ can be written in the form $g = \binom{\mathfrak{m}_1(g)}{\mathfrak{m}_2(g)}^*$ where $\mathfrak{m}_1(g) \in \mathsf{GL}(2,\mathbb{R})$ and $\mathfrak{m}_2(g) \in \mathsf{GL}(2,\mathbb{R})$.

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$$E_{\mathcal{P}_{2,2}, \phi_1 \otimes \phi_2}(g, s) = \sum_{\substack{\gamma \in \begin{pmatrix} * & * & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \\ * & * & * \end{pmatrix}} \phi_1(\mathfrak{m}_1(\gamma g)) \phi_2(\mathfrak{m}_2(\gamma g)) \cdot |\gamma g|_{\mathcal{P}_{2,2}}^{s+(1,-1)}$$

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Standard Parabolic Subgroup

$$\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} := \left\{ \begin{pmatrix} \mathsf{GL}(n_1) & * & \cdots & * \\ 0 & \mathsf{GL}(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathsf{GL}(n_r) \end{pmatrix} \right\}.$$

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Unipotent radical of ${\cal P}$

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$$\boxed{\Phi(u\mathfrak{m}k) := \prod_{i=1}^r \phi_i(\mathfrak{m}_i), \qquad (u \in N^{\mathcal{P}}, \mathfrak{m} \in M^{\mathcal{P}}, k \in K)}$$

where
$$\mathfrak{m} \in M^{\mathcal{P}}$$
 has the form $\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_r \end{pmatrix}$, with $\mathfrak{m}_i \in \mathsf{GL}(n_i, \mathbb{R})$.

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Remark: The ρ -function is introduced as a normalizing factor (a shift in the s variable) in Eisenstein series to simplify later formulae.





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$$E_{\mathcal{P},\Phi}(g,s) := \sum_{\gamma \in (\Gamma_n \cap \mathcal{P}) \setminus \Gamma_n} \Phi(\gamma g) \cdot |\gamma g|_{\mathcal{P}}^{s+\rho_{\mathcal{P}}}.$$

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Then the Borel Eisenstein series is

$$E_{\mathcal{B}}(g,s) = \sum_{\gamma \in (\Gamma_n \cap \mathcal{B}) \setminus \Gamma_n} |\gamma g|_{\mathcal{B}}^{s+\rho_{\mathcal{B}}}.$$

Unipotent Group

$$U_n(\mathbb{R}) := \left\{ \begin{pmatrix} 1 & * & * & \cdots & * \\ 1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & 1 & * \end{pmatrix} \right\} \subset GL(n, \mathbb{R})$$

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Definition (Completed Whittaker function)

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We define the completed Whittaker function $W_{\alpha}^{(n)}:\mathfrak{h}^n\to\mathbb{C}$, with Langlands parameter α , by the integral

$$W_{\alpha}^{(n)}(g) = \left(\prod_{1 \leq j < k \leq n} \frac{\Gamma\left(\frac{1+\alpha_{j}-\alpha_{k}}{2}\right)}{\pi^{\frac{1+\alpha_{j}-\alpha_{k}}{2}}}\right) \int_{U_{n}(\mathbb{R})} \left|w_{n} \cdot ug\right|_{\mathcal{B}}^{s+\rho_{\mathcal{B}}} \overline{\psi_{1,1,\ldots,1}(u)} du,$$

where w_n is the long element of the Weyl group for $GL(n,\mathbb{R})$, and $|*|_{\mathcal{B}}^s$ is the power function for the Borel \mathcal{B} .

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Suppose each ϕ_j in Φ has Langlands parameter $(\alpha_{j,1},\ldots,\alpha_{j,n_j})$, with the convention that if $n_i=1$ then $\alpha_{i,1}=0$.

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Then the first coefficient $A_{\mathcal{P},\Phi}((1,\ldots,1),s)$ is given by

$$\prod_{\substack{k=1\\ n_k \neq 1}}^r L^* \big(1, \operatorname{Ad} \phi_k \big)^{-\frac{1}{2}} \prod_{1 \leq j < \ell \leq r} L^* \big(1 + s_j - s_\ell, \; \phi_j \times \phi_\ell \big)^{-1}$$

up to a non-zero constant factor with absolute value depending only on n.

Action of the symmetric group S_r on \mathcal{P}, Φ, s

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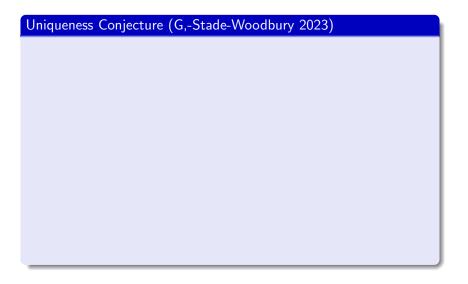
Suppose $\sigma \in S_r$ acts on \mathcal{P} , Φ , and s as above. Then we have the functional equation

$$E_{\mathcal{P},\Phi}^*(g,s) = E_{\sigma\mathcal{P},\sigma\Phi}^*(g,\sigma s)$$

for all $g \in GL(n, \mathbb{R})$.

Uniqueness of functional equations

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Uniqueness Conjecture (G,-Stade-Woodbury 2023)

Let $\sigma' \in S_r$. Suppose $E^*_{\mathcal{P},\Phi}(g,s)$ satisfies a functional equation of the form

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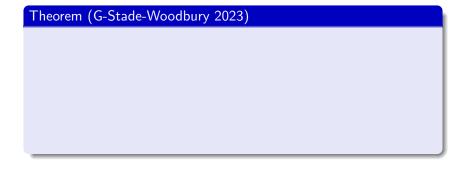
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where $a_{ij}, b_i \in \mathbb{R}$ for all $1 \leq i, j \leq r$.

Then, in fact, $\mu = \sigma$ for some $\sigma \in S_r$ for which $\sigma \mathcal{P} = \sigma' \mathcal{P}$ and $\sigma \Phi = \sigma' \Phi$.



Theorem (G-Stade-Woodbury 2023)

The uniqueness conjecture holds in the special case that

$$\phi_1 = \phi_2 = \cdots = \phi_r$$

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In this case, every permutation $\sigma \in S_r$ has the property that $\sigma \mathcal{P} = \mathcal{P}$ and $\sigma \Phi = \Phi$.

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Recall the uniqueness conjecture:

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Assume the Maass form Conjecture.

If the transformation μ given in the Uniqueness Conjecture is linear, i.e., $b_i = 0$ for each i = 1, 2, ..., r. Then the Uniqueness conjecture holds.