Langlands Eisenstein series for $\mathrm{SL}(n, \mathbb{Z})$

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The generalized Upper Half Plane $\mathfrak{h}^n$

Definition:

For $n = 2, 3, 4, \ldots$ we define the generalized upper half plane

$$\boxed{\mathfrak{h}^n = \mathrm{GL}(n, \mathbb{R}) / \left(\mathrm{O}(n, \mathbb{R}) \cdot \mathbb{R}^\times \right).}$$

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Example 1

$$\mathfrak{h}^2 = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R},\ y > 0 \right\}.$$

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Example 2

$$\mathfrak{h}^n = \left\{ \begin{pmatrix} 1 & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ & 1 & x_{2,3} & \cdots & x_{2,n} \\ & & \ddots & & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 \cdots y_{n-1} & & & & \\ & y_1 y_2 \cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix} \;\middle|\; x_{i,j} \in \mathbb{R},\ y_i > 0 \right\}$$

Automorphic functions for SL($n, \mathbb{Z}$)

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Definition: (Automorphic function)

An automorphic function for $\mathrm{SL}(n, \mathbb{Z})$ is a smooth function $\phi : \mathfrak{h}^n \to \mathbb{C}$ satisfying:

$$\boxed{\phi(\gamma g) = \phi(g)}$$

for all $\gamma \in \mathrm{SL}(n, \mathbb{Z})$ and all $g \in \mathfrak{h}^n$.

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Remark: Note that ϕ defined on $\mathfrak{h}^n$ means that $\boxed{\phi(dgk) = \phi(g)}$ for all $g \in \mathrm{GL}(n, \mathbb{R})$, $k \in O(n, \mathbb{R})$ and all matrices d in the center of $\mathrm{GL}(n, \mathbb{R})$.

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Notation for Langlands Eisenstein series

For $g \in \mathfrak{h}^n$, the Eisenstein series is denoted: $\boxed{E_{\mathcal{P},\Phi}(g,s).}$

Power function on $\mathfrak{h}^2$

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Motivating idea of a power function for $\mathfrak{h}^2$

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• This can be realized by letting matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$ act on the power function via matrix multiplication.

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The condition $\sum_{i=1}^{r} n_i s_i = 0$ assures that the above power function is invariant under multiplication by elements of the center of $\mathrm{GL}(n, \mathbb{R})$.

Examples of Langlands Eisenstein series of small rank

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$$E_{\mathcal{P}_{1,1}}(g,s) = \sum_{\gamma \in \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \backslash \mathrm{SL}(2,\mathbb{Z})} \left|\gamma g\right|^{s+(1/2,-1/2)}_{\mathcal{P}_{1,1}} = \sum_{\gamma \in \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \backslash \mathrm{SL}(2,\mathbb{Z})} (\mathrm{Im}\,\gamma z)^{s_1+1/2},$$

where $g = \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $z = x + iy$.

Fourier expansion and functional equation of $E_{\mathcal{P}_{1,1}}(g, s)$

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The Fourier expansion

$$E_{\mathcal{P}_{1,1}}(g,s) = y^{s_1+\frac{1}{2}} + \phi(s_1+\tfrac{1}{2})y^{\frac{1}{2}-s_1} + \frac{1}{\zeta^*(2s_1+1)} \sum_{m\neq 0} \sigma_{2s_1}(m)|m|^{-s_1}\sqrt{y}\,K_{s_1}(2\pi|m|y)e^{2\pi imx},$$

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where

$$\zeta^*(s_1) = \pi^{-s_1/2}\,\Gamma(\tfrac{s_1}{2})\,\zeta(s_1), \quad \phi(s_1) = \frac{\zeta^*(2s_1-1)}{\zeta^*(2s_1)}, \quad \sigma_{s_1}(n) = \sum_{\substack{d|n \\ d>0}} d^{s_1}.$$

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Functional equation of $E_{\mathcal{P}_{1,1}}(g,s)$

Shifting s by $\frac{1}{2}$ simplifies the functional equation which is given by

$$\boxed{E^*_{\mathcal{P}_{1,1}}(g,s) := \zeta^*(2s_1+1)E_{\mathcal{P}_{1,1}}(g,s) = E^*_{\mathcal{P}_{1,1}}(g,-s).}$$

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Note that when multiplying by $\zeta^*(2s_1+1)$ we are clearing the denominator in the Fourier expansion.

The Borel Eisenstein series $E_{\mathcal{P}_{1,1,1}}(g, s)$ for $\mathrm{SL}(3, \mathbb{Z})$

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$$\mathfrak{h}^3 = \left\{ xy = \begin{pmatrix} 1 & x_1 & x_3 \\ 0 & 1 & x_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x_1, x_2, x_3 \in \mathbb{R},\ y_1, y_2 > 0 \right\}.$$

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Let $s = (s_1, s_2, s_3) \in \mathbb{C}^3$ with $s_1 + s_2 + s_3 = 0$. Then the power function is given by

$$\left| dxyk \right|^s_{\mathcal{P}_{1,1,1}} := \left| \begin{pmatrix} y_1y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|^s_{\mathcal{P}_{1,1,1}} = (y_1y_2)^{s_1} y_1^{s_2},$$

where d is in the center of $\mathrm{GL}(3,\mathbb{R})$ and $k \in \mathrm{O}(n,\mathbb{R})$.

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Definition: (The Eisenstein series $E_{\mathcal{P}_{1,1,1}}(g,s)$)

Then for $g \in \mathrm{GL}(3,\mathbb{R})$ we have

$$\boxed{E_{\mathcal{P}_{1,1,1}}(g,s) = \sum_{\gamma \in \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \backslash \mathrm{SL}(3,\mathbb{Z})} \left|\gamma g\right|^{s+(1,0,-1)}_{\mathcal{P}_{1,1,1}}}$$

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$$\left|dxyk\right|^{s}_{\mathcal{P}_{1,1,1}} := \left| \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right|^{s}_{\mathcal{P}_{1,1,1}} = (y_1 y_2)^{s_1} y_1^{s_2},$$

where d is in the center of $\mathrm{GL}(3,\mathbb{R})$ and $k \in \mathrm{O}(n,\mathbb{R})$.

Definition: (The Eisenstein series $E_{\mathcal{P}_{1,1,1}}(g,s)$)

Then for $g \in \mathrm{GL}(3,\mathbb{R})$ we have

$$E_{\mathcal{P}_{1,1,1}}(g,s) = \sum_{\gamma \in \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \Big\backslash \mathrm{SL}(3,\mathbb{Z})} \left|\gamma g\right|^{s+(1,0,-1)}_{\mathcal{P}_{1,1,1}}$$

The shift by $(1,0,-1)$ makes the form of the functional equations as simple as possible.

The Functional Equation of $E_{\mathcal{P}_{1,1,1}}(g, s)$

Functional Equation (Selberg, Langlands, Bump)

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$$E^*_{\mathcal{P}_{1,1,1}}(g,s) = \Big(\prod_{1 \le j < \ell \le 3} \zeta^*(1 + s_j - s_\ell) \Big) \cdot E_{\mathcal{P}_{1,1,1}}(g,s).$$

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Then $E^*_{\mathcal{P}_{1,1,1}}(g,s)$ satisfies the functional equation

$$\boxed{E^*_{\mathcal{P}_{1,1,1}}(g, s_1, s_2, s_3) = E^*_{\mathcal{P}_{1,1,1}}\big(g,\ s_{\sigma(1)}, s_{\sigma(2)}, s_{\sigma(3)}\big)}$$

for any $\sigma \in S_3$.

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Theorem (G, Stade, Woodbury, 2023)

This functional equation is unique in that, if μ is any real affine transformation of s such that

$$E^*_{\mathcal{P}_{1,1,1}}(g, s_1, s_2, s_3) = E^*_{\mathcal{P}_{1,1,1}}(g, \mu(s)),$$

then $\mu(s)$ is a permutation of s_1, s_2, s_3.

The Eisenstein series $E_{\mathcal{P}_{1,2},\, 1\otimes\phi}(g, s)$

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The functional equation for $E_{\mathcal{P}_{1,2},\, 1\otimes\phi}(g, s)$

Let ϕ be a Maass cusp form for $\mathrm{SL}(2,\mathbb{Z})$ whose completed L-function $L^*(s,\phi)$ satisfies the functional equation

$$L^*(s,\phi) = \pi^{-s}\,\Gamma\left(\tfrac{s+\alpha_1}{2}\right)\Gamma\left(\tfrac{s+\alpha_2}{2}\right)L(s,\phi) = L^*(1-s,\phi).$$

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If μ is any real linear transformation of $s=(s_1,s_2)$ such that

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A *"Langlands parameter"* for $\mathrm{GL}(n)$ is an n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{C}^n$ satisfying $\sum_{i=1}^{n} \alpha_i = 0$.

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Definition (Langlands Parameter for an automorphic form)

Let $F : \mathfrak{h}^n \to \mathbb{C}$ be a smooth $\mathrm{SL}(n, \mathbb{Z})$ invariant function. Suppose F is an eigenfunction of all $\mathrm{GL}(n, \mathbb{R})$-invariant differential operators on $\mathfrak{h}^n$.

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where $\mathcal{B}$ denotes the partition $n = 1 + 1 + \cdots + 1$, and

$$\rho_{\mathcal{B}} = \left(\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{1-n}{2} \right).$$

Ingredients for the Eisenstein series

The Eisenstein series $E_{\mathcal{P}_{2,2},\ \phi_1 \otimes \phi_2}(g, s)$ for $\mathrm{SL}(4, \mathbb{Z})$

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Recall the Rankin-Selberg L-function $L^*(s, \phi_1 \times \phi_2)$. The completed L-function for this is given by
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Parabolic subgroups of $\mathrm{GL}(n, \mathbb{R})$

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Standard Parabolic Subgroup

$$\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & * & \cdots & * \\ 0 & \mathrm{GL}(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}.$$

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Unipotent radical of $\mathcal{P}$

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}, \qquad \big(I_k = k \times k \text{ identity matrix}\big)$$

Parabolic subgroups of $\mathrm{GL}(n, \mathbb{R})$

Standard Parabolic Subgroup

$$\mathcal{P} := \mathcal{P}_{n_1, n_2, \ldots, n_r} := \left\{ \begin{pmatrix} \mathrm{GL}(n_1) & * & \cdots & * \\ 0 & \mathrm{GL}(n_2) & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathrm{GL}(n_r) \end{pmatrix} \right\}.$$

Unipotent radical of $\mathcal{P}$

$$N^{\mathcal{P}} := \left\{ \begin{pmatrix} I_{n_1} & * & \cdots & * \\ 0 & I_{n_2} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I_{n_r} \end{pmatrix} \right\}, \qquad \left(I_k = k \times k \text{ identity matrix}\right)$$

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- $\phi_i : \mathrm{GL}(n_i, \mathbb{R}) \to \mathbb{C}$ be either the constant function 1 (if $n_i = 1$) or a Maass cusp form for $\mathrm{SL}(n_i, \mathbb{Z})$ (if $n_i > 1$).

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$$\boxed{\Phi(u\mathfrak{m}k) := \prod_{i=1}^{r} \phi_i(\mathfrak{m}_i),} \qquad (u \in N^{\mathcal{P}}, \mathfrak{m} \in M^{\mathcal{P}}, k \in K)$$

where $\mathfrak{m} \in M^{\mathcal{P}}$ has the form $\mathfrak{m} = \begin{pmatrix} \mathfrak{m}_1 & 0 & \cdots & 0 \\ 0 & \mathfrak{m}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathfrak{m}_r \end{pmatrix}$, with $\mathfrak{m}_i \in \mathrm{GL}(n_i,\mathbb{R})$.

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For $g \in \mathcal{P}$, with diagonal block entries $\mathfrak{m}_i \in \mathrm{GL}(n_i, \mathbb{R})$, recall the power function:

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Let $\mathcal{P}$ be the parabolic subgroup $\mathcal{P} := \mathcal{P}_{n_1,n_2,\ldots,n_r}$. Then we define

$$\rho_{\mathcal{P}}(j) := \begin{cases} \frac{n-n_1}{2}, & j = 1, \\ \frac{n-n_j}{2} - n_1 - \cdots - n_{j-1}, & j \geq 2, \end{cases}$$

and $\rho_{\mathcal{P}} = (\rho_{\mathcal{P}}(1), \rho_{\mathcal{P}}(2), \ldots, \rho_{\mathcal{P}}(r))$.

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Remark: The ρ-function is introduced as a normalizing factor (a shift in the s variable) in Eisenstein series to simplify later formulae.

Langlands Eisenstein series for SL($n, \mathbb{Z}$)

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$$E_{\mathcal{P},\Phi}(g,s) := \sum_{\gamma\in(\Gamma_n\cap\mathcal{P})\backslash\Gamma_n} \Phi(\gamma g)\cdot|\gamma g|_{\mathcal{P}}^{s+\rho_{\mathcal{P}}}.$$

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$$\mathcal{B} := \mathcal{P}_{1,1,\ldots,1} := \left\{ \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix} \right\}.$$

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Then the Borel Eisenstein series is

$$\boxed{E_{\mathcal{B}}(g, s) = \sum_{\gamma \in (\Gamma_n \cap \mathcal{B}) \backslash \Gamma_n} |\gamma g|_{\mathcal{B}}^{s + \rho_{\mathcal{B}}}.}$$

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We define the *completed Whittaker function* $W_\alpha^{(n)} : \mathfrak{h}^n \to \mathbb{C}$, *with Langlands parameter* α, by the integral

$$W_\alpha^{(n)}(g) = \left(\prod_{1 \le j < k \le n} \frac{\Gamma\left(\frac{1+\alpha_j - \alpha_k}{2}\right)}{\pi^{\frac{1+\alpha_j-\alpha_k}{2}}} \right) \int\limits_{U_n(\mathbb{R})} \left| w_n \cdot ug \right|_{\mathcal{B}}^{s+\rho_{\mathcal{B}}} \overline{\psi_{1,1,\ldots,1}(u)} \, du,$$

where w_n is the long element of the Weyl group for $\mathrm{GL}(n, \mathbb{R})$, and $|*|_{\mathcal{B}}^{s}$ is the power function for the Borel $\mathcal{B}$.

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$$\int_0^1 \cdots \int_0^1 E_{\mathcal{P},\Phi}(ug, s)\, \overline{\psi_M(u)} \prod_{1\le i<j\le n} du_{i,j} \;=\; \frac{A_{\mathcal{P},\Phi}(M,s)}{\prod_{k=1}^{n-1} m_k^{k(n-k)/2}}\; W_{\alpha_{\mathcal{P},\Phi}(s)}\big(Mg\big)$$

where $A_{\mathcal{P},\Phi}(M,s) = A_{\mathcal{P},\Phi}\big((1,\ldots,1), s\big) \cdot \lambda_{\mathcal{P},\Phi}(M,s)$,

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$$\lambda_{\mathcal{P},\Phi}\big((m,1,\ldots,1),s\big) = \sum_{\substack{c_1,c_2,\ldots,c_r \in \mathbb{Z}_{>0} \\ c_1c_2\cdots c_r = m}} \lambda_{\phi_1}(c_1)\cdots\lambda_{\phi_r}(c_r)\cdot c_1^{s_1}\cdots c_r^{s_r}$$

is the $(m,1,\ldots,1)^{th}$ Hecke eigenvalue of $E_{P,\Phi}$.

The first coefficient $A_{\mathcal{P},\Phi}\big((1,\ldots,1),s\big)$ of $E_{\mathcal{P},\Phi}$

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Theorem (G-Stade-Woodbury (2023))

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Then the first coefficient $A_{\mathcal{P},\Phi}\big((1,\ldots,1),s\big)$ is given by

$$\boxed{\prod_{\substack{k=1\\ n_k\neq 1}}^{r} L^*\big(1, \mathrm{Ad}\ \phi_k\big)^{-\frac{1}{2}} \prod_{1\le j<\ell\le r} L^*\big(1+s_j-s_\ell,\ \phi_j\times\phi_\ell\big)^{-1}}$$

up to a non-zero constant factor with absolute value depending only on n.

Functional equation of Langlands Eisenstein series $E_{\mathcal{P},\Phi}$

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Theorem (Langlands 1976)

Suppose $\sigma \in S_r$ acts on $\mathcal{P}$, Φ, and s as above. Then we have the functional equation

$$\boxed{E^*_{\mathcal{P},\Phi}(g, s) = E^*_{\sigma\mathcal{P},\,\sigma\Phi}(g, \sigma s)}$$

for all $g \in \mathrm{GL}(n, \mathbb{R})$.

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for some affine transformation

$$\mu\left(\begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{bmatrix}\right) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & & & \\ a_{11} & a_{12} & \cdots & a_{r1} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_r \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{bmatrix},$$

where $a_{ij}, b_i \in \mathbb{R}$ for all $1 \le i, j \le r$.

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where $a_{ij}, b_i \in \mathbb{R}$ for all $1 \le i, j \le r$.

Then, in fact, $\mu = \sigma$ for some $\sigma \in S_r$ for which $\sigma\mathcal{P} = \sigma'\mathcal{P}$ and $\sigma\Phi = \sigma'\Phi$.

Uniqueness of functional equations continued

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Theorem (G-Stade-Woodbury 2023)

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Theorem (G-Stade-Woodbury 2023)

The uniqueness conjecture holds in the special case that

$$\phi_1 = \phi_2 = \cdots = \phi_r$$

is the same automorphic form with $r \geq 2$.

In this case, every permutation $\sigma \in S_r$ has the property that $\sigma\mathcal{P} = \mathcal{P}$ and $\sigma\Phi = \Phi$.

Uniqueness of functional equations continued

Theorem (G-Stade-Woodbury 2023)

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Recall the uniqueness conjecture:

$$E^*_{\mathcal{P},\Phi}(g,s) = E^*_{\sigma'\mathcal{P},\sigma'\Phi}(g,\mu(s))$$

with

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If the transformation μ given in the Uniqueness Conjecture is linear, i.e., $b_i = 0$ for each $i = 1, 2, \ldots, r$. Then the Uniqueness conjecture holds.