Motohashi formula and its generalization in a non-split case

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Joint work with Valentin Blomer and Paul D. Nelson

Motohashi formula: moments

- $V \in C_c^\infty(\mathbb{R})$:

$$\int_{\mathbb{R}} |\zeta(\tfrac{1}{2}+it)|^4 V(t)dt = \text{ main term}$$

$$+ \sum_j \frac{L(\frac{1}{2},\psi_j)^3}{L(1,\psi_j,\mathrm{Ad})} \check{V}(t_j) + \text{continuous.}$$

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- $V \rightsquigarrow \breve{V}$ certain integral transform given in terms of hypergeometric functions.
- $\{\psi_j\}$ an orthonormal basis of cusp forms for $\mathrm{SL}_2(\mathbb{Z})$.

Motohashi formula: shifted convolution

- V and ψ as before.

$$\sum_{n\geqslant 1}\tau(n)\tau(n+b)V\left(\frac{n}{b}\right) = \text{ main term} + \sum_j \frac{b^{1/2}\lambda_j(b)L(\frac{1}{2},\psi_j)^2}{L(1,\psi_j,\mathrm{Ad})}\widetilde{V}(t_j) + \text{continuous}$$

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- $V \rightsquigarrow \widetilde{V}$ certain integral transform.
- τ: divisor sum, λ_j: Hecke eigenvalues.

Motohashi formula: integral transform

$$\widetilde{V}(t) = \int_0^\infty \mathcal{K}_1(t,y)V(y)dy$$

and

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where

$$\mathcal{K}_1(t,y) := \frac{1}{2}\sum_{\pm}\left(1 \pm \frac{i}{\sinh(\pi t)}\right) y^{-\frac{1}{2}\mp it} \frac{\Gamma(\frac{1}{2}\pm it)^2}{\Gamma(1\pm 2it)}$$
$$F\left(\tfrac{1}{2}\pm it, \tfrac{1}{2}\pm it, 1\pm 2it, -1/y\right)$$

and

$$\mathcal{K}_2(t,y) := 2\int_0^\infty \left(\int_{-\infty}^\infty \cos\left(y \log\frac{x+1}{x}\right)\right)\mathcal{K}_1(t,x)\frac{dx}{\sqrt{x(1+x)}}.$$

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- No apparent reason for the existence of a relation between the fourth moment of GL(1) and the cubic moment of GL(2) L-functions.
- The integral transforms look "artificial".

Questions

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1. Can one replace $\tau(n)$ by other Hecke eigenvalues?
2. How about replacing $|\zeta(\frac{1}{2}+it)|^2$ by $L(\frac{1}{2}+it,\pi)$ for some GL(2)-automorphic π?
3. Can one give more structural interpretations of the integral transforms from a representation theoretic point of view?

Prior work

- Blomer–Harcos (2008): For $h \in \mathbb{C}_c^\infty((\mathbb{R}^\times)^2)$ and π_1, π_2 cuspidal representations for GL(2), there exists

$$\{W_\pi : \mathbb{R}^\times \to \mathbb{C} \mid \pi \text{ unitary aut. rep. for } \mathrm{GL}(2)\}$$

such that for $b > 0$

$$\sum_{n_1+n_2=b} \frac{\lambda_{\pi_1}(|n_1|)\lambda_{\pi_2}(|n_2|)}{\sqrt{|n_1 n_2|}} h(n_1, n_2) = \int_{\pi \neq \mathrm{triv}} \frac{\lambda_\pi(b)}{\sqrt{b}} W_\pi(b)$$

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- Growth properties of W_π are obtained that are enough for certain applications.

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- Michel–Venkatesh (2010): formally discussed a period-theoretic approach. Sheds some light on the local transforms.
- Wu (2022): Somewhat different period theoretic approach.

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- Nelson (2019+), Bolkanova–Frolenkov–Wu (2021+): Period theoretic approach.
- Kwan (2023): Cuspidal variants of the cubic moment.

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1. Replace $\tau(n)$ by $\lambda_\pi(n)$ in the shifted convolution problem.
2. Replace $|\zeta(\frac{1}{2}+it)|^2$ by $L(\frac{1}{2}+it,\pi)$
3. Structural description of the local integral transforms.
4. Integral transforms in the Maass cusp form case match with Motohashi's.

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- Motohashi's proof does not generalize right away!
- Factorization for $\tau(n)$ (equivalently, $|\zeta|^2$) is not available for λ (equivalently, L).
- Use: Michel–Venkatesh/Nelson's period approach.
- Integral transform: from local viewpoint. Serious convergence/regularization issue!

Global result (shifted convolution)

Theorem (BJN, 2024+)

Let π_1, π_2 *be two Maass-cusp forms for* $\mathrm{SL}_2(\mathbb{Z})$*. We have*

$$\sum_{n_1 - n_2 = b} \frac{\lambda_{\pi_1}(|n_1|)\lambda_{\pi_2}(|n_2|)}{\sqrt{|n_1 n_2|}} h(n_1, n_2) = \int_{\pi \neq \mathrm{triv}} \frac{\lambda_\pi(b)}{\sqrt{b}} c(\pi) h^\vee(\pi, b)\, \mathrm{d}\pi$$

where

$$|c(\pi)| = \frac{\sqrt{L(\frac{1}{2}, \pi_1 \otimes \pi_2 \otimes \pi)}}{L(1, \mathrm{Ad}, \pi)}$$

and $h \rightsquigarrow h^\vee$ *is a certain integral transform.*

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- Generalization to arbitrary number fields, ramifications, and cuspidal representations.

Local result (shifted convolution)

Theorem (BJN, 2024+)

We have

$$h^\vee(\pi, b) := \int_{\mathbb{R}^\times} h(tb, (t-1)b)\kappa(t, \pi; \pi_1, \chi_2)\,\mathrm{d}^\times t$$

where

$$\kappa(t, \pi; \pi_1, \chi_2) := \overline{\chi_2}\left(\frac{t-1}{t}\right)\left|\frac{t}{t-1}\right|^{1/2} \times \int_{\Re(\chi)=\sigma} \gamma(\tfrac{1}{2}, \pi \otimes \chi)\gamma(1, \pi_1 \otimes \overline{\chi_2}^{-1} \otimes \chi^{-1})\chi^{-1}(t)\,\mathrm{d}\chi$$

where χ_2 is the character of $\mathbb{R}^\times$ that induces π_2.

Global result (Moments of L-functions)

Theorem (BJN, 2024+)

Let

$$\omega(t) := \int_{\mathbb{R}^\times}\int_{\mathbb{R}^\times} h(z,yz)|y|^{it}\,\mathrm{d}^\times y\,\mathrm{d}^\times z, \quad h^\sharp(\pi) := \int_{\mathbb{R}^\times} h^\vee(\pi,y)\,\mathrm{d}^\times y.$$

Then

$$\int_{\mathbb{R}} L(\tfrac{1}{2}+it,\pi_1)L(\tfrac{1}{2}-it,\pi_2)\omega(t)\,\mathrm{d}t = \mathcal{M} + \int_{\pi\neq\mathrm{triv}} L(\tfrac{1}{2},\pi)c(\pi)h^\sharp(\pi)\,\mathrm{d}\pi$$

where

$$\mathcal{M} := \lim_{s\to 0}\sum_{\pm}\frac{L(1\pm s,\pi_1\otimes\pi_2)}{\zeta(1\pm 2s)}\int_{\mathbb{R}^\times} h(z,z)J^{\pm}_{\pi_1,\pi_2}(z)|z|^s\,\mathrm{d}^\times z$$

and $J^{\pm}_{\pi_1,\pi_2}$ are the Bessel distributions attached to $\pi_1\otimes\pi_2$.

Proof idea: Moments

- Kirillov theory: $\varphi_j \in \pi_j$ with

$$W_1(a(y_1))\overline{W_2(a(y_2))} = h(y_1, y_2).$$

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- Mellin expansion + Hecke theory of $\mathrm{GL}(2) \times \mathrm{GL}(1)$:

$$\mathrm{LHS} = \int_{\mathbb{R}} \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \varphi_1(a(y))|y|^{it} \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \overline{\varphi_2(a(y))}|y|^{-it} \,\mathrm{d}t$$
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- Spectral decomposition:

$$\text{LHS} \quad \text{``$=$''} \quad \int_{\pi} \sum_{\varphi \in \mathcal{B}(\pi)} \langle \varphi_1 \overline{\varphi_2}, \varphi \rangle \left(\int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \varphi \right) \mathrm{d}\pi.$$

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- Watson–Ichino + Hecke theory of $\mathsf{GL}(2) \times \mathsf{GL}(1) \rightsquigarrow \mathrm{RHS}$

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- Watson–Ichino + Hecke theory of $\mathrm{GL}(2) \times \mathrm{GL}(1) \rightsquigarrow \mathrm{RHS}$
- The last integral does not converge if π is Eisenstein!

Proof ideas

- Regularization:

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- Re-work with

$$\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} (\varphi_1\overline{\varphi_2})_{\text{reg}}(a(y))|y|^s + \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} (\varphi_1\overline{\varphi_2})_0(a(y))|y|^s.$$

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- Meromorphic continuation to $\Re(s) = 0$; Residues + Degenerate term $= \mathcal{M}$.

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- Then

$$h^\sharp(\pi) := \sum_{W \in \mathcal{B}(\pi)} \left(\int_{\mathbb{R}^\times} W(a(y)) \right) \int_{N \backslash G} W_1 \overline{f_2 W}.$$

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- Local representation theory (Mellin theory of Bessel distribution):

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- Can we find the inverse transform?

Local result

Theorem (BJN, 2024+)

If

$$h^\vee(\pi, y) := \sum_{W \in \mathcal{B}(\pi)} W(a(y)) \int_{N \backslash G} W_1 \overline{f_2 W},$$

then we have

$$h(y_1, y_2) = \frac{|y_2|}{|y_1 - y_2|} \int_{\widehat{G}} h^\vee(\pi, y_1 - y_2) \kappa \left(\tfrac{y_1}{y_1 - y_2}, \pi; \pi_1, \chi_2^{-1} \right) \mathrm{d}\pi$$

where $\widehat{G}$ *is the class of unitary irreducible representations of* $\mathrm{PGL}_2(\mathbb{R})$.

Local result

What we have so far:

$$\begin{array}{ccc} \mathrm{Whit}(\pi_1) \otimes \mathrm{Whit}(\pi_2) & \longrightarrow & \mathrm{Whit}(\pi_1) \otimes \mathrm{Ind}(\chi_2) \\ \big\downarrow & & \big\downarrow \\ \{h : \mathbb{R}^\times \times \mathbb{R}^\times \to \mathbb{C}\} & \longleftarrow\!\!\dashrightarrow & \{h^\vee : \widehat{G} \times \mathbb{R}^\times \to \mathbb{C}\}. \end{array}$$

An application

Theorem (BJN, 2024+)

Let $1 \leqslant Y \leqslant X/10$, $1 \leqslant b \leqslant X/2$ *an integer, and* $V \in C^\infty([1,2])$. *Then*

$$\mathcal{S}(X,Y,b) := \sum_n \lambda_{\pi_1}(n+b)\lambda_{\pi_2}(n) V\left(\frac{n-X}{Y}\right)$$

$$\ll_{V,\pi_1,\pi_2,\varepsilon} \frac{X^{1+\varepsilon}}{Y}(Y^{1/2}+b^{1/2}) \min\left(b^\theta, 1+\frac{Yb^{1/4}}{X}\right)$$

and

$$\int_X^{2X} |\mathcal{S}(x,Y,b)|^2 \,\mathrm{d}x \ll_{V,\pi_1,\pi_2,\varepsilon} b^{2\theta} X^{2+\varepsilon}\left(1+\frac{b}{Y}\right)$$

with $\theta \leqslant 7/64$.

Thanks for your attention!