One level density of a large orthogonal family of L-functions

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Shanks Conference

The Riemann zeta function

Let $\zeta(s)$ denote the usual Riemann zeta function.

- The Riemann zeta function can be analytically continued to the whole complex plane except a simple pole at $s = 1$.
- There are countably infinite number of zeros (non-trivial zeros of $\zeta(s)$) in the critical strip $0 \leq \mathrm{Re}(s) \leq 1$.

The Riemann hypothesis (RH)

All non-trivial zeros of $\zeta(s)$ lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Pair correlation of zeros of the Riemann zeta function

Assume RH, and let $\rho = 1/2 + i\gamma$ be nontrivial zeros of the Riemann zeta function.

Let $N(T)$ be the number of nontrivial zeros ρ, where $0 < \gamma < T$. (The number of zeros up to height T.)

$$N(T) \sim \frac{T}{2\pi} \log T.$$

If γ, γ' are consecutive ordinates in $[0, T]$,

$$\text{Average of } |\gamma - \gamma'| \sim \frac{2\pi}{\log T} =: \mathcal{AS}$$

on average.

Montgomery's work

How often is $|\gamma - \gamma'| < \frac{\mathcal{AS}}{2}$? (Related to Siegel zeros - see Conrey-Iwaniec.)

Montgomery studied a quantity of the form

$$\sum_{\substack{0<\gamma_{j_1},\gamma_{j_2}\leq T \\ j_1\neq j_2}} f\left((\gamma_{j_1} - \gamma_{j_2})\frac{\log T}{2\pi}\right),$$

where f is a suitable test function whose Fourier transform $\hat{f}$ is supported in (-1, 1). (Bandwidth limited.)

Bandwidth limit is a serious impediment.

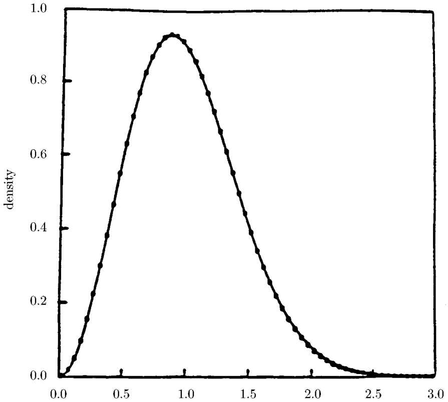
Montgomery showed that as $T \to \infty$

$$\frac{1}{N(T)} \sum_{0<\gamma\neq\gamma'\leq T} f\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) \to \int_{-\infty}^{\infty} f(x)\left(1-\left(\frac{\sin \pi x}{\pi x}\right)^2\right) dx,$$

and conjectured that this holds for all nice f.

Freeman Dyson pointed out to Montgomery that the factor $1-\left(\frac{\sin \pi x}{\pi x}\right)^2$ is the same as the distribution of the spacings of eigenvalues of the Gaussian unitary ensemble (GUE) distribution from random matrix theory.

Odlyzko's graph



Nearest neighbor spacings among 70 million zeroes beyond the 10^{20}-th zero of zeta, verses GUE. Source: *Zeros of zeta functions and symmetry*, Katz and Sarnak. Bulletin of AMS.

Statistics of L-functions: Katz-Sarnak philosophy

- Heuristic: statistics of families of L-functions should match analogous statistics from classical compact groups of random matrices. Katz and Sarnak proved that such heuristics hold in many examples of zeta and L-functions over function fields.
- The t-aspect pair correlation corresponds to unitary symmetry. Examples should exist for other symmetry groups (orthogonal and symplectic) for L-functions over number fields.

Other families: Iwaniec, Luo and Sarnak

Iwaniec, Luo, and Sarnak studied the one level density of low lying zeros of different families of L-functions.

The Katz-Sarnak philosophy predicts that the distribution of the low-lying zeros is governed by some underlying symmetry group.

Iwaniec, Luo and Sarnak verified this for certain families with bandwidth restrictions.

Holomorphic Hecke eigencuspforms

1. Let $S_k(q)$ be the space of cusp forms of fixed weight k and level q.
2. Let $\mathcal{H}_k(q) \subset S_k(q)$ be an orthogonal basis of the space of newforms consisting of Hecke cusp newforms, normalized so that the first Fourier coefficient is 1.
3. For $f \in \mathcal{H}_k(q)$, assume that the L-function $L(s, f)$ satisfies GRH and write the non-trivial zeros as $1/2 + i\gamma_f$.

One level density

Now let $\Phi(x)$ be an even Schwartz class function, and let $\widehat{\Phi}(t)$ be the usual Fourier transform.
Let

$$\mathscr{O}(q) := \sum_{f\in\mathcal{H}_k(q)}^{h} \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi}\log q\right).$$

Imagine that we want to know how many zeros of $L(s,f)$ are near $1/2$. Then we would want to set Φ to approximate the indicator function of a short interval centered at $1/2$.

Density conjecture and result of Iwaniec, Luo, and Sarnak

The *Density Conjecture* from the Katz-Sarnak philosophy predicts that

$$\lim_{q\to\infty} \mathscr{O}(q) = \int_{-\infty}^{\infty} \Phi(x)\left(1 + \frac{1}{2}\delta_0(x)\right)\, dx.$$

Iwaniec, Luo, and Sarnak verify that this holds for squarefree q, assuming GRH, and that the support of $\widehat{\Phi}$ is compact and contained in $(-2, 2)$.

We study a larger family of orthogonal $GL(2)$ L-functions by including an average over q. Fix a nice smooth function Ψ compactly supported on $\mathbb{R}_{>0}$ and let

$$\mathscr{O}\mathscr{L}(Q) := \frac{1}{N(Q)} \sum_{q} \Psi\left(\frac{q}{Q}\right) \sum_{f \in \mathcal{H}_k(q)}^{h} \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi} \log q\right),$$

where

$$N(Q) := \sum_{q} \Psi\left(\frac{q}{Q}\right) \sum_{f \in \mathcal{H}_k(q)}^{h} 1.$$

Main result

Joint with S. Baluyot and V. Chandee:

Theorem

Assume GRH. Let Φ be an even Schwartz function with $\widehat{\Phi}$ compactly supported in $(-4, 4)$. Then with notation as before,

$$\lim_{Q\to\infty} \mathscr{O}\mathscr{L}(Q) = \int_{-\infty}^{\infty} \Phi(x)\left(1 + \frac{1}{2}\delta_0(x)\right)\, dx,$$

where $\delta_0(x)$ is the Dirac δ distribution at $x = 0$.

Comments

The size of the support of $\widehat{\Phi}$ is doubled in our result compared to the result of Iwaniec, Luo and Sarnak.

We have taken advantage of the additional average over all levels q around size Q. This family of around Q^2 forms has elements with conductors around Q.

The ideas lead (after overcoming significant difficulties) to sixth and eighth moment asymptotics, critical line theorem, etc.

It is instructive to compare this with results on the similar family of Dirichlet L-functions attached to Dirichlet characters χ mod q with q around size Q. (Also a family of size Q^2 with conductor around Q.)

Comparison with Dirichlet L-functions

Fiorilli and Miller studied the family of Dirichlet L-functions over all characters modulo q, where $q \asymp Q$ and wanted to understand

$$\sum_{q \asymp Q} \sum_{\chi \bmod q} \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi} \log q\right).$$

Hughes and Rudnick studided verified the analogous density conjecture (for a smaller family of Dirichlet L-functions) with the support of $\widehat{\Phi}$ restricted within $[-2, 2]$.

Fiorilli and Miller were able to extend the support to $(-4, 4)$ assuming both GRH and a very strong "de-averaging hypothesis" on the variance of primes in residue classes.

Comparison with Dirichlet L-functions II

Drappeau, Pratt and Radziwiłł considered the one level density for the large family of Dirichlet L-functions and showed that the support of $\widehat{\Phi}$ can be extended to be within $(-2-50/1093, 2+50/1093)$ unconditionally.

Extending the support to something more like $(-4, 4)$ conditionally on GRH appears challenging.

One way to explain the difference:

1. When averaging over m, n and modulus $q \asymp Q$, the asymptotic large sieve from Conrey, Iwaniec, Soundararajan depends on a "complementary divisor trick" that allows us to switch the modulus to $\frac{m \pm n}{Q}$.
2. In our family of cusp forms, we use a "complementary level trick", which allows us to switch to level $\frac{\sqrt{mn}}{Q}$.
3. The geometric mean $\sqrt{mn}$ is the same as the arithmetic mean $\frac{m+n}{2}$ if and only if $m = n$, and is far smaller when m and n are far apart. In the context of one level density, we essentially have $m \ll Q^{4-\epsilon}$ and $n = 1$.

Initial setup

The explicit formula relates a sum over zeros to a sum over primes via $-\frac{L'}{L}(s)$, and reduces the problem to understanding

$$\sum_{q} \Psi\left(\frac{q}{Q}\right) \sum_{f \in \mathcal{H}_k(q)}^{h} \sum_{p} \frac{\log p \lambda_f(p)}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log q}\right)$$

We now apply a version of Petersson's formula for newforms (extra work); for simplicity, let's apply the usual Petersson's formula.

Ideas and ingredients

We are led to consider

$$\sum_{q} \Psi\left(\frac{q}{Q}\right) \sum_{p} \frac{\log p}{\sqrt{p}} \widehat{\Phi}\left(\frac{\log p}{\log q}\right) \sum_{c \geq 1} S(p, 1; cq) J_{k-1}\left(\frac{4\pi\sqrt{p}}{cq}\right). \tag{1}$$

The sum in (1) is easy to bound if $\widehat{\Phi}$ is supported on $(-2, 2)$ by an application of the additive Large Sieve.

We need to understand (1) when $\widehat{\Phi}$ has support extended to $(-4, 4)$. To do this, we extract non-trivial cancellation in the sum over Kloosterman sums.

For simplicity, suppose we are in transition region of the Bessel function, so that $cq \asymp \sqrt{p}$.

The support of $\widehat{\Phi}$ is compact and contained in $(-4, 4)$, so the sum over p is restricted to $p \leq Q^{4-\delta}$ for some $\delta > 0$ depending on Φ. Hence, we see that in this range

$$c \ll Q^{1-\delta/2}.$$

Complementary level

This motivates us to write (1) as

$$\frac{1}{N(Q)} \sum_{p} \frac{\log p}{\sqrt{p}} \sum_{c \geq 1} \mathcal{S}(p, c)$$

where

$$\mathcal{S}(p, c) = \sum_{q} S(p, 1; cq) \Psi\left(\frac{q}{Q}\right) \widehat{\Phi}\left(\frac{\log p}{\log q}\right) J_{k-1}\left(\frac{4\pi\sqrt{p}}{cq}\right)$$

can be transformed into a sum of forms of level c via Kuznetsov's formula.

We started with a sum over forms of level $q \asymp Q$ and in applying Kuznetsov's formula, we have lowered the conductor from $q \asymp Q$ to $c \ll Q^{1-\delta/2}$.

We have morally that

$$\sum_{p \asymp P} \frac{\log p}{\sqrt{p}} \lambda_g(p) \ll \log^2 Q$$

where λ_g is some type of Hecke eigenvalue associated to a holomorphic form, a Maass form, or an Eisenstein series, and $P \asymp Q^{4-\delta}$.

For holomorphic forms and Maass forms, we reduce the problem to bounding these sums by choosing a special basis based on Atkin-Lehner theory.

For Eisenstein series, we use recent explicit calculations of Kiral and Young.

Eisenstein contribution

In the case of an Eisenstein series, we may morally replace $\lambda_g(p)$ with sums involving $\chi(p)$ for certain Dirichlet characters χ.

When χ is non-principal, the approach from the last slide holds.

When $\chi = \chi_0$ is principal, the sum over p is genuinely large and is essentially $P^{1/2-it}\tilde{V}(1/2 - it)$ for some rapidly decaying function $\tilde{V}$, where t is the spectral parameter and P can be as large as $Q^{4-\delta}$.

Eisenstein contribution

This contribution turns out to be small by taking advantage of the average over the spectral parameter. This is in contrast to other situations (e.g. Duke-Friedlander-Iwaniec), where a truly large contribution arises from the Eisenstein series.

To be more precise, this naturally leads to something like

$$\int_{(1/2)} \frac{P^s \tilde{V}(s)}{L(2s, \chi)\zeta(2s)} ds.$$

Thank you for your attention!