Towards a New Shimura Lift

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Plan of This Talk

Plan:

1. Shimura Correspondences (overview)
2. A Conjecture of Bump-Friedberg-Ginzburg (2001)
3. Approach via the Relative Trace Formula
4. The Fundamental Lemma

For more details: arXiv:2202.01247 (not final version), to appear in *JEMS*. A second paper is in preparation.

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  - the *nonvanishing of a central L-value* $L(1/2, \pi, \chi)$ for some quadratic character χ, or equivalently by
  - the *nonvanishing of a certain period*, namely the integral of an automorphic form in the space of π over a cycle.

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  - the *nonvanishing of a certain period*, namely the integral of an automorphic form in the space of π over a cycle.
- Using this period, the Shimura correspondence may also be established by a relative trace formula (Jacquet).

Generalizations of the Shimura Lift, I

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To explain, let

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Then Kubota showed that the map $\kappa : \Gamma(n^2) \subset SL_2(\mathcal{O}) \to \mathbb{C}^\times$ given by

$$\kappa\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \left(\frac{c}{d}\right)_n$$

is a homomorphism.

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Remark: When $n = 2$, this is the multiplier system that arises from theta series over F.

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- Let G be any split connected reductive algebraic group over a global field F. Then one may define an n-fold metaplectic cover $\widetilde{G}^{(n)}(\mathbb{A})$ (or simply $\widetilde{G}^{(n)}$) of the adelic points $G(\mathbb{A})$:

$$1 \longrightarrow \mu_n \longrightarrow \widetilde{G}^{(n)} \longrightarrow G(\mathbb{A}) \longrightarrow 1$$

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  (Matsumoto, Bass-Milnor-Serre, Brylinski-Deligne).
- Interesting directions (but not for today's talk): Eisenstein series, theta correspondences, L-functions via generalized twisted doubling. Surprise: Quantum groups play a role.

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- Flicker and Kazhdan, Savin implies: There is a **conjectural global Shimura lift** that is known locally at almost all places.

Generalizations of the Shimura Lift, IV: Double Covers

- For the double cover, Jacquet conjectured (1991) that the Shimura lift from the double cover of GL_r to GL_r is related to the value of the standard L-function at $1/2$, and the image of the lift can be characterized by a period along an orthogonal group.

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- Do (4 papers from 2011 to 2021) studied the double cover of GL_n, developing the relative trace formula by first treating positive characteristic and then transferring the Fundamental Lemma to characteristic zero using the approach of Cunningham, Cluckers, Hales and Loeser.
- These works are consistent with the existence of a generalized Shimura correspondence but do not yet prove it.

Generalizations of the Shimura Lift, V: The Cubic Shimura Lift for SL_2

Besides the work of Flicker, there are two additional approaches to the Shimura lift for $\widetilde{SL}_2^{(3)}$. The first is due to Ginzburg, Rallis and Soudry (1997):

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- GRS observed that SL_2 and its 3-fold cover $\widetilde{SL}_2^{(3)}$ form a dual pair in the 3-fold cover of the exceptional group G_2.

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- They then used an exceptional theta correspondence (using the minimal representation on the cubic cover of G_2, constructed by Savin (1991)) to obtain a lifting of genuine cuspidal automorphic representations on $\widetilde{SL}_2^{(3)}$ to automorphic representations on $SL_2(\mathbb{A})$.

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- They also used their construction to determine the image of the map by a period.

Generalizations of the Shimura Lift, VI: The Cubic Shimura Lift for SL_2 (continued)

Let $\mathrm{Sym}^3 : SL_2 \to Sp_4$ be the symmetric cube map, and Θ_{Sp_4} be the theta representation on the metaplectic double cover of $Sp_4(\mathbb{A})$. This cover splits on the image of Sym^3.

Generalizations of the Shimura Lift, VI: The Cubic Shimura Lift for SL_2 (continued)

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Theorem (Ginzburg, Rallis and Soudry)

An irreducible cuspidal automorphic representation τ of $SL_2(\mathbb{A})$ is in the image of the rank-one cubic Shimura map if and only if the period integral

$$\int_{SL_2(F)\backslash SL_2(\mathbb{A})} \varphi(g)\, \theta(Sym^3(g))\, dg$$

is nonzero for some φ in the space of τ and some θ in the space of Θ_{Sp_4}.

Generalizations of the Shimura Lift, VII: The Cubic Shimura Lift for SL_2 (continued)

- The second approach to the cubic Shimura lifting in the rank one case is due to Mao and Rallis (1999). They used the period of Ginzburg, Rallis and Soudry to establish the Shimura lift from $\widetilde{SL_2}^{(3)}$ to $SL_2(\mathbb{A})$ via a relative trace formula.

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$SL_2^{(2)}(\mathbb{A})$	lifts to	$PGL_2(\mathbb{A})$	Shimura, Waldspurger,....
$SL_2^{(3)}(\mathbb{A})$	lifts to	$SL_2(\mathbb{A})$	Flicker, GRS, Mao-Rallis.

Generalizations of the Shimura Lift, VIII

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$$SL_2^{(3)}(\mathbb{A}) \quad \text{lifts to} \quad SL_2(\mathbb{A}) \quad \text{Flicker, GRS, Mao-Rallis.}$$

Based on Savin's work, one expects

$$SL_n^{(n)}(\mathbb{A}) \quad \text{lifts to} \quad PGL_n(\mathbb{A}).$$

$$SL_n^{(r)}(\mathbb{A}) \quad \text{lifts to} \quad SL_n(\mathbb{A}) \quad \text{if (n,r)=1.}$$

Summary: Where We Stand

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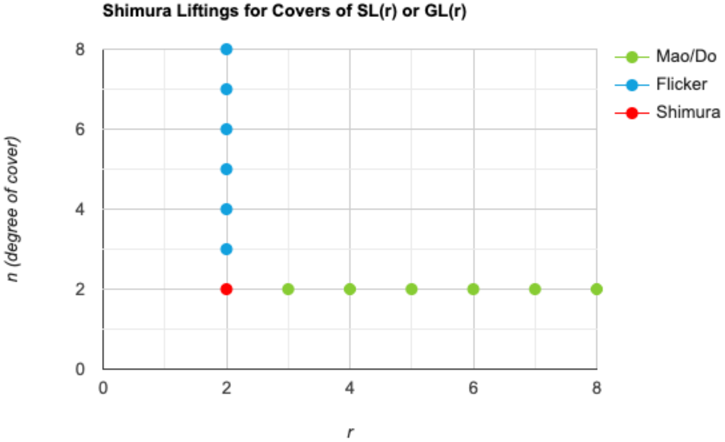
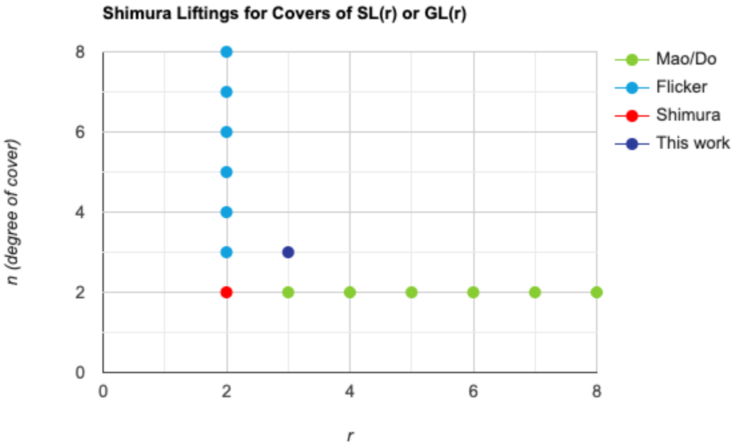


This Work

This talk concerns a new Shimura lift:



A Conjecture of Bump-Friedberg-Ginzburg, I

In a paper in 2001 in the *Israel J. of Math.*, Bump, Ginzburg and I looked at the cubic Shimura lift for SL_3. Based on Savin's work on the local Shimura correspondence,

$$SL_3^{(3)}(\mathbb{A}) \quad \text{should lift to} \quad PGL_3(\mathbb{A}).$$

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We made a conjecture about how to characterize the image of this hypothetical lift and provided some evidence.
Key ingredients:

- Let Θ_{SO_8} be the *automorphic minimal representation* on the split special orthogonal group $\mathrm{SO}_8(\mathbb{A})$. This representation was constructed by Ginzburg, Rallis and Soudry as a multi-residue of a Borel Eisenstein series on SO_8.
- Let Ad denote the Adjoint representation $\mathrm{Ad} : PGL_3 \to \mathrm{SO}_8$ (realized as explained below).

A Conjecture of Bump-Friedberg-Ginzburg, II

Supposing that a Shimura lift exists in this case, we made the following Conjecture.

Conjecture (Bump, Friedberg, Ginzburg)

Let π be an irreducible cuspidal automorphic representation of $PGL_3(\mathbb{A})$. Then π is in the image of the cubic Shimura correspondence from $\widetilde{SL}_3^{(3)}(\mathbb{A})$ if and only if the period

$$\int_{PGL_3(F)\backslash PGL_3(\mathbb{A})} \varphi(g)\,\theta(\mathrm{Ad}(g))\,dg$$

is nonzero for some φ in the space of π and some θ in Θ_{SO_8}.

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We remark that we do not know of an extension of this conjecture to higher rank special linear groups or higher degree covers.

A Conjecture of Bump-Friedberg-Ginzburg, III

In our 2001 paper, we presented two pieces of evidence for this conjecture.

- If π is not cuspidal but rather an Eisenstein series induced from cuspidal data τ on $SL_2(\mathbb{A})$, then formally unfolding the PGL_3 period in this case, we showed that the resulting integral is nonvanishing for some choice of data if and only if the GRS period for τ is nonvanishing for some choice of data. This involved a rather remarkable identity between theta series on the double covers of Sp_8 and Sp_4.

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- We gave evidence from finite fields, looking at families of Deligne-Lusztig characters in general position. (Note: there is no metaplectic cover over finite fields, but the n-th order Shimura map should correspond to the n-th power map on induction data.) We also showed that this approach could be used to predict the GRS period as well.

Approach via the Relative Trace Formula, I

Our goal, following the path of Jacquet and of Mao-Rallis, is to establish a relative trace formula that will

- prove the existence of the Shimura map from $SL_3^{(3)}(\mathbb{A})$ to $PGL_3(\mathbb{A})$ and
- prove the period conjecture of Bump, Friedberg and Ginzburg.

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The key step in the relative trace formula is a *comparison of distributions*. We will describe them on the next slides.

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First, we introduce some standard general notation:

- For any algebraic group H defined over F, denote by $[H] = H(F)\backslash H(\mathbb{A})$ the automorphic quotient.
- For an affine variety X defined over F denote by $\mathcal{S}(X(\mathbb{A}))$ the space of Schwartz-Bruhat functions on $X(\mathbb{A})$.

Approach via the Relative Trace Formula, II: the relative distribution

We describe the distribution that involves a period.

- Let $G = \mathrm{PGL}(3)$ considered as an algebraic group defined over F.
- Let N be the standard maximal unipotent subgroup of G of upper triangular 3×3 unipotent matrices.
- Let ψ be a (fixed) non-trivial character of $F\backslash\mathbb{A}$. We also use ψ for the generic character $\psi(n) = \psi(n_{1,2} + n_{2,3})$ of $[N]$.

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For $\Theta \in \Theta_{\mathrm{SO}_8}$ we consider the distribution $I(\Theta)$ on $G(\mathbb{A})$ defined by

$$I(f, \Theta) = \int_{[N]} \int_{[G]} \left\{ \sum_{\gamma \in G(F)} f(g^{-1}\gamma n) \right\} \Theta(\mathrm{Ad}(g))\psi(n)\, dg\, dn. \qquad (1)$$

Here $f \in \mathcal{S}(G(\mathbb{A}))$.

Approach via the Relative Trace Formula, III: the Kuznetsov distribution

We describe the Kuznetsov distribution on the 3-fold cover.

- The group $SL_3(F)$ splits in $\widetilde{SL}_3^{(3)}(\mathbb{A})$; we continue to denote its image in this group by $SL_3(F)$.
- The group $N(\mathbb{A})$ also splits in $\widetilde{SL}_3^{(3)}(\mathbb{A})$ and we continue to denote by $N(\mathbb{A})$ the image of this splitting.

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The Kuznetsov trace formula is the distribution on $\widetilde{SL}_3^{(3)}(\mathbb{A})$ defined by

$$J(f') = \int_{[N]} \int_{[N]} \Big\{ \sum_{\gamma \in SL_3(F)} f'(n_1^{-1} \gamma n_2) \Big\} \psi(n_1 n_2) \, dn_1 \, dn_2.$$

Here $f' \in \mathcal{S}(\widetilde{SL}_3^{(3)}(\mathbb{A}))$ is genuine.

Approach via the Relative Trace Formula, IV: outline of steps

Our goal is to *establish an equality* of the two distributions $I(f, \Theta)$ and $J(f')$ for suitable matching functions $f \leftrightarrow f'$.

The steps are as follows:

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With the comparison established, one then uses spectral expansions to deduce the desired global consequences.

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3. Compare the orbital integrals for the full spherical Hecke algebras.
4. Establish the local matching for smooth functions $f_v \leftrightarrow f'_v$.

With the comparison established, one then uses spectral expansions to deduce the desired global consequences.

Our first paper handles steps 1 and 2. We have just completed step 3; a paper (working title *On the cubic Shimura lift to PGL(3): Hecke correspondences and applications*) is in preparation.

The Kuznetsov Distribution: reduction to local

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- Using the Bruhat decomposition, one sees that representatives Ξ_{rel} for the relevant orbits are:

$$(i)\ \zeta I_3, \zeta \in \mu_3; \quad (ii)\ \begin{pmatrix} & a^{-2} \\ aI_2 & \end{pmatrix}, a \in F^*; \quad (iii)\ \begin{pmatrix} & aI_2 \\ a^{-2} & \end{pmatrix}, a \in F^*;$$

$$(iv)\ \begin{pmatrix} & & b^{-1} \\ & -a^{-1}b & \\ a & & \end{pmatrix}, a,\ b \in F^* \quad \text{(the big cell)}$$

The last orbit gives the main orbital integral.

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- One has the equality

$$J(f') = \sum_{\xi \in \Xi_{\mathrm{rel}}} \mathcal{O}(\xi, f')$$

where $\mathcal{O}(\xi, f')$ is given by an adelic integration, so for factorizable test functions f' it is the product of local integrals.

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- To address this, we make use of the dual pair (SL_2, SO_8) inside Sp_{16}, and realize the automorphic minimal representation Θ_{SO_8} as the theta lift of the trivial representation from $SL_2(\mathbb{A})$ to $\mathrm{SO}_8(\mathbb{A})$ (Ginzburg, Rallis, Soudry).

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- This is related to Kudla and Rallis's work on Siegel-Weil identities, and requires a suitably regularized theta lift.

The Relative Trace Formula, II: reduction to local (continued)

- This realization allows us to do an unfolding and to express the distribution $I(f,\Theta)$, $f \in \mathcal{S}(\mathrm{PGL}_3(\mathbb{A}))$, as a sum of factorizable orbital integrals

$$I(f,\Theta_\phi) = \sum_{\xi \in \Xi_{\mathrm{rel}}(\mathrm{PGL}(3))} \mathcal{O}(\xi, f*\phi), \quad \phi \in \mathcal{S}(\mathbb{A}^8).$$

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where ω is the Weil representation on $\mathrm{Sp}_{16}^{(2)}(\mathbb{A})$, and for $\xi\in F^8$,

$$\mathcal{O}(\xi,\phi)=\int_{(\mathrm{SL}_2\times N)_\xi(\mathbb{A})\backslash\mathrm{SL}_2(\mathbb{A})\times N(\mathbb{A})}(\omega(h,n)\phi)(\xi)\psi(n)\,dn.$$

The Relative Trace Formula, III: reduction to local (continued)

- Working with the Weil representation, we determine a set of representatives $\Xi_{\mathrm{rel}}(\mathrm{PGL}(3)) \subset F^8$ of relevant orbits.

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- This allows us to establish a matching of relevant orbits on the two sides and so reduce the problem to a comparison of local orbital integrals.
- Next, we formulate this matching precisely for the big cell orbital integrals.

The Fundamental Lemma, I

We use the following (standard) notation:

- F a non-archimedean local field
- $\mathcal{O}$ the ring of integers in F
- $\mathfrak{p}$ its maximal ideal, ϖ a uniformizer in F

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Assume that F contains a primitive cube root of unity ρ and let $\mu_3 = \langle \rho \rangle$.

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We now describe the *local integrals for the big cell relevant orbits* on the two sides. Each is parameterized by $F^* \times F^*$.

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After the use of the theta lifting described above and an unfolding, we show that the first "big cell" orbital integral arising from the period side of the RTF is given by the expression

$$I(a,b) = \int_F \int_F \int_F \int_F \int_{F^*} \mathbf{1}_{\mathcal{O}^8}[(0, t, t^{-1}a, t(x+as), t^{-1}b, t(bs-y), t^{-1}(xb+y\rho^2 a), t[(xb+y\rho^2 a)s - (xy+z\rho)])] \, |t|^2 \, \psi[x+y+2a(xy-z)y\rho^2 - 2bxz\rho] \, d^*t \, ds \, dx \, dy \, dz$$

with $a, b \in F^*$. Here $\mathbf{1}_{\mathcal{O}^8}$ denotes the characteristic function of $\mathcal{O}^8$.

The Fundamental Lemma, III: first orbital integral, continued

- Here we realize the Adjoint representation of $PGL(3)$ as the map $PGL(3) \to GL(\mathfrak{sl}(3))$ defined by conjugation. Considering $\mathfrak{sl}(3)$ as a quadratic space with respect to the bilinear form $< x, y >= \mathrm{Tr}(xy)$, the image lies in the special orthogonal group $SO(\mathfrak{sl}(3))$.

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- With respect to the basis

$$\{e_{1,3}, e_{2,3}, e_{1,2}, e_{1,1} - e_{2,2}, e_{2,2} - e_{3,3}, e_{2,1}, e_{3,2}, e_{3,1}\}$$

of $\mathfrak{sl}(3)$ where the $e_{i,j}$ are the standard elementary matrices, $SO(\mathfrak{sl}(3))$ is isomorphic to $SO(J)$ where

$$J = \begin{pmatrix} & & & w_3 \\ & 2 & -1 & \\ & -1 & 2 & \\ w_3 & & & \end{pmatrix} \quad \text{with} \quad w_3 = \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}.$$

The Fundamental Lemma, III: first orbital integral, continued

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- If F contains the cube roots of unity, then $SO(J)$ is split over F and J can be conjugated to w_8.

The Fundamental Lemma, IV: first orbital integral, continued

Carrying this out explicitly leads to the expression

$$\mathrm{Ad}\begin{pmatrix}1 & x & z\\ & 1 & y\\ & & 1\end{pmatrix} =$$

$$\begin{pmatrix}1 & x & -y & -(xy+\rho z) & -(xy+\rho^2 z) & x(xy-z) & -zy & z(xy-z)\\ & 1 & 0 & \rho^2 y & \rho y & xy-z & -y^2 & y(xy-z)\\ & & 1 & x & x & -x^2 & z & -zx\\ & & & 1 & 0 & -x & -\rho y & \rho xy+\rho^2 z\\ & & & & 1 & -x & -\rho^2 y & \rho^2 xy+\rho z\\ & & & & & 1 & 0 & y\\ & & & & & & 1 & -x\\ & & & & & & & 1\end{pmatrix}$$

Computing with this, we obtain with the expression above for the orbital integral.

The Fundamental Lemma, V: second orbital integral

To give the second orbital integral, we need further notation involving the metaplectic group $\widetilde{\mathrm{SL}}_3^{(3)}(F)$.

- We realize $\widetilde{\mathrm{SL}}_3^{(3)}(F)$ as the set $\mathrm{SL}_3(F) \times \mu_3$ with the group operation given by
$$(g_1, z_1)(g_2, z_2) = (g_1 g_2, z_1 z_2 \sigma(g_1, g_2))$$
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- Let $K = \mathrm{SL}_3(\mathcal{O})$. The group K also a splitting in $\widetilde{\mathrm{SL}}_3^{(3)}(F)$, but this requires a nontrivial section: there is a map $\kappa : K \to \mu_3$ such that the map $g \mapsto (g, \kappa(g))$ embeds K into $\widetilde{\mathrm{SL}}_3^{(3)}(F)$.

The Fundamental Lemma, VI: second orbital integral, continued

Fix an identification of μ_3 as a subgroup of $\mathbb{C}^*$ and let $f_0 : \widetilde{\mathrm{SL}}_3^{(3)}(F) \to \mathbb{C}$ be the unit element in the genuine spherical Hecke algebra, defined by

$$f_0((g,z)) = \begin{cases} z\kappa(g)^{-1} & g \in K \\ 0 & \text{otherwise} \end{cases} \quad g \in \mathrm{SL}_3(F),\ z \in \mu_3.$$

The Fundamental Lemma, VI: second orbital integral, continued

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The second family of integrals is defined by

$$J(a,b) = \int_{N(F)} \int_{N(F)} f_0((n_1 g_{a,b} n_2, 1))\psi(n_1 n_2)\ dn_1\ dn_2$$

where

$$g_{a,b} = \begin{pmatrix} & & b^{-1} \\ & -a^{-1}b & \\ a & & \end{pmatrix} \in \mathrm{SL}_3(F).$$

The Fundamental Lemma, VII: second orbital integral, continued

To give a sense of the complexity of this orbital integral, let $(\cdot,\cdot)_3 : F^* \times F^* \to \mu_3$ be the cubic Hilbert symbol. Then:

Lemma

Let $g = n_1 g_{a,b} n_2 \in K$ with $a,\ b \in F^*$ and $n_1,\ n_2 \in N(F)$.

1. *if $|a| = 1 = |b|$ then $\kappa(g) = 1$.*
2. *if $|a| = 1 > |b|$ then $\kappa(g) = (b,a)_3(a^{-1}b, y_2)_3$.*
3. *if $|a|, |b| < 1$ then at most one of ay_1 and ax_2 is in $\mathcal{O}^*$ and*
    1. *if $|ay_1| = 1$ then $\kappa(g) = (b,a)_3(y_1y_2, ab^{-1})_3(y_2, y_1)_3$*
    2. *if $|ax_2| = 1$ then $\kappa(g) = (b, ax_2)_3(z_2x_2^{-1} - y_2, b^{-1}ax_2)_3$*
    3. *if $|ay_1|, |ax_2| < 1$ then $ay_1x_2 - a^{-1}b \in \mathcal{O}^*$ and if in addition $y_1 \neq 0$ then*

$$\kappa(g) = (b,a)_3(y_1, ab^{-1})_3(y_1a(x_2y_2 - z_2), ay_1x_2 - a^{-1}b)_3(b, x_2y_2 - z_2)_3.$$

This is proved using an algorithm presented in Bump and Hoffstein (1987).

The Fundamental Lemma, VIII

Our main result is this:

Theorem (The Fundamental Lemma for the Big Cell Orbital Integrals)

Assume $p > 3$. For any a, $b \in F^*$ we have

$$I(a,b) = (c,d)_3 \; J(c,d), \quad \textit{where} \quad c = -54a, \; d = 54b.$$

We also establish matching for the other relevant orbits, up to transfer factors that become trivial on taking the product over all places.

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We also establish matching for the other relevant orbits, up to transfer factors that become trivial on taking the product over all places.

We have recently extended this matching to the *full spherical Hecke algebras* on the two groups (again for all relevant orbits).

The Fundamental Lemma, IX: the number theoretic facts behind the proofs

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- As Duke and Iwaniec remark, the analogous fact at a real archimedean place is Nicholson's formula for the Airy integral

$$\int_0^\infty \cos(t^3 + ty)\, dt = \tfrac{1}{3} y^{1/2} K_{1/3}(2(y/3)^{3/2}) \quad y > 0.$$

The Fundamental Lemma, X: the number theoretic facts behind the proofs, continued

- The result of Duke and Iwaniec applies when the additive character is of conductor 1, that is, to exponential sums of the form

$$\sum_{x \in k_F} \exp\left(2\pi i \frac{\mathrm{tr}(ax^3 + bx)}{p}\right).$$

These sums are shown to be equal to certain Kloosterman sums for k_F^* with a cubic character.

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  These sums are shown to be equal to certain Kloosterman sums for k_F^* with a cubic character.

- We show that a similar relation is true for exponential sums that involve additive characters of higher conductor. For higher conductor, the proof is based on *the method of stationary phase*.

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  These sums are shown to be equal to certain Kloosterman sums for k_F^* with a cubic character.
- We show that a similar relation is true for exponential sums that involve additive characters of higher conductor. For higher conductor, the proof is based on *the method of stationary phase*.
- In our situation, the orbital integrals frequently reduce to integrals of pairs of Kloosterman integrals with cubic characters. Using this identity twice to make them into integrals of pairs of cubic exponential integrals, we are able to effect the desired comparison.

The Fundamental Lemma, XI: the number theoretic facts behind the proofs, continued

Here is the extension of the identity of Duke and Iwaniec. For $t \in F^*$ and $a, b, c, d \in F$, define the integrals

$$\mathcal{C}(a,b) = \int_{\mathcal{O}} \psi(ax+bx^3)\, dx \quad \text{and} \quad \mathcal{K}(t;c,d) = \int_{\mathcal{O}^*} (t,u)_3\, \psi(cu+du^{-1})\, du.$$

The Fundamental Lemma, XI: the number theoretic facts behind the proofs, continued

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Proposition

We have

$$\mathcal{C}(a, -3^{-3}c^{-1}d^{-1}a^3) = (t, c^{-1}d)_3\mathcal{K}(t;c,d)$$

whenever

- *either* $|a| = |c| = |d| = q$ *and* $3 \nmid \mathrm{val}(t)$
- *or* $|a| = |c| = |d| > q$.

The Fundamental Lemma, XII: the number theoretic facts behind the proofs, continued

- We also need an identity that we apply exactly exactly for the cases where the computation of $J(a, b)$ involves Kloosterman sums without a cubic character and so the result of Duke-Iwaniec is not applicable.

The Fundamental Lemma, XII: the number theoretic facts behind the proofs, continued

- We also need an identity that we apply exactly exactly for the cases where the computation of $J(a, b)$ involves Kloosterman sums without a cubic character and so the result of Duke-Iwaniec is not applicable.
- This is given as follows. For $\ell \in \mathbb{Z}$ let

$$\mathcal{C}_\ell(a, b) = \int_{\mathrm{val}(x)=\ell} \psi(ax + bx^3)\ dx.$$

Lemma

For $|a| = |b| \leq q^{-3}$ we have

$$3+|a|^{-1}\sum_{k=0}^{2}\mathcal{C}_{\mathrm{val}(a)-1}(b^{-1}+a^{-1}\rho^k, -3^{-3}a^{-1}b^{-1}) = \begin{cases} q & -(ab^{-1})^3 \in 1+\mathfrak{p} \\ 0 & -(ab^{-1})^3 \notin 1+\mathfrak{p}. \end{cases}$$

Thanks!

Thank you for listening, and thanks to the organizers for putting together this exciting conference!