A modular framework for generalized Hurwitz class numbers

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Joint work with Andreas Mono

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Outline

- Half-integral weight modular forms - definition and background.
- Hurwitz class numbers and holomorphic modular forms.
- Non-holomorphic modular forms: harmonic and sesquiharmonic Maass forms.
- Hurwitz class numbers and non-holomorphic modular forms.
- Proof outline.
- Open questions.

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Modular forms notation

$\mathbb{H} := \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$.

$q := e^{2\pi iz}$ for $z = x + iy \in \mathbb{H}$.

$\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is a congruence subgroup.

$$\Gamma_0(N) := \{\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c\}.$$

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For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, $k \in \frac{1}{2}\mathbb{Z}$, and $f : \mathbb{H} \to \mathbb{C}$,

$$(f\big|_k \gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right).$$

The Jacobi theta function

$$\theta(z) := \sum_{n\in\mathbb{Z}} q^{n^2}.$$

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$,

$$\nu_\theta(\gamma) := (cz+d)^{-\frac{1}{2}} \frac{\theta(\gamma z)}{\theta(z)} = \left(\frac{c}{d}\right) \epsilon_d^{-1},$$

where

$$\epsilon_d := \begin{cases} 1, & d \equiv 1 \pmod 4, \\ i, & d \equiv 3 \pmod 4. \end{cases}$$

Holomorphic modular forms

Let χ be a Dirichlet character mod N.

Definition

$f : \mathbb{H} \to \mathbb{C}$ is a weight $k \in \frac{1}{2}\mathbb{Z}$ modular form with respect to $\Gamma_0(N)$ if:

1. For all $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N)$, we have

$$f\big|_k \gamma = \begin{cases} \chi(d) f & k \in \mathbb{Z} \\ \nu_\theta^{2k}(\gamma)\chi(d) f & k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$

2. f is holomorphic on $\mathbb{H}$.
3. $f(z)$ has at most polynomial growth at cusps.

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2. f is holomorphic on $\mathbb{H}$.
3. $f(z)$ has at most polynomial growth at cusps.

- $M_k(N, \chi)$ is the space of such functions, $S_k(N, \chi)$ is the subspace of cusp forms, and $E_k(N, \chi) = S_k(N, \chi)^\perp$ is the Eisenstein subspace.
- Suppress χ when $\chi = \chi_0$ is the trivial character.

Kohnen's Plus Space

Kohnen (1981)

Let N be odd and square-free, χ a Dirichlet character mod $4N$.

$$M^{+}_{k+\frac{1}{2}}(4N,\chi) :=$$
$$\left\{ \sum a_n q^n \in M_{k+\frac{1}{2}}(4N,\chi) : a_n = 0 \text{ if } n\chi(-1)(-1)^k \equiv 2,3 \pmod{4} \right\}$$

If $\chi^2 = 1$, the newform subspace of $S^{+}_{k+\frac{1}{2}}(4N,\chi)$ is isomorphic to $S_{2k}(N,\chi)$ under a linear combination of Shimura maps.

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The Plus-Space Projector

Kohnen (1981)

If $\chi^2 = 1$, we have the projection map $\mathrm{pr}^+ : M_{k+\frac{1}{2}}(4N, \chi) \to M^+_{k+\frac{1}{2}}(4N, \chi)$ given by:

$$f|\mathrm{pr}^+ := \frac{f}{3} + \frac{(-1)^{\lfloor \frac{k+1}{2} \rfloor}}{3\sqrt{2}} \sum_{\nu \pmod 4} f|_{k+\frac{1}{2}}(BA^*_\nu).$$

Here

$$A^*_\nu := \left(\begin{pmatrix} 1 & 0 \\ 4N\nu & 1 \end{pmatrix}, (4N\nu z + 1)^{k+\frac{1}{2}} \right), \qquad B := \left(\begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}, i \right),$$

elements of the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$.

Notation

Throughout, N is odd and square-free, and D is a discriminant.

$$\chi_D := \left(\frac{D}{\cdot}\right)$$

$$L_{\mathcal{N}}(\chi, s) := L(\chi, s) \prod_{p \mid \mathcal{N}} \left(1 - \chi(p) p^{-s}\right) = \prod_{p \nmid \mathcal{N}} \frac{1}{1 - \chi(p) p^{-s}}$$

$$= \sum_{\substack{n \geq 1 \\ \gcd(n, \mathcal{N}) = 1}} \frac{\chi(n)}{n^s}$$

with

$$\sigma_{N,1}(r) := \sum_{\substack{0 < d \mid r \\ \gcd(d, N) = 1}} d, \qquad \sigma_{\ell,N,1}(r) := \sum_{\substack{d \mid r \\ \gcd(d, \ell) = 1 \\ \gcd\left(\frac{r}{d}, \frac{N}{\ell}\right) = 1}} d$$

Cohen-Eisenstein series

Let $r \in \mathbb{Z}^+$, $n = (-1)^r Df^2$, D a fundamental discriminant.

$$H(r,n) = L(1-r, \chi_D) \sum_{d|f} \mu(d)\chi_D(f) d^{r-1} \sigma_{2r-1}(f/d)$$

- **Cohen (1975)**: $\sum_{n=0}^{\infty} H(r,n) q^n \in M^+_{r+\frac{1}{2}}(4)$.

Pei and Wang's generalized class numbers

$$H_{k,\ell,N}(n) :=$$

$$\begin{cases} L_\ell\left(1-k,\chi_t\right) \prod\limits_{p \mid \frac{N}{\ell}} \frac{1-\chi_t(p)p^{-k}}{1-p^{-2k}} \sum\limits_{\substack{a \mid m \\ \gcd(a,N)=1}} \mu(a)\chi_t(a)\sigma_{\ell,N,1}\left(\frac{m}{a}\right) & \begin{array}{l}\text{if } \ell \neq N \\ -n = tm^2, \\ t \text{ fundamental,}\end{array} \\ L_N\left(1-2k,\mathrm{id}\right) & \text{if } n=0, \ell = N \\ L_N(0,\chi_t) \sum\limits_{\substack{a \mid m \\ \gcd(a,N)=1}} \mu(a)\chi_t(a)a^{k-1}\sigma_{N,2k-1}\left(\frac{m}{a}\right) & \begin{array}{l}\text{if } \ell = N \\ -n = tm^2, \\ t \text{ fundamental,}\end{array} \\ 0 & \text{else.} \end{cases}$$

Pei and Wang's work

Pei and Wang (2003)

Weights $\geq \frac{5}{2}$:
For $k \geq 2$, the set $\{\sum_{n\geq 0} H_{k,\ell,N}(n)q^n : \ell \mid N\}$ is a basis of $E^+_{k+\frac{1}{2}}(4N)$.

Weight $\frac{3}{2}$:
The set $\{\mathscr{H}_{\ell,N}(z) := \sum_{n\geq 0} H_{1,\ell,N}(n)q^n : \ell \mid N, \ell \neq 1\}$ is a basis of $E^+_{1+\frac{1}{2}}(4N)$.

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- In this case we have $\ell \neq 1$ because $\mathscr{H}_{1,N}(z)$ is not modular.
- Set $H_{\ell,N}(n) := H_{1,\ell,N}(n)$.
- $H_{1,1}(n) = H(n) =$ the nth Hurwitz class number.
- We call the $H_{\ell,N}(n)$ Generalized Hurwitz Class Numbers.

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Binary Quadratic Forms

For discriminants n, let

$$\mathcal{Q}_n := \{ax^2 + bxy + cy^2 : a, b, c \in \mathbb{Z}, b^2 - 4ac = n\}.$$

We have the group action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{Q}_n$ given by

$$Q^\gamma(x, y) = Q(ax + by, cx + dy),$$

for $\gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.

Definition of the Hurwitz Class Numbers $H(n)$

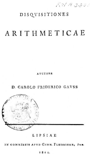
$H(n) := 0$ if $-n$ is not a discriminant.

For $n > 0$ such that $-n$ is a discriminant,

$$H(n) := \sum_{Q \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathcal{Q}_{-n}} \frac{2}{|\operatorname{Stab}(Q)|}.$$

$H(0) := -\frac{1}{12}$.

Gauss (1798)



DISQVISITIONES

ARITHMETICAE

AVCTORE

D. CAROLO FRIDERICO GAVSS

LIPSIAE

- Formulated conjectures about the size of $\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{Q}_{-n} \to$ "Class Number One Problem, Gauss Conjectures".
- Counted the number of ways of writing n as a sum of three squares. The number of ways is

$$r_3(n) = 12(H(4n) - 2H(n))$$

Formula for $H(n)$

Let $n = -Df^2 > 0$, where D is a negative fundamental discriminant.

$$H(n) = \frac{h(D)}{w(D)} \sum_{d|f} \mu(d) \left(\frac{D}{d}\right) \sigma\left(\frac{f}{d}\right)$$
$$= L(0, \chi_D) \sum_{d|f} \mu(d) \left(\frac{D}{d}\right) \sigma\left(\frac{f}{d}\right)$$

- $h(D)$ is the size of the ideal class group of $\mathbb{Q}(\sqrt{D})$,
- $w(D)$ is half the number of roots of unity in $\mathbb{Q}(\sqrt{D})$,
- μ is the Möbius function,
- σ is the sum of divisors function,

Connections to half-integral weight modular forms

- **Gauss:**

$$\Theta^3(z) = \sum_{n\geq 0} r_3(n)q^n = 12\sum_{n\geq 0}(H(4n) - 2H(n))q^n \in M_{\frac{3}{2}}(4)$$

- **Bringmann and Kane:** For odd primes ℓ,

$$\sum_{n\geq 0}(H(\ell n) - \ell H(n))q^n \in M_{\frac{3}{2}}(4\ell, \chi_\ell)$$

New Examples

Theorem 1 (B, Mono 2024)

Let $N > 1$ be odd and square-free. Then

$$\prod_{p|N} \frac{1}{p+1} \sum_{n\geq 0} H(n)q^n - \frac{1}{N} \sum_{n\geq 1} H_{1,N}(n)q^n \in E_{\frac{3}{2}}^{+}(4N).$$

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If N is an odd prime then

$$\frac{1}{1-N} \mathscr{H}_{N,N}(z) = \sum_{n \geq 0} H(n) q^n - \frac{N+1}{N} \sum_{n \geq 1} H_{1,N}(n) q^n.$$

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- These functions arise from the $\ell = 1$ case of the Generalized Hurwitz Class numbers of Pei and Wang.

Examples

$N = 5$:

$$3\mathscr{H}_{5,5}(z) = -12\left(\sum_{n\geq 0} H_{1,1}(n)q^n - \frac{6}{5}\sum_{n\geq 1} H_{1,5}(n)q^n\right)$$

$$= \sum_{(a,b,c)\in\mathbb{Z}^3} q^{Q_5(a,b,c)}, \quad Q_5(x,y,z) := 7x^2 + 3y^2 + 7z^2 + 2xy - 6xz + 2yz.$$

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$N = 7$:

$$2\mathscr{H}_{7,7}(z) = -12\left(\sum_{n\geq 0} H_{1,1}(n)q^n - \frac{8}{7}\sum_{n\geq 1} H_{1,7}(n)q^n\right)$$
$$= \sum_{(a,b,c)\in\mathbb{Z}^3} q^{Q_7(a,b,c)}, \quad Q_7(x,y,z) := 4x^2 + 7y^2 + 8z^2 - 4xz.$$

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Differential operators

The Bruinier-Funke Operator:

$$\xi_k = 2iy^k \overline{\frac{\partial}{\partial \overline{z}}} = iy^k \overline{\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)} = iy^k \left(\overline{\frac{\partial}{\partial x}} - i\overline{\frac{\partial}{\partial y}}\right)$$

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Important Properties:

- $\xi_k(f|_k\gamma) = \xi_k(f)|_{2-k}\gamma$.

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- $\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \xi_{2-k}\xi_k$

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- $\xi_k f = 0 \iff f$ is holomorphic.

Harmonic Maass forms

A real-analytic $f : \mathbb{H} \to \mathbb{C}$ is a weight $k \in \frac{1}{2}\mathbb{Z}$ harmonic Maass form if:

1. For all $\gamma \in \Gamma$, we have
$$f\big|_k \gamma = \begin{cases} f & k \in \mathbb{Z} \\ \nu_\theta^{2k}(\gamma) f & k \in \frac{1}{2} + \mathbb{Z}. \end{cases}$$
2. $\Delta_k f(z) = 0$.
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One can replace (2) with the condition that $f(z)$ is of the form

$$f(z) = f^+(z) + c'(0) y^{1-k} + \sum_{n \ll \infty} \Gamma(1-k, 4\pi|m|y) c(m) q^m$$

with $f^+(z)$ holomorphic and $\Gamma(s,x) := \int_x^\infty t^{s-1} e^{-t} dt$.

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If $f^+(z) + c'(0)y^{1-k} + \sum_{n \ll \infty} \Gamma(1-k, 4\pi|m|y)c(m)q^m$ is a harmonic Maass form and f^+ is holomorphic, then f^+ is a mock modular form.

Sesquiharmonic Maass Forms

A real-analytic function $f : \mathbb{H} \to \mathbb{C}$ is a weight $k \in \frac{1}{2}\mathbb{Z}$ sesquiharmonic Maass form with respect to Γ if:

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Examples:

- **The Kronecker Limit Formula:**
$E(z,s) = \frac{\pi}{s-1} + 2\pi(\gamma - \log 2 - \log \sqrt{y}|\eta(z)|^2) + O(s-1)$.

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- **Bringmann, Diamantis, Raum (2014)**: Coefficients involve non-critical modular L-values.
- **Ahlgren-Andersen-Samart (2017):** Eta multiplier version of Duke-Imamoḡlu-Tóth's example.

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Mock modular properties of $H(n)$

Zagier (1975)

The function

$$\mathcal{H}(z) := \sum_{n=0}^{\infty} H(n)q^n + \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{n\geq 1} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 y\right) q^{-n^2}$$

is a weight $\frac{3}{2}$ harmonic Maass form on $\Gamma_0(4)$.

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Zagier (1975)

The function

$$\mathcal{H}(z) := \sum_{n=0}^{\infty} H(n)q^n + \frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}} \sum_{n\geq 1} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 y\right) q^{-n^2}$$

is a weight $\frac{3}{2}$ harmonic Maass form on $\Gamma_0(4)$.

- Thus $\sum_{n=0}^{\infty} H(n)q^n$ is a mock modular form.

Higher level analogue

Theorem 2 (B, Mono 2024)

Let $N > 1$ be odd and square-free. Then the function

$$\frac{1}{N}\sum_{n\geq 1} H_{1,N}(n)q^n + \Big(\prod_{p|N}\frac{1}{p+1}\Big)\left(\frac{1}{8\pi\sqrt{y}} + \frac{1}{4\sqrt{\pi}}\sum_{n\geq 1} n\Gamma\left(-\frac{1}{2}, 4\pi n^2 y\right) q^{-n^2}\right)$$

is a weight $\frac{3}{2}$ harmonic Maass form on $\Gamma_0(4N)$. Its image under $\xi_{\frac{3}{2}}$ is given by

$$-\frac{1}{16\pi}\Big(\prod_{p|N}\frac{1}{p+1}\Big)\theta(z).$$

Positive discriminants

Set

$$R(d) := \begin{cases} 2\log e_d & d > 0,\ \text{non-square} \\ \log d & d \text{ square} \end{cases}$$

where e_d is the smallest unit > 1 of norm 1 in the quadratic order $\mathcal{O}$ of discriminant d, and

$$h^*(d) := \frac{1}{2\pi} \sum_{\substack{\ell^2 | d \\ d/\ell^2 \text{ a discriminant}}} R(d/\ell^2) h(d/\ell^2).$$

Duke, Imamoḡlu, Tóth (2011)

The function $Z(z)$ with Fourier expansion given by

$$Z(z) = \sum_{d>0} \frac{h^*(d)}{\sqrt{d}} q^d + \sum_{d<0} \frac{H(|d|)}{\sqrt{|d|}} \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2}, \pi 4|d|y) q^d + \sum_{n \neq 0} \alpha(4n^2 y) q^{n^2}$$
$$+ \left(\frac{\gamma - \log(16\pi)}{4\pi} \right) - \frac{\log y}{4\pi} + \frac{\sqrt{y}}{3}$$

is a sesquiharmonic Maass form of weight $\frac{1}{2}$ for $\Gamma_0(4)$, where

$$\alpha(y) := \frac{\sqrt{y}}{4\pi} \int_0^\infty t^{-1/2} e^{-\pi y t} \log(1+t) dt.$$

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- The positive square index coefficients were computed by Ahlgren, Andersen, and Samart, who also computed an eta multiplier system analogue.

Higher level analogue of $Z(z)$

$$\mathcal{F}_{k,4N}(z,s) := \sum_{\gamma \in (\Gamma_0(4N))_\infty \backslash \Gamma_0(4N)} \nu_\theta(\gamma)^{-2k} y^{s-\frac{k}{2}}\Big|_k \gamma, \qquad \mathrm{Re}(s) > 1 - \frac{k}{2}.$$

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$$\mathcal{F}^+_{k,4N}(z,s) := \mathrm{pr}^+ \mathcal{F}_{k,4N}(z,s) = \sum_{j=-1}^{\infty} f_j(z) \left(s - \frac{3}{4}\right)^j$$

- Let $\mathcal{G}(z) = f_0(z)$.

Properties of $\mathcal{G}$

Theorem 3 (B, Mono 2024)

The function $\mathcal{G}$ is a weight $\frac{1}{2}$ sesquiharmonic Maass form on $\Gamma_0(4N)$, and

$$\mathcal{G}(z) = \frac{2}{3}y^{\frac{1}{2}} - \frac{\log(16y)}{2\pi}\prod_{p|N}\frac{1}{p+1}$$

$$+\frac{1}{\pi}\prod_{p|N}\frac{1}{p+1}\sum_{m\geq 1}\left(\gamma + \log\left(\pi m^2\right) + \alpha\left(4m^2 y\right)\right)q^{m^2} +$$

$$\frac{2}{3}(1-i)\pi \sum_{\substack{n\geq 0 \\ n\equiv 0,1 \pmod 4}} \mathfrak{c}(n)q^n + \frac{2}{3}(1-i)\pi^{\frac{1}{2}} \sum_{\substack{n<0 \\ n\equiv 0,1 \pmod 4}} \mathfrak{c}(n)\Gamma\left(\frac{1}{2}, 4\pi|n|y\right)q^n.$$

- The coefficients $\mathfrak{c}(n)$ are given in terms of the Kloosterman zeta function on the next slide.

The coefficients $\mathfrak{c}(n)$

$$\mathscr{K}_k(m,n;s)=\sum_{c>0}\frac{1+\left(\frac{4}{c}\right)}{(4Nc)^s}\sum_{\substack{r \pmod{4Nc}\\ \gcd(4Nc,r)=1}}\left(\frac{4Nc}{r}\right)\varepsilon_r^{2k}e^{2\pi i\frac{mr^*+nr}{4Nc}},\qquad \mathrm{Re}(s)>1.$$

- $\mathscr{K}_k(m,n;s)=$ "Kloosterman zeta function" w.r.t. ν_θ^{2k}.

$$\mathfrak{c}(n):=\frac{\partial}{\partial s}\left[\left(s-\frac{3}{4}\right)\mathscr{K}_{\frac{1}{2}}(0,n;2s)\right]\Bigg|_{s=\frac{3}{4}}.$$

Shadow of $\mathcal{G}$

Theorem 3 continued (B, Mono 2024)

We have

$$\frac{1}{4}\xi_{\frac{1}{2}}\mathcal{G}(z) = -\Big(\prod_{p|N}\frac{1}{p+1}\Big)\mathcal{H}(z) - \frac{1}{N}\sum_{\substack{\ell|N\\ \ell>1}}\ell\Big(\prod_{p|\ell}\frac{1}{1-p}\Big)\mathscr{H}_{\ell,N}(z)$$
$$+\Big(\prod_{p|N}\frac{1}{p+1}\Big)\sum_{n\geq 0}H_{1,1}(n)q^n - \frac{1}{N}\sum_{n\geq 1}H_{1,N}(n)q^n.$$

Outline

- Half-integral weight modular forms - definition and background.
- Hurwitz class numbers and holomorphic modular forms.
- Non-holomorphic modular forms: harmonic and sesquiharmonic Maass forms.
- Hurwitz class numbers and non-holomorphic modular forms.
- Proof outline.
- Open questions.

Fourier series calculation

$$\mathcal{F}^{+}_{k,4N}(z,s) = \frac{2}{3}\left(y^{s-\frac{k}{2}} + \frac{4^{1-s}\pi i^{-k}\Gamma(2s-1)}{\Gamma\left(s+\frac{k}{2}\right)\Gamma\left(s-\frac{k}{2}\right)}\mathscr{K}_k(0,0;2s)y^{1-s-\frac{k}{2}} \right)$$

$$+\frac{2}{3}i^{-k}\pi^s y^{-\frac{k}{2}} \sum_{\substack{n\neq 0 \\ (-1)^{k-\frac{1}{2}} n\equiv 0,1 \pmod 4}} \frac{\mathscr{K}_k(0,n;2s)\,|n|^{s-1}}{\Gamma\left(s+\mathrm{sgn}(n)\frac{k}{2}\right)} W_{\mathrm{sgn}(n)\frac{k}{2},s-\frac{1}{2}}(4\pi\,|n|\,y)e^{2\pi i n x}.$$

where

$$\mathscr{K}_k(m,n;s) = \sum_{c>0} \frac{1+\left(\frac{4}{c}\right)}{(4Nc)^s} \sum_{\substack{r \pmod{4Nc} \\ \gcd(4Nc,r)=1}} \left(\frac{4Nc}{r}\right) \varepsilon_r^{2k} e^{2\pi i \frac{mr^*+nr}{4Nc}}, \qquad \mathrm{Re}(s) > 1.$$

- $\mathscr{K}_k(m,n;s)$ = "Kloosterman zeta function" w.r.t. ν_θ^{2k}.
- These Kloosterman sums appear in Kohnen's work.

The Kloosterman Zeta function at $k = \frac{1}{2}$

- $\mathscr{K}_{\frac{1}{2}}(0, n; s)$ can be written as a combination of Kloosterman zeta functions appearing in work by Shimura and Sturm.

$$\mathscr{K}_{\frac{1}{2}}\left(0, n; s + \frac{1}{2}\right) = \frac{L_{4N}\left(s, \chi_{4N}^2 \chi_t\right)}{L_{4N}(2s, \chi_{4N}^2)} T_{4N,1-s}^{\chi_t}(m)$$

$$\left(\sum_{(4N)|M|(4N)^\infty} + \sum_{\substack{(4N)|M|(4N)^\infty \\ \nu_2(M)=2}}\right) \frac{1}{M^{s+\frac{1}{2}}} \sum_{r=1}^{M} \varepsilon_r \left(\frac{M}{r}\right) e^{2\pi i \frac{nr}{M}}.$$

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$$\left(\sum_{(4N)|M|(4N)^\infty} + \sum_{\substack{(4N)|M|(4N)^\infty \\ \nu_2(M)=2}}\right) \frac{1}{M^{s+\frac{1}{2}}} \sum_{r=1}^{M} \varepsilon_r \left(\frac{M}{r}\right) e^{2\pi i \frac{nr}{M}}.$$

- The sums were studied by Pei (1982).

The Kloosterman Zeta function - product formula

$$\mathscr{K}_{\frac{1}{2}}\left(0, n; s+\frac{1}{2}\right)$$

$$= \frac{L_{4N}(s, \chi_{4N}^2\chi_t)}{L_{4N}(2s, \chi_{4N}^2)} T_{4N,1-s}^{\chi_t}(m) \left(\prod_{p|N} \sum_{j=1}^{\infty} \frac{a(p^j, n)}{p^{j(s+\frac{1}{2})}}\right) \cdot \left(\sum_{j=2}^{\infty} \frac{a(2^j, n)}{2^{j(s+\frac{1}{2})}} + \frac{a(2^2, n)}{2^{2(s+\frac{1}{2})}}\right).$$

where

$$a\left(p^j, n\right) := \begin{cases} \sum_{r=1}^{2^j} \left(\frac{2^j}{r}\right) \varepsilon_r e^{2\pi i \frac{nr}{2^j}} & \text{if } p = 2, \\ \varepsilon_{p^j}^{-1} \sum_{r=1}^{p^j} \left(\frac{r}{p^j}\right) e^{2\pi i \frac{nr}{p^j}} & \text{if } p > 2, \end{cases}$$

Simplifying the factors

- We show that the $a(p^j, n)$ are very closely related to numbers studied by Maass (1937), which leads to (for $n \neq 0$):

If $p > 2$, and $\mathcal{D}(n) =$ discriminant of $\mathbb{Q}(\sqrt{n})$, then

$$\sum_{j \geq 0} \frac{a\left(p^j, n\right)}{p^{j\left(s+\frac{1}{2}\right)}} = \begin{cases} \left(1 - p^{-2s}\right) \displaystyle\sum_{j=0}^{\frac{\nu_p(n)-1}{2}} p^{j(1-2s)} & p \mid \mathcal{D}(n), \\ \left(1 + \left(\frac{\mathcal{D}(n)}{p}\right) p^{-s}\right) & \\ \cdot \left(p^{(1-2s)\frac{\nu_p(n)}{2}} + \left(1 - \left(\frac{\mathcal{D}(n)}{p}\right) p^{-s}\right) \displaystyle\sum_{j=0}^{\frac{\nu_p(n)}{2}-1} p^{j(1-2s)}\right) & p \nmid \mathcal{D}(n). \end{cases}$$

$p = 2$ case

- Maass also computed the $p = 2$ case:

$$\sum_{j\geq 0} \frac{a\left(2^{j+1}, n\right)}{2^{j\left(s+\frac{1}{2}\right)}} = e^{\frac{\pi i}{4}} \times \begin{cases} 1 + \left(\frac{\mathcal{D}(n)}{2}\right) 2^{-s} & 2 \nmid \mathcal{D}(n), \\ 1 - 2^{-2s} & 2 \mid \mathcal{D}(n), \end{cases}$$

$$\times \begin{cases} \frac{1}{1+2^{-s}} \left(\sum_{j=0}^{\frac{\nu_2(n)-1}{2}} 2^{j(1-2s)} + 2^{1-s} \sum_{j=0}^{\frac{\nu_2(n)-3}{2}} 2^{j(1-2s)} \right) & 2 \nmid \nu_2(n), \\ -\frac{2^s}{1+2^{-s}} + 2^s \sum_{j=0}^{\frac{\nu_2(n)}{2}} 2^{j(1-2s)} & 2 \mid \nu_2(n),\ \frac{n}{2^{\nu_2(n)}} \equiv 3 \pmod 4, \\ \left(1 + 2^{1-s}\right) \sum_{j=0}^{\frac{\nu_2(n)}{2}} 2^{j(1-2s)} & 2 \mid \nu_2(n),\ \frac{n}{2^{\nu_2(n)}} \equiv 5 \pmod 8, \\ \frac{2}{1+2^{-s}} + \left(2^{1-s} - 1\right) \sum_{j=0}^{\frac{\nu_2(n)}{2}} 2^{j(1-2s)} & 2 \mid \nu_2(n),\ \frac{n}{2^{\nu_2(n)}} \equiv 1 \pmod 8. \end{cases}$$

Sketch of Proofs of Theorems 1 and 2

Theorem 3 $\implies$ Theorem 2:

- $\xi_{\frac{1}{2}}\mathcal{G}$ is a weight $\frac{3}{2}$ harmonic Maass form on $\Gamma_0(4N)$.
- Subtracting a linear combination of $\mathcal{H}$ and the Pei-Wang holomorphic functions produces $\mathcal{H}_{1,N}(z)$.

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Theorem 2 $\implies$ Theorem 1:

- The non-holomorphic parts of $\mathcal{H}_{1,N}(z)$ and $c_N\mathcal{H}(z)$ are equal, so that $\mathcal{H}_{1,N}(z) - c_N\mathcal{H}(z)$ is holomorphic.

Open Questions

1. (Griffin) Can we obtain a higher level class number relations along the lines of Mertens' thesis?
2. What are the Shimura lifts of our weight $\frac{3}{2}$ modular forms?
3. Are there more explicit connections to representation numbers of ternary quadratic forms?
4. (Rolen) Are there interesting congruences satisfied by the coefficients of our modular forms?
5. Can we compute the square indexed coefficient as a regularized inner product?
6. Are there versions of these results using the η-multiplier involving Andrews' mock modular generating function of the $\mathrm{spt}(n)$ function?

The End

Thanks for listening!