Hypergeometric Functions, Galois Representations, and Modular Forms

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Hypergeometric data and hypergeometric functions

- A hypergeometric datum is $HD = \{\alpha, \beta\}$, where $\alpha = \{a_1, ..., a_n\}$, $\beta = \{b_1 = 1, b_2, ..., b_n\}$ with $a_i, b_j \in \mathbb{C}$.
- Associate the hypergeometric function for $z \in \mathbb{C}$

$$F(HD; z) = {}_nF_{n-1}\left[\begin{matrix} a_1 & a_2 & \cdots & a_n \\ & b_2 & \cdots & b_n \end{matrix} ; z\right] = \sum_{r \geq 0} \frac{(a_1)_r \cdots (a_n)_r}{(b_1)_r \cdots (b_n)_r} \, z^r$$

whenever it converges. Here $(a)_r$ is the Pochhammer symbol

$$(a)_r = a(a+1)\cdots(a+r-1) = \frac{\Gamma(a+r)}{\Gamma(a)}.$$

- ${}_nF_{n-1}(HD; z)$ satisfies an nth order linear ODE with three regular sigularities at $0, 1, \infty$. Solutions at nonsingular points form a local system of rank n on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. They give rise to a monodromy representation of the fundamental group $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}, *)$.

Hypergeometric data and Galois representations

- Assume $a_i, b_j \in \mathbb{Q}^\times$ and HD is primitive, i.e., $a_i - b_j \notin \mathbb{Z}$ for all i, j.
- $M = lcd(\alpha \cup \beta)$ is the least common denominator of a_i, b_j.
- Katz introduced an ℓ-adic rank-n hypergeometric sheaf $\mathcal{H}(HD)_\ell$ on $\mathbb{G}_m$ with explicit action of the Galois group $Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_M)) =: G(M)$. Its action on the fiber at $\lambda \in \mathbb{Q}(\zeta_M)^\times$, denoted $\rho_{HD,\lambda,\ell}$, has Frob. traces equal to hypergeometric character sums, which are finite field analog of ${}_nF_{n-1}(HD; \lambda)$ studied by Katz, Fuslier-Long-Ramakrishna-Swisher-Tu, etc.

The BCM representation

- HD is defined over $\mathbb{Q}$ if $\{\begin{pmatrix} a_i \\ b_i \end{pmatrix} \mod \mathbb{Z}\}$ is invariant under multiplication by $(\mathbb{Z}/M\mathbb{Z})^\times$.
- When HD is defined over $\mathbb{Q}$, the Katz representation can be extended to $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. One extension is $\rho^{BCM}_{HD,\ell}$ studied by Beukers-Cohen-Mellit, with explicit $Frob_p$ traces given by $H_p(HD;\lambda)$ for $\lambda \in \mathbb{Q}^\times$. When $p \equiv 1 \mod M$, it is

$$H_p(HD;\lambda) = \frac{1}{1-p}\sum_{k=0}^{p-2}\prod_{j=1}^{n}\frac{\mathfrak{g}(\omega^{(p-1)a_j}\omega^k)}{\mathfrak{g}(\omega^{(p-1)a_j})}\frac{\mathfrak{g}(\overline{\omega}^{(p-1)b_j}\overline{\omega}^k)}{\mathfrak{g}(\overline{\omega}^{(p-1)b_j})}\omega^k((-1)^n\lambda).$$

Here ω generates $\widehat{\mathbb{F}_p^\times}$, and $H_p(HD;\lambda) \in \mathbb{Q}$ is indep. of ω.

Theorem [Katz, Beukers-Cohen-Mellit]

Suppose $HD = \{\alpha, \beta\}$ *is primitive, of length* n*, defined over* $\mathbb{Q}$*, with* $M = lcd(\alpha \cup \beta)$*. Assume that exactly* m *elements in* β *are integers. Let* $\lambda \in \mathbb{Z}[1/M] \setminus \{0\}$*. Then for each prime* ℓ*,*

- *for primes* $p \nmid M\ell$ *such that* $ord_p \lambda = 0$*, we have*

$$Tr\rho^{BCM}_{HD,\lambda,\ell}(Frob_p) = \left(\frac{-1}{p}\right)^{\delta} p^{\frac{n-m}{2}} H_p(HD; \frac{1}{\lambda}) \in \mathbb{Z}.$$

- *The degree* d *of* $\rho^{BCM}_{HD,\lambda,\ell}$ *is* n *for* $\lambda \neq 1$*, and* $n-1$ *for* $\lambda = 1$*.*
- $\rho^{BCM}_{HD,\lambda,\ell}$ *has image in* $GO_d(\mathbb{Q}_\ell)$ *for* n *odd and* $GSp_d(\mathbb{Q}_\ell)$ *for* n *even.*

Here $n-m$ is even, $\delta = 0$ unless $\sum a_i \equiv 1/2 \mod \mathbb{Z}$ and $2||M$, in which case $\delta = 1$.

An example

$HD = \{\alpha, \beta\}$ where $\alpha = \{\frac{1}{2}, \frac{1}{2}\}$, $\beta = \{1, 1\}$ is primitive, defined over $\mathbb{Q}$ with $M = lcd(\alpha \cup \beta) = 2$. Given $\lambda \in \mathbb{Q}^\times$, at primes $p \nmid lcd(\frac{1}{2}, \lambda, \frac{1}{\lambda})$,

$$H_p(HD; \lambda) = -\sum_{x \in \mathbb{F}_p} \phi_p(x(x-1)(1-\lambda x)),$$

where ϕ_p is the quadratic character of $\mathbb{F}_p^\times$ and $\phi_p(0) = 0$.

In this case

$$\mathrm{Tr}\rho^{BCM}_{HD,\lambda,\ell}(Frob_p) = H_p(HD; \frac{1}{\lambda}),$$

showing that at $\lambda \neq 1$, $\rho^{BCM}_{HD,\lambda,\ell}$ is the Galois rep'n on the ℓ-adic Tate module of the elliptic curve $y^2 = x(x-1)(1-\frac{x}{\lambda})$ and hence is modular; at $\lambda = 1$, $\rho^{BCM}_{HD,\lambda,\ell}$ is the trivial rep'n.

Relation between the fundamental and Galois groups

Let $X = \mathbb{P}^1 \backslash \{0, 1, \infty\}$ over $\mathbb{Q}(\zeta_M)$. In the language of schemes, the algebraic points λ come from specializing a generic point x of X, and the Galois actions come from a rep'n of the *étale fundamental group* $\pi_1(X_x, *)$ of the generic fiber. Have the exact sequence

$$0 \to \pi_1(X_{\mathbb{C}}, *) \to \pi_1(X_x, *) \to G(M) \to 0.$$

Here $\pi_1(X_{\mathbb{C}}, *)$ is the profinite completion of $\pi_1(X(\mathbb{C}), *)$.

We shall explore this relation and study automorphy of the BCM/Katz representations.

The Clausen formulas

The classical Clausen formula over $\mathbb{C}$:

$$
{}_2F_1\left[\begin{matrix} a & b \\ & a+b+\frac{1}{2} \end{matrix} ; z\right] {}_2F_1\left[\begin{matrix} 1-a & 1-b \\ & \frac{3}{2}-a-b \end{matrix} ; z\right] = (1-z)^{-\frac{1}{2}} \, {}_3F_2\left[\begin{matrix} \frac{1}{2} & a-b+\frac{1}{2} & b-a+\frac{1}{2} \\ & a+b+\frac{1}{2} & \frac{3}{2}-a-b \end{matrix} ; z\right].
$$

Clausen formula over $\mathbb{F}_q$ by Evans-Greene: for $2s \equiv a+b \mod \mathbb{Z}$,

$$
H_q(\{-\frac{1}{2}+b-s, s\}, \{1, b\}; t, \omega) H_q(\{-\frac{1}{2}-b+s, 1-s\}, \{1, 1-b\}; t, \omega) = \phi_q(1-t) H_q(\{\frac{1}{2}, a, 1-a\}, \{1, b, 1-b\}; t, \omega) + q^{\delta(b)},
$$

when $t \in \mathbb{F}_q \setminus \{0, 1\}$. Here $\delta(b) = 1$ if $b \in \mathbb{Z}$ and 0 otherwise.

When $t = 1$, we have

$$H_q\left(\{\frac{1}{2}, a, 1-a\}, \{1, b, 1-b\}; 1; \omega\right)$$
$$= \frac{1}{q^{1-\delta(b)}} \frac{J_\omega(a+b, b-a)}{J_\omega(\frac{1}{2}, -b)} \left(J_\omega(s-b, \frac{1}{2}-s)^2 + J_\omega(\frac{1}{2}+s-b, s)^2\right),$$

if $\omega^{(q-1)(a+b)}$ is a square in $\widehat{\mathbb{F}_q^\times}$; otherwise

$$H_q\left(\{\frac{1}{2}, a, 1-a\}, \{1, b, 1-b\}; 1; \omega\right) = 0.$$

Here $J_\omega(s,t)$ is a Jacobi sum.

Thus $\rho_{\{\frac{1}{2},a,1-a\},\{1,b,1-b\},\lambda,\ell}$ is the *symmetric square* of a degree-2 rep'n up to twist when $\lambda \neq 1$; when $\lambda = 1$, it is induced from a character of a quadratic extension.

Some low degree automorphy results

Suppose $HD = \{\alpha, \beta\}$ has length 3, is primitive and defined over $\mathbb{Q}$.

Theorem. [L-Liu-Long]

(i) *The degree-2* $\rho^{BCM}_{HD,1,\ell}$ *is modular and has CM.*

(ii) *For* $\lambda \in \mathbb{Q} \setminus \{0,1\}$, *the degree-3* $\rho^{BCM}_{HD,\lambda,\ell}$, *up to twist by a character, is the symmetric square of a degree-2 modular representation of* $G_{\mathbb{Q}}$, *hence it is automorphic.*

Next consider representations $\rho = \rho_{HD,\lambda,\ell}^{BCM}$ of $G_{\mathbb{Q}}$ with image in $GO_4(\mathbb{Q}_\ell)$. So α, β in $HD = \{\alpha, \beta\}$ have length 5 and $\lambda = 1$.

By Liu-Yu, there is a field F of degree ≤ 2 over $\mathbb{Q}$ so that $\rho|_{G_F} = \rho_1 \otimes \rho_2$, where ρ_i have degree-2.

Theorem. [L-Liu-Long]

(i) *If ρ is induced from a character or $F = \mathbb{Q}$, then ρ is automorphic.*

(ii)*If F is real quadratic, then ρ is potentially automorphic.*

Example. For any prime $p > 3$, we have

$$H_p\left(\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\},\{1,1,1,1,1\};1\right)$$
$$= H_p\left(\{\frac{1}{2},\frac{1}{2}\},\{1,1\};-1\right)\left(H_p\left(\{\frac{1}{4},\frac{3}{4},\frac{1}{4},\frac{3}{4}\},\{1,1,1,1\};1\right)-p\right).$$

With $\alpha = \{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$ and $\beta = \{1,1,1,1,1\}$, this shows that

$$\chi_{-1}\otimes\rho^{BCM}_{HD,1,\ell}\simeq\rho_{f_{32.2.a.a}}\otimes\rho_{f_{32.4.a.a}}.$$

A Whipple ${}_7F_6$ formula

The well-known Whipple ${}_7F_6(1)$ formula asserts that

$$
{}_7F_6\left[\begin{matrix} a & 1+\frac{a}{2} & c & d & e & f & g \\ & \frac{a}{2} & 1+a-c & 1+a-d & 1+a-e & 1+a-f & 1+a-g \end{matrix} ;1\right]
$$
$$
= \frac{\Gamma(1+a-e)\Gamma(1+a-f)\Gamma(1+a-g)\Gamma(1+a-e-f-g)}{\Gamma(1+a)\Gamma(1+a-f-g)\Gamma(1+a-e-f)\Gamma(1+a-e-g)} \times
$$
$$
\times\ {}_4F_3\left[\begin{matrix} a & e & f & g \\ & e+f+g-a & 1+a-c & 1+a-d \end{matrix} ;\ 1\right],
$$

when both sides terminate.

We specialize it to the following self-dual form with a prime p :

$$
{}_7F_6\left[\begin{matrix} \frac{1}{2} & \frac{5}{4} & c & 1-c & \frac{1-p}{2} & f & 1-f \\ & \frac{1}{4} & \frac{3}{2}-c & \frac{1}{2}+c & 1+\frac{p}{2} & \frac{3}{2}-f & \frac{1}{2}+f \end{matrix} ;\ 1\right] =
$$
$$
\frac{\Gamma(\frac{p}{2})^2\Gamma(\frac{3}{2}-f)\Gamma(\frac{1}{2}+f)}{\Gamma(\frac{1}{2})^2\Gamma(1+\frac{p}{2}-f)\Gamma(\frac{p}{2}+f)} \times \left(p\cdot {}_4F_3\left[\begin{matrix} \frac{1}{2} & \frac{1-p}{2} & f & 1-f \\ & 1-\frac{p}{2} & \frac{3}{2}-c & \frac{1}{2}+c \end{matrix} ;\ 1\right]\right).
$$

We give a Galois representation theoretic interpretation of the Whipple ${}_7F_6$ identity and study the “automorphy”of the Galois representations.

This is joint work with Ling long and Fang-Ting Tu

The associated HD are

$$HD_1(c,f) :=$$
$$\left\{\alpha_6(c,f) = \{\frac{1}{2}, c, 1-c, \frac{1}{2}, f, 1-f\}, \beta_6(c,f) = \{1, \frac{3}{2}-c, \frac{1}{2}+c, 1, \frac{3}{2}-f, \frac{1}{2}+f\}\right\};$$

$$HD_2(c,f) := \left\{\alpha_4(f) = \{\frac{1}{2}, \frac{1}{2}, f, 1-f\}, \beta_4(c) = \{1, 1, \frac{3}{2}-c, \frac{1}{2}+c\}\right\}.$$

A Galois representation theoretic interpretation of the Whipple ${}_7F_6$ formula

Theorem.[L-Long-Tu] *For* (c,f) *such that both* $HD_1(c,f)$ *and* $HD_2(c,f)$ *are primitive, let* $N(c,f) = lcd(\frac{1+2f-2c}{4}, \frac{3-2f-2c}{4})$*. Then* $M(HD_i(c,f))|N(c,f)$ *for* $i = 1,2$*. Given any prime* ℓ*,*

$$\rho_{HD_1(c,f),1,\ell}{}^{ss}|_{G(N(c,f))} \cong (\epsilon_\ell \otimes \rho_{HD_2(c,f),1,\ell})|_{G(N(c,f))} \oplus \sigma_{sym,\ell},$$

where ϵ_ℓ *is the* ℓ*-adic cyclotomic character, and* $\sigma_{sym,\ell}$ *is a 2-dim'l rep'n of* $G(N(c,f))$ *that can be computed explicitly.*

Moreover, when $HD_1(c,f)$ *is defined over* $\mathbb{Q}$*,* $\rho_{HD_1(c,f),1,\ell}$ *is semi-simple and hence can be decomposed as above.*

Seven pairs of (c, f)

- Only seven un-ordered pairs of $(c,f) \in (\mathbb{Q}^\times)^2$ are such that $HD_1(c,f)$ is defined over $\mathbb{Q}$ and primitive:

$$\left(\frac{1}{2},\frac{1}{2}\right),\left(\frac{1}{2},\frac{1}{3}\right),\left(\frac{1}{2},\frac{1}{6}\right),\left(\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{6},\frac{1}{6}\right),\left(\frac{1}{5},\frac{2}{5}\right),\left(\frac{1}{10},\frac{3}{10}\right).$$

- Among them $HD_2(c,f)$ is primitive for all pairs, but is defined over $\mathbb{Q}$ only for the first five pairs.
- $M(c,f) = lcd(\alpha_6(c,f) \cup \beta_6(c,f)) = lcd(\alpha_4(c,f) \cup \beta_4(c,f))$.
- For each of the seven pairs, the repn's can be extended to $G_{\mathbb{Q}}$, denoted by $\rho^{BCM}_{HD_1(c,f),1,\ell}$ and $\rho^{BCM}_{HD_2(c,f),1,\ell}$, with Frobenius traces in $\mathbb{Z}$.

Modularity of $\rho^{BCM}_{HD_2(c,f),1,\ell}$ and $\rho^{BCM}_{HD_1(c,f),1,\ell}$ for the seven pairs of (c,f)

Katz and Beukers-Cohen-Mellit:

$$\rho^{BCM}_{HD_2(c,f),1,\ell} = \rho^{BCM,prim}_{HD_2(c,f),1,\ell} \oplus \rho^{BCM,1}_{HD_2(c,f),1,\ell} \quad \text{of degree 2 and 1;}$$

$$\rho^{BCM}_{HD_1(c,f),1,\ell} = \rho^{BCM,prim}_{HD_1(c,f),1,\ell} \oplus \rho^{BCM,1}_{HD_1(c,f),1,\ell} \quad \text{of degree 4 and 1.}$$

Theorem.[L-Long-Tu] *For each of the seven pairs* (c, f), $\rho^{BCM,prim}_{HD_2(c,f),1,\ell}$ *and* $\rho^{BCM,1}_{HD_2(c,f),1,\ell}$ *are modular, given as follows:*

(c, f)	$\rho^{BCM,prim}_{HD_2(c,f),1,\ell}$	$\rho^{BCM,1}_{HD_2(c,f),1,\ell}$
$(\frac{1}{2}, \frac{1}{2})$	$\rho_{f_{8.4.a.a}}$	ϵ_ℓ
$(\frac{1}{2}, \frac{1}{3})$	$\rho_{f_{36.4.a.a}} = \chi_{-3} \otimes \rho_{f_{12.4.a.a}}$	$\chi_3 \cdot \epsilon_\ell$
$(\frac{1}{3}, \frac{1}{3})$	$\rho_{f_{6.4.a.a}} = \chi_{-3} \otimes \rho_{f_{18.4.a.a}}$	$\chi_{-3} \cdot \epsilon_\ell$
$(\frac{1}{2}, \frac{1}{6})$	$\rho_{f_{72.4.a.b}} = \chi_{-3} \otimes \rho_{f_{24.4.a.a}}$	ϵ_ℓ
$(\frac{1}{6}, \frac{1}{6})$	$\epsilon_\ell \otimes \rho_{f_{24.2.a.a}} = \chi_{-3}\epsilon_\ell \otimes \rho_{f_{72.2.a.a}}$	$\chi_{-3} \cdot \epsilon_\ell$
$(\frac{1}{5}, \frac{2}{5})$	$\rho_{f_{50.4.a.b}} = \chi_5 \otimes \rho_{f_{50.4.a.d}}$	ϵ_ℓ or $\chi_5 \cdot \epsilon_\ell$
$(\frac{1}{10}, \frac{3}{10})$	$\epsilon_\ell \otimes \rho_{f_{200.2.a.b}} = \chi_5\epsilon_\ell \otimes \rho_{f_{200.2.a.d}}$	ϵ_ℓ or $\chi_5 \cdot \epsilon_\ell$

Theorem. [L-Long-Tu] *For each of the seven pairs* (c, f), $\rho^{BCM,prim}_{HD_1(c,f),1,\ell}$ *decomposes as a sum of two 2-dimensional* $G_{\mathbb{Q}}$-*modules, all subrepresentations are modular.*

(c, f)	$\rho^{BCM}_{HD_1(c,f),1,\ell}$
$(\frac{1}{2}, \frac{1}{2})$	$\rho_{f_{8.6.a.a}} \oplus (\epsilon_\ell \otimes \rho_{f_{8.4.a.a}}) \oplus \chi_{-1}\epsilon_\ell^2$
$(\frac{1}{2}, \frac{1}{3})$	$\rho_{f_{4.6.a.a}} \oplus (\epsilon_\ell \otimes \rho_{f_{12.4.a.a}}) \oplus \chi_3\epsilon_\ell^2$
$(\frac{1}{3}, \frac{1}{3})$	$\rho_{f_{6.6.a.a}} \oplus (\epsilon_\ell \otimes \rho_{f_{18.4.a.a}}) \oplus \chi_{-1}\epsilon_\ell^2$
$(\frac{1}{2}, \frac{1}{6})$	$(\epsilon_\ell \otimes \rho_{f_{8.4.a.a}}) \oplus (\epsilon_\ell \otimes \rho_{f_{24.4.a.a}}) \oplus \chi_3\epsilon_\ell^2$
$(\frac{1}{6}, \frac{1}{6})$	$(\epsilon_\ell^2 \otimes \rho_{f_{24.2.a.a}}) \oplus (\epsilon_\ell^2 \otimes \rho_{f_{72.2.a.a}}) \oplus \chi_{-1}\epsilon_\ell^2$
$(\frac{1}{5}, \frac{2}{5})$	$(\epsilon_\ell \otimes \rho_{f_{10.4.a.a}}) \oplus (\epsilon_\ell \otimes \rho_{f_{50.4.a.d}}) \oplus \chi_{-5}\epsilon_\ell^2$
$(\frac{1}{10}, \frac{3}{10})$	$(\epsilon_\ell^2 \otimes \rho_{f_{40.2.a.a}}) \oplus (\epsilon_\ell^2 \otimes \rho_{f_{200.2.a.b}}) \oplus \chi_{-5}\epsilon_\ell^2$

Traces of Hecke operators

Joint work with Hoffman, Long, and Tu.

Interested in an explicit formula for the trace of Hecke operators T_p on the space $S_{k+2}(\Gamma)$ of weight $k+2$ modular/cusp forms for congruence subgroups Γ of $\mathrm{SL}_2(\mathbb{R})$ such that $X_\Gamma = \Gamma\backslash\mathfrak{H}^*$ is either an elliptic modular curve or a Shimura curve defined over $\mathbb{Q}$. Hence k is even if $-I \in \Gamma$.

Previous results: Ahlgren for $\Gamma_0(4)$, Ahlgren-Ono for $\Gamma_0(8)$, Frechette-Ono-Papanikolas for newforms of $\Gamma_0(8)$, Ihara and Fuselier for $\mathrm{SL}_2(\mathbb{Z})$, and Lennon for $\Gamma_0(3)$ and $\Gamma_0(9)$; all used the Selberg trace formula.

Our goal: express $\mathrm{Tr}(T_p \mid S_{k+2}(\Gamma))$ in terms of hypergeometric character sums.

Our geometric approach

For each integer $k \geq 1$ and a prime ℓ, denote by $V^k(\Gamma)_\ell$ the standard ℓ-adic sheaf on $X_\Gamma \otimes \bar{\mathbb{Q}}$ from the moduli interpretation, first for Γ torsion-free using universal elliptic curves/abelian surfaces with QM, then for Γ with torsion by push-forward.

Theorem. [Deligne and Ohta] *Given ℓ and $k \geq 1$, for almost all $p \neq \ell$,*

$$\mathrm{T}r(T_p \mid S_{k+2}(\Gamma)) = \mathrm{T}r(Frob_p \mid H^1_{et}(X_\Gamma \otimes \bar{\mathbb{Q}}, V^k(\Gamma)_\ell)).$$

Combined with the Lefschetz fixed point formula, we get

$$-\mathrm{T}r(T_p \mid S_{k+2}(\Gamma)) = \sum_{\lambda \in X_\Gamma(\mathbb{F}_p)} \mathrm{T}r(Frob_\lambda \mid (V^k(\Gamma)_\ell)_{\bar{\lambda}}).$$

Idea: Replace $V^2(\Gamma)_\ell$ or $V^1(\Gamma)_\ell$ by a twist of the hypergeometric sheaf associated to $HD(\Gamma) = \{\alpha(\Gamma), \beta(\Gamma)\}$ constructed by Katz. Use Katz's rigidity theorem and the comparison theorem to determine the twists needed to achieve isomorphism.

Γ	$\Gamma_1(4)$ (∞,∞,∞)	$\Gamma_1(3)$ $(3,\infty,\infty)$	$\Gamma_0(2)$ $(2,\infty,\infty)$	$\mathrm{SL}_2(\mathbb{Z})$ $(2,3,\infty)$	$\Gamma_0(2)^+$ $(2,4,\infty)$	$\Gamma_0(3)^+$ $(2,6,\infty)$	$(2,4,6)$ $(2,4,6)$
$\alpha(\Gamma)$	$\{\frac{1}{2},\frac{1}{2}\}$	$\{\frac{1}{3},\frac{2}{3}\}$	$\{\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$	$\{\frac{1}{2},\frac{1}{6},\frac{5}{6}\}$	$\{\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$	$\{\frac{1}{2},\frac{1}{3},\frac{2}{3}\}$	$\{\frac{1}{2},\frac{1}{4},\frac{3}{4}\}$
$\beta(\Gamma)$	$\{1,1\}$	$\{1,1\}$	$\{1,1,1\}$	$\{1,1,1\}$	$\{1,1,1\}$	$\{1,1,1\}$	$\{1,\frac{7}{6},\frac{5}{6}\}$

Let $B = \left(\frac{-1,3}{\mathbb{Q}}\right)$ be the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant 6; O_B^1 the norm 1 subgroup of a maximal order O_B. The group $(2,4,6)$ is generated by O_B^1 and the Atkin Lehner involutions w_2, w_3.

Explicit trace formula

Theorem. [Hoffman-L-Long-Tu]

(I) *Let* Γ *be one of the last five. Fix a prime* ℓ *and even* $k \geq 2$*. For almost all* $p \neq \ell$*, the contributions from* $Tr(Frob_\lambda)$ *at generic* $\lambda \in X_\Gamma(\mathbb{F}_p)$ *can be expressed in terms of* $H_p(HD(\Gamma); 1/\lambda)$*; the contributions from* λ *corresponding to elliptic points can be expressed by Jacobi sums, and each* λ *corresponding to a cusp contributes* 1*.*

(II) *For* $\Gamma = \Gamma_1(4)$ *and* $\Gamma_1(3)$*, denote by* N_Γ *the level of* Γ*.*

(i)For even $k \geq 2$*, similar results as in Theorem 1 hold;*

(ii)For odd $k \geq 1$*,* $Tr(T_p|S_{k+2}(\Gamma)) = 0$ *for primes* $p \equiv -1 \mod N_\Gamma$*. For* $p \equiv 1 \mod N_\Gamma$*, similar contributions from* $\lambda \in X_\Gamma(\mathbb{F}_p)$ *generic and elliptic; the contribution at* λ *corresponding to a regular (resp. irregular) cusp is 1 (resp. -1).*

Example: $S_8(2,4,6)$

For $\Gamma=(2,4,6)$ the lowest weight with $S_{k+2}(\Gamma)\neq 0$ is $S_8(\Gamma)=\langle h_4^2\rangle$, where $S_4(O_B^1)=\langle h_4\rangle$. For any prime $p>5$, the eigenvalue of T_p on h_4^2, denoted $a_p(h_4^2)$, is expressed in terms of

$$a_\Gamma(\lambda,p)=\mathrm{T}r(Frob_\lambda|(V^2(\Gamma))_{\bar\lambda})=\left(\frac{-3(1-1/\lambda)}{p}\right)pH_p(HD(\Gamma),\frac{1}{\lambda})$$

as follows:

$$-a_p(h_4^2)=\sum_{\lambda\in\mathbb{F}_p,\neq 0,1}\Big(a_\Gamma(\lambda,p)^3-2pa_\Gamma(\lambda,p)^2-p^2a_\Gamma(\lambda,p)+p^3\Big)$$

$$+p((pH_p(HD(\Gamma);1))^2-p^2)+\left(\left(\frac{-1}{p}\right)+\left(\frac{-3}{p}\right)+\left(\frac{-6}{p}\right)\right)p^3.$$

Remark. $h_4^2\leftrightarrow f_{6.8.a.a}$ (LMFDB label) by JL, $a_p(h_4^2)=a_p(f_{6.8.a.a})$.

Some applications of the explicit trace formula

1. Get explicit eigenvalues of T_p on $S_{k+2}(\Gamma)$.

2. Get explicit Hecke traces on $S_{k+2}(\Gamma')$ for subgroups Γ' of Γ whenever there is an explicit covering map $X_{\Gamma'} \to X_\Gamma$ over $\mathbb{Q}$.

3. Get explicit hypergeometric values, e.g. by the work of Yang, $a_7(h_4^2)$ yields

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2} & \frac{1}{4} & \frac{3}{4} \\ & \frac{5}{6} & \frac{7}{6} \end{matrix} ; \frac{2^{10}\cdot 3^3\cdot 5^6\cdot 7}{11^4\cdot 23^4}\right] = \frac{11\cdot 23}{140\sqrt{3}}\cdot\frac{2^{1/3}(4+2\sqrt{2})}{7^{7/6}}\frac{\Gamma(7/6)\Gamma(13/24)\Gamma(19/24)}{\Gamma(5/6)\Gamma(17/24)\Gamma(23/24)}. $$

4. The coefficients of $\Delta(z) = \eta^{24}(z) = \sum_{n\geq 1}\tau(n)e^{2\pi i n z}$ satisfy

$$ \tau(p) \equiv \begin{cases} 2 \mod 10 & \text{if } p \equiv 1, 3 \mod 10; \\ 6 \mod 10 & \text{if } p \equiv 7 \mod 10; \\ 0 \mod 10 & \text{if } p \equiv 9 \mod 10. \end{cases} $$

THANK YOU !!