Averages of Long Dirichlet Polynomial Approximations of Primitive Dirichlet L-functions

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Moments of the Riemann zeta-function

$$\boxed{M_k(T) := \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt}$$

Hardy & Littlewood initiated the study of $M_k(T)$.

- Lindelöff Hypothesis: For any $\varepsilon>0$,

$$|\zeta(\tfrac{1}{2}+it)| \ll_\varepsilon t^\varepsilon.$$

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$$\left|\zeta\left(\tfrac{1}{2}+it\right)\right| \ll_\varepsilon t^{\varepsilon}.$$

- LH $\Leftrightarrow$ for any $\varepsilon > 0$, $M_k(T) \ll_\varepsilon T^{1+\varepsilon}$ for all $k \in \mathbb{N}$.

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- study extreme values of $|\zeta(\frac{1}{2}+it)|$

Folklore Conjecture

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \sim c_k T (\log T)^{k^2}$$

where c_k is a constant

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$$T(\log T)^{k^2} \ll \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\,dt \ll T(\log T)^{k^2}$$

Lower Bound: holds unconditionally for $k \geq 0$

- Radziwiłł - Soundararajan (2013)
- Heap - Soundararajan (2020)

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Upper Bound : holds unconditionally for :

- $k = \frac{1}{n}, n \in \mathbb{N}$ Heath-Brown (1981)
- $k = 1 + \frac{1}{n}, n \in \mathbb{N}$ Bettin-Chandee-Radziwiłł (2017)
- $0 \leq k \leq 2$ Heap-Radziwiłł-Soundararajan (2019)

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Upper Bound : holds on RH for $k \geqslant 0$

- Soundararajan (2007)
- Harper (2013)

Asymptotics

$$M_k(T) = \int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt$$

- Hardy + Littlewood (1918): $M_1(T) \sim T \log T$
- Ingham (1926): $M_2(T) \sim \frac{T}{2\pi^2} \log^4 T$

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For $k \geq 3$, no asymptotic formula has been proven unconditionally.

- Ng (2016): $M_3(T)$ with a power saving error term, assuming a ternary additive divisor conjecture.

- Ng-Shen-Wong (2022): $M_4(T)$ assuming RH and a quaternary additive divisor conjecture.

Why are asymptotics difficult for large k?

credit to Fai Chandee for this overview

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt = \int_0^T \zeta^k\left(\tfrac{1}{2}+it\right)\overline{\zeta^k\left(\tfrac{1}{2}+it\right)}\, dt$$

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For $\mathrm{Re}(s)>1$,

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}$$

where $d_k(n)$ is the k-th divisor function:

$$d_k(n) = \sum_{m_1\cdots m_k = n} 1 = \#\left\{(m_1,\ldots,m_k)\in\mathbb{N}^k : m_1\cdots m_k = n\right\}$$

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt = \int_0^T \zeta^k\left(\tfrac{1}{2}+it\right) \overline{\zeta^k\left(\tfrac{1}{2}+it\right)}\, dt$$

We expect that

$$\zeta^k\left(\tfrac{1}{2}+it\right) \approx \sum_{n \le t^k} \frac{d_k(n)}{n^{1/2+it}},$$

so

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \approx \int_0^T \sum_{m,n \le T^{k/2}} \frac{d_k(m) d_k(n)}{(mn)^{1/2}} \left(\frac{m}{n}\right)^{-it} dt$$

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Note:

$$\int_0^T \left(\frac{m}{n}\right)^{-it} dt = \begin{cases} T & \text{if } m = n \\ \dfrac{\sin(T \log(m/n))}{\log(m/n)} & \text{if } m \ne n \end{cases}$$

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^{2k} dt \approx T \sum_{n \le T^{k/2}} \frac{d_k(n)^2}{n}$$

$$+ \sum_{\substack{m,n \le T^{k/2} \\ m \ne n}} \frac{d_k(m)\, d_k(n)}{\sqrt{m}\sqrt{n}} \cdot \frac{\sin(T \log(m/n))}{\log(m/n)}$$

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$\log\left(\frac{m}{n}\right)$ is small if m is close to n

$\therefore$ terms with $m = n + r$ and r small contribute to the main term

Example: $k=3$

$$\sum_{\substack{m,n\leq T^{3/2}\\ m\neq n}} \frac{d_3(m)d_3(n)}{\sqrt{m}\sqrt{n}}\cdot\frac{\sin(T\log(m/n))}{\log(m/n)}$$

- For $m = T^{5/4} + T^{1/4}$ and $n = T^{5/4}$

$$\log\left(\frac{m}{n}\right) \asymp \log\left(1+\frac{1}{T}\right) \asymp \frac{1}{T}$$

$$\boxed{\frac{\sin(T\log(m/n))}{\log(m/n)} \asymp T}$$

Example: $k=3$

$$\sum_{\substack{m,n \le T^{3/2} \\ m \neq n}} \frac{d_3(m)\, d_3(n)}{\sqrt{m}\sqrt{n}} \cdot \frac{\sin(T \log(m/n))}{\log(m/n)}$$

- This leads to the difficult problem of additive divisor sums:

$$\sum_{n \le x} d_k(n)\, d_k(n+r).$$

* $k=2$ ✓ Motohashi
$k \geqslant 3$ no asymptotics

Folklore Conjecture

$$M_k(T) \sim \frac{g_k}{(k^2)!} a_k T (\log T)^{k^2}$$

where

- a_k : defined via $\displaystyle\sum_{n \le T} \frac{d_k(n)^2}{n} \sim \frac{a_k}{(k!)^2} (\log T)^{k^2}$

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- can show that

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \binom{k-1}{j}^2 \frac{1}{p^j}.$$

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- g_k : some constant; for $k \geqslant 3$, we have conjectures for its value.

Folklore Conjecture

$$M_k(T) \sim \frac{g_k}{(k^2)!} a_k T (\log T)^{k^2}$$

Conjectures for g_k:

- Conrey & Ghosh (1996) Dirichlet polynomials + AFE $g_3 = 42$
- Conrey & Gonek (1998) Dirichlet polynomials + AFE $g_4 = 24024$
- Keating & Snaith (1998) RMT, $\mathrm{Re}(k) \geq -\frac{1}{2}$ $g_k = (k^2)! \prod_{j=1}^{k-1} \frac{j!}{(j+k)!}$
- Diaconu-Goldfeld-Hoffstein (2000) mult. Dirichlet series, $k \in \mathbb{N}$
- Conrey-Farmer-Keating-Rubenstein-Snaith (2000) recipe, $k \in \mathbb{N}$

Where does the conjectured combinatorial structure come from?

The CFKRS recipe for shifted moments of $\zeta(s)$

$$\boxed{\mathcal{M}_{A,B}(T) := \int_0^T \prod_{\alpha\in A} \zeta(\tfrac{1}{2}+\alpha+it) \prod_{\beta\in B} \zeta(\tfrac{1}{2}+\beta-it)\,dt}$$

- "shifts" α, β are small complex numbers ($\ll 1/\log T$)

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Basic recipe (conjectures lower order terms & their coefficients too)

(1) use the approximate functional equation:

$$\zeta(s) \approx \sum_m \frac{1}{m^s} + \chi(s) \sum_n \frac{1}{n^{1-s}}$$

where

$$\chi(s) = \left(\frac{t}{2\pi}\right)^{\frac{1}{2}-s} e^{it+\pi i/4}\left(1+O\left(\tfrac{1}{t}\right)\right)$$

(2) multiply out

(3) Ignore terms where the product of χ-factors is oscillating rapidly

(4) Ignore off-diagonal contributions of what's left.

$$\mathcal{M}_{A,B}(T) := \int_0^T \prod_{\alpha\in A} \zeta(\tfrac{1}{2}+\alpha+it) \prod_{\beta\in B} \zeta(\tfrac{1}{2}+\beta-it)\,dt$$

Conjecture (CFKRS, 2000)

$$\mathcal{M}_{A,B}(T) \sim$$

$$\sum_{\substack{U\subseteq A,\, V\subseteq B \\ |U|=|V|}} \int_0^T \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha - \sum_{\beta\in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A\setminus U\cup V^-}(n)\,\tau_{B\setminus V\cup U^-}(n)}{n}\,dt$$

$$\prod_{\alpha\in A} \zeta(\alpha+s) =: \sum_{n=1}^{\infty} \frac{\tau_A(n)}{n^s} \qquad \tau_A(n) = \sum_{m_1 m_2\cdots m_k = n} m_1^{-\alpha_1} m_2^{-\alpha_2} \cdots m_k^{-\alpha_k}$$

$$U^- := \{-\alpha : \alpha\in U\}$$

- We call the cardinality $|U| = |V|$ the number of "swaps."

Example: The recipe prediction for the fourth moment

CFKRS Recipe Prediction

$$\int_0^T \prod_{\alpha\in A} \zeta(\tfrac{1}{2}+\alpha+it) \prod_{\beta\in B} \zeta(\tfrac{1}{2}+\beta-it)\,dt$$

$$\sim \sum_{\substack{U\subseteq A,\, V\subseteq B \\ |U|=|V|}} \int_0^T \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha - \sum_{\beta\in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A\setminus U\cup V^-}(n)\,\tau_{B\setminus V\cup U^-}(n)}{n}\,dt$$

For the fourth moment, take $|A| = |B| = 2$:

$A = \{\alpha_1, \alpha_2\}$

$U \subseteq A$: $\emptyset$, $\{\alpha_1\}$,

$\{\alpha_2\}$, $\{\alpha_1, \alpha_2\}$

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For the fourth moment, take $|A| = |B| = 2$:

$A = \{\alpha_1, \alpha_2\}$

$U \subseteq A$: ϕ, $\{\alpha_1\}$, $\{\alpha_2\}$, $\{\alpha_1, \alpha_2\}$

$B = \{\beta_1, \beta_2\}$

$V \subseteq B$: ϕ, $\{\beta_1\}$, $\{\beta_2\}$, $\{\beta_1, \beta_2\}$

Example: The recipe prediction for the fourth moment

CFKRS Recipe Prediction

$$M_{\alpha,\beta}(T) \sim \sum_{\substack{U\subseteq A, V\subseteq B \\ |U|=|V|}} \int_0^T \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha - \sum_{\beta\in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A\setminus U\cup V^-}(n)\,\tau_{B\setminus V\cup U^-}(n)}{n}\, dt$$

"0-swap" $(U = V = \emptyset)$:

$$\sum_{n=1}^{\infty} \frac{\tau_A(n)\tau_B(n)}{n} = \frac{\zeta(1+\alpha_1+\beta_1)\zeta(1+\alpha_1+\beta_2)\zeta(1+\alpha_2+\beta_1)\zeta(1+\alpha_2+\beta_2)}{\zeta(2+\alpha_1+\alpha_2+\beta_1+\beta_2)}$$

$$=: Z(\alpha_1,\alpha_2;\beta_1,\beta_2)$$

Example: The recipe prediction for the fourth moment

CFKRS Recipe Prediction

$$M_{\alpha,\beta}(T) \sim \sum_{\substack{U \subseteq A,\, V \subseteq B \\ |U|=|V|}} \int_0^T \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha \in U}\alpha - \sum_{\beta \in V}\beta} \sum_{n=1}^{\infty} \frac{\tau_{A \setminus U \cup V^-}(n)\, \tau_{B \setminus V \cup U^-}(n)}{n}\, dt$$

$$Z(\alpha_1, \alpha_2; \beta_1, \beta_2) := \frac{\zeta(1+\alpha_1+\beta_1)\zeta(1+\alpha_1+\beta_2)\zeta(1+\alpha_2+\beta_1)\zeta(1+\alpha_2+\beta_2)}{\zeta(2+\alpha_1+\alpha_2+\beta_1+\beta_2)}$$

"1-swap" example: $U = \{\alpha_1\},\ V = \{\beta_2\}$

$$A \setminus U \cup V^- = \{-\beta_2, \alpha_2\} \qquad B \setminus V \cup U^- = \{\beta_1, -\alpha_1\}$$

$$\left(\frac{t}{2\pi}\right)^{-\alpha_1-\beta_2} \sum_{n=1}^{\infty} \frac{\tau_{\{-\beta_2,\alpha_2\}}(n)\, \tau_{\{\beta_1,-\alpha_1\}}(n)}{n} = \left(\frac{t}{2\pi}\right)^{-\alpha_1-\beta_2} Z(-\beta_2, \alpha_2; \beta_1, -\alpha_1)$$

CFKRS Recipe Prediction

$$\int_0^T \prod_{\alpha\in A}\zeta(\tfrac12+\alpha+it)\prod_{\beta\in B}\zeta(\tfrac12+\beta-it)\,dt$$

$$\sim \sum_{\substack{U\subseteq A,\, V\subseteq B\\ |U|=|V|}} \int_0^T \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha\in U}\alpha-\sum_{\beta\in V}\beta}\sum_{n=1}^{\infty} \frac{\tau_{A\setminus U\cup V^-}(n)\,\tau_{B\setminus V\cup U^-}(n)}{n}\,dt$$

Theorem (CFKRS, 2000)

$$\int_0^T \zeta(\tfrac12+\alpha_1+it)\zeta(\tfrac12+\alpha_2+it)\zeta(\tfrac12+\beta_1-it)\zeta(\tfrac12+\beta_2-it)\,dt \sim$$

$$\int_0^T \Bigg(\overbrace{Z(\alpha_1,\alpha_2;\beta_1,\beta_2)}^{\text{0-swap}}$$

1-swaps

$$+\left(\frac{t}{2\pi}\right)^{-\alpha_1-\beta_1} Z(-\beta_1,\alpha_2;-\alpha_1,\beta_2) + \left(\frac{t}{2\pi}\right)^{-\alpha_1-\beta_2} Z(-\beta_2,\alpha_2;\beta_1,-\alpha_1)$$

$$+\left(\frac{t}{2\pi}\right)^{-\alpha_2-\beta_1} Z(\alpha_1,-\beta_1;-\alpha_2,\beta_2) + \left(\frac{t}{2\pi}\right)^{-\alpha_2-\beta_2} Z(\alpha_1,-\beta_2;\beta_1,-\alpha_2)$$

2-swap

$$+\left(\frac{t}{2\pi}\right)^{-\alpha_1-\alpha_2-\beta_1-\beta_2} Z(\underline{-\beta_1,-\beta_2};\underline{-\alpha_1,-\alpha_2}) \Bigg)\,dt$$

letting shifts $\to 0$
agrees with Heath-Brown

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- The CFKRS recipe conjectures are consistent with proven theorems from random matrix theory, where we also see the swapping phenomenon.

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- Katz-Sarnak philosophy - behind each family of L-functions is a symmetry type.

What is guiding the CFKRS heuristic?

Theorem (CFKRS, 2003)

Let $U(N)$ be the group of $N \times N$ unitary matrices.

Then integrating with respect to the Haar measure gives

$$\int_{U(N)} \prod_{\alpha \in A} \det(1 - e^{-\alpha} M) \prod_{\beta \in B} (1 - e^{-\beta} M^{-1})\, dM$$

$$= \sum_{\substack{U \subseteq A,\, V \subseteq B \\ |U| = |V|}} (e^N)^{-\sum_{\alpha \in U} \alpha - \sum_{\beta \in V} \beta} Z(A \setminus U \cup V^-, B \setminus V \cup U^-),$$

where $Z(A,B) := \prod_{\alpha \in A, \beta \in B} (1 - e^{-\alpha - \beta})^{-1}$.

A new approach to proving high moments

Conrey - Keating: The general idea is to estimate

(series of 5 papers, 2015-2019)

$$\mathcal{M}_{A,B}(T) := \int_0^T \prod_{\alpha\in A} \zeta(\tfrac{1}{2}+\alpha+it) \prod_{\beta\in B} \zeta(\tfrac{1}{2}+\beta-it)\,dt$$

using the approximation

$$\int_0^T \sum_{m\le X} \frac{\tau_A(m)}{m^{\frac{1}{2}+it}} \sum_{n\le X} \frac{\tau_B(n)}{n^{1/2-it}}\,dt$$

$$\tau_A(n) = \sum_{m_1 m_2 \cdots m_k = n} m_1^{-\alpha_1} m_2^{-\alpha_2} \cdots m_k^{-\alpha_k}$$

How does X affect the accuracy of the approximation?

- CFKRS recipe predicts an asymptotic formula for

$$\int_0^T \prod_{\alpha \in A} \zeta\left(\tfrac{1}{2}+\alpha+it\right) \prod_{\beta \in B} \zeta\left(\tfrac{1}{2}+\beta-it\right) dt$$

with lower order terms.

- The terms in the formula are categorized by certain shared combinatorial properties.

How does X affect the accuracy of the approximation?

- CFKRS "recipe" predicts an asymptotic formula for

$$\int_0^T \prod_{\alpha \in A} \zeta\left(\tfrac{1}{2}+\alpha+it\right) \prod_{\beta \in B} \zeta\left(\tfrac{1}{2}+\beta-it\right) dt$$

with lower order terms.

- The terms in the formula are categorized by certain shared combinatorial properties.

- The categories are called "ℓ-swaps."

- If $|A| = |B| = k$, there are "ℓ-swap terms" for

$$\ell \in \{0, 1, 2, 3, \ldots, k\}.$$

How does X affect the accuracy of the approximation?

$$\int_0^T \sum_{m \le X} \frac{\tau_A(m)}{m^{1/2+it}} \sum_{n \le X} \frac{\tau_B(n)}{n^{1/2-it}} \approx \int_0^T \prod_{\alpha \in A} \zeta(\tfrac{1}{2}+\alpha+it) \prod_{\beta \in B} \zeta(\tfrac{1}{2}+\beta-it)\, dt$$

CFKRS predicts:	CFKRS predicts:
0-swaps, 1-swaps, 2-swaps, ⋮, k-swaps $\Big\}$ for the truncations	0-swaps, 1-swaps, 2-swaps, ⋮, k-swaps $\Big\}$ for the full 2k-th moment

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CFKRS predicts:

$$\left.\begin{array}{l} 0\text{-swaps} \\ 1\text{-swaps} \\ 2\text{-swaps} \\ \vdots \\ k\text{-swaps} \end{array}\right\} \text{for the truncations}$$

CFKRS predicts:

$$\left.\begin{array}{l} 0\text{-swaps} \\ 1\text{-swaps} \\ 2\text{-swaps} \\ \vdots \\ k\text{-swaps} \end{array}\right\} \text{for the full } 2k\text{-th moment}$$

Conrey-Keating: If $X \gg T^{\ell}$ then the ℓ-swap terms for the truncations on the LHS are precisely the ℓ-swap terms for the full $2k$'th moment on the RHS.

* As X increases, more ℓ-swaps match up. *

- Conrey-Keating (2015) found that the "1-swap" terms for zeta are the consequence of formulas for correlations of divisor sums.

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- This connection has been made rigorous by A. Hamieh & N. Ng.

Theorem (Hamieh and Ng, 2021)

Assume the expected asymptotic formula for correlations of divisor sums.

If $X = T^{\eta}$ with $1 < \eta < 2$, then as $T \to \infty$,

$$\int_0^T \sum_{m \le X} \frac{\tau_A(m)}{m^{1/2+it}} \sum_{n \le X} \frac{\tau_B(n)}{n^{1/2-it}}\, dt \sim \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} \frac{X^{z+w}}{zw} \sum_{\substack{U \subseteq A,\, V \subseteq B \\ 0 \le |U| = |V| \le 1}}$$

$$\times \int_0^T \left(\frac{t}{2\pi}\right)^{-\sum_{\alpha \in U}(\alpha+z) - \sum_{\beta \in V}(\beta+w)} \sum_{\substack{1 \le m,n < \infty \\ m=n}} \frac{\tau_{A_z \setminus U_z \cup V_w^-}(m)\, \tau_{B_w \setminus V_w \cup U_z^-}(n)}{\sqrt{mn}}\, dt\, dw\, dz.$$

Adapting the Conrey-Keating approach

Family of all Dirichlet L-functions of modulus q

$$M_k(q) = \frac{1}{\varphi^*(q)} \sum_{\chi \pmod q}^{*} |L(1/2, \chi)|^{2k}$$

$\sum^*$: the sum is over all primitive characters

$\varphi^*(q)$: the number of primitive characters mod q.

- Like $\zeta(s)$, progress on $M_k(q)$ is limited for large k.

study in t-aspect

study in q-aspect

What is known?

Bounds: (q prime)

$$q(\log q)^{k^2} \ll \sum_{\chi \bmod q}^{*} |L(1/2,\chi)|^{2k} \ll q(\log q)^{k^2}$$

Lowerbound: all $k>0$

$k \geq 1$, Radziwiłł - Soundararajan (2012)

$0<k<1$, Heap - Soundararajan (2020)
Gao (2021)

Upperbound: all $k>0$ under GRH

Soundararajan (2009), Harper (2013)

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Asymptotics:

- Paley (1934): $M_1(q) \sim \log q$
- Heath-Brown (1981), Soundararajan (2007), Young (2010) gives:

$$M_2(q) \sim 2b_2 \frac{(\log q)^4}{4!}$$

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Asymptotics for $k\geq 3$?

Conjecture:

$$M_k(q) \sim g_k b_k \frac{(\log q)^{k^2}}{k^2!}$$

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}$$

Introducing extra averaging over q

Using the large sieve inequality to obtain upper bound:

Huxley (1970): $$\sum_{q\le Q}\ \sideset{}{^*}\sum_{\chi \bmod q} |L(1/2,\chi)|^{2k} \ll Q^2(\log Q)^{k^2},$$ where $k=3,4$

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Using the asymptotic large sieve (for asymptotics!)

- Conrey-Iwaniec-Soundararajan (2012): 6th moment w/ small averaging over t
- Chandee-Li-Matomaki-Radziwiłł (2023+): $\displaystyle\sum_{q \le Q} \sum_{\chi(q)}^{*} |L(1/2, \chi)|^6 \sim 42 \tilde{c}_3 Q^2 \frac{(\log Q)^9}{9!}$
- Chandee-Li-Matomaki-Radziwiłł (2023+): 8th moment w/ small averaging over t
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  - error term is size $Q^2 (\log Q)^{15+\varepsilon}$

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Asympotic large sieve
- a framework that harnesses the extra averaging to work with off-diagonal terms

Adapting the Conrey-Keating approach

The 2kth moment: $|A|=|B|=k$

$$\sideset{}{^b}\sum_{\chi \bmod q} \prod_{\alpha\in A} L\left(\tfrac{1}{2}+\alpha, \chi\right) \prod_{\beta\in B} L\left(\tfrac{1}{2}+\beta, \overline{\chi}\right)$$

Approximate by:

$$\sum_{q\leq Q} \sideset{}{^b}\sum_{\chi \bmod q} \sum_{m\leq X} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} \sum_{n\leq X} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}}$$

b: primitive, even

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The twisted 2kth moment, averaged over q:

$$\sum_{q \le Q} \sum_{\chi \bmod q}^{b} \chi(h)\overline{\chi}(k) \sum_{m \le X} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} \sum_{n \le X} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}}$$

b: primitive, even

Twisted moment of Dirichlet polynomial approx.

$$S(h,k) := \sum_{q=1}^{\infty} W\left(\frac{q}{Q}\right) \sideset{}{^\flat}\sum_{\chi \bmod q} \chi(h)\overline{\chi}(k)$$

$$\times \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right).$$

Here:

- W, V are smooth cut-off functions
- $\flat$ denotes that the sum is over even, primitive characters modulo q

What does the CFKRS recipe predict for $S(h,k)$?

Notation: gathering the ingredients

$$\mathcal{I}_\ell(h,k) := \sum_{\substack{q=1 \\ (q,hk)=1}}^{\infty} W\left(\frac{q}{Q}\right) \sum_{x \bmod q}^{b} \frac{1}{(2\pi i)^2} \int_{(\varepsilon)} \int_{(\varepsilon)} X^{s_1+s_2} \tilde{V}(s_1)\tilde{V}(s_2)$$

$$\times \sum_{\substack{\mathcal{U}\subseteq A, V\subseteq B \\ |\mathcal{U}|=|V|=\ell}} \prod_{\alpha\in\mathcal{U}} \frac{\mathcal{X}\left(\frac{1}{2}+\alpha+s_1\right)}{q^{\alpha+s_1}} \prod_{\beta\in V} \frac{\mathcal{X}\left(\frac{1}{2}+\beta+s_2\right)}{q^{\beta+s_2}}$$

$$\times \sum_{\substack{1\le m,n<\infty \\ mh=nk \\ (mn,q)=1}} \frac{\tau_{A_{s_1}\setminus \mathcal{U}_{s_1}\cup V^-_{s_2}}(m)\, \tau_{B_{s_2}\setminus V_{s_2}\cup \mathcal{U}^-_{s_1}}(n)}{\sqrt{mn}}\, ds_2\, ds_1$$

the sum of all the ℓ-swap terms from the recipe prediction.

$$\tilde{V}(s) := \int_0^\infty V(x)\, x^{s-1}\, dx$$

(Mellin transform of V)

$$\mathcal{X}(s) := \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1}{2}-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}s\right)}$$

The recipe conjecture

Recipe Conjecture : Let $A = \{\alpha_1, \dots, \alpha_k\}$, $B = \{\beta_1, \dots, \beta_k\}$ with $\alpha_i, \beta_j \ll 1/\log Q$. For all $X > 0$, where X is the length of the L-function approximations,

$$S(h,k) \sim \sum_{\ell=0}^{K} I_\ell(h,k)$$

as $Q \to \infty$.

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Roughly : the 2k-th moment of the Dirichlet polynomial approximations of the L-functions with length $X>0$ is asymptotic to the sum of the predicted $0,1,\dots,K$-swap terms of the approximations.

Main Result

Theorem (S.Baluyot and C.T-B, 2022+)

Let Q be a large parameter and $X = Q^\eta$ with $1 < \eta < 2$. Let $A = \{\alpha_1, \ldots, \alpha_k\}$, $B = \{\beta_1, \ldots, \beta_k\}$ with $\alpha_i, \beta_j \ll 1/\log Q$. Then, assuming the Generalized Lindelöf Hypothesis, we have

$$S(h,k) \sim \mathcal{I}_0(h,k) + \mathcal{I}_1(h,k).$$

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Roughly: the $2k$-th moment of the Dirichlet polynomial approximations of the L-functions with lengths Q^η, $1<\eta<2$, is asymptotic to the sum of the predicted $0,1$-swap terms of the approximations.

Interpretation of result

- the 1-swap terms predicted by the CFKRS recipe are correct for this family of L-functions, averaged over q.

- For the general 2kth moment, this gives the first rigorous proof of the validity of the CFKRS heuristic "beyond the diagonal" for this family of L-functions.

Overview of proof

- Start with

$$S(h,k) := \sum_{q=1}^{\infty} W\left(\frac{q}{Q}\right) \sum_{\chi \bmod q}^{b} \chi(h)\overline{\chi}(k)$$

$$\times \sum_{m=1}^{\infty} \frac{\tau_A(m)\chi(m)}{\sqrt{m}} V\left(\frac{m}{X}\right) \sum_{n=1}^{\infty} \frac{\tau_B(n)\overline{\chi}(n)}{\sqrt{n}} V\left(\frac{n}{X}\right).$$

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- Bring in the sum over χ, and use the standard lemma

$$\sideset{}{^b}\sum_{\chi \bmod q} \chi(mh)\overline{\chi(nk)} = \frac{1}{2}\left(\sum_{\substack{q=dc \\ d \mid (mh+nk)}} \varphi(d)\mu(c) + \sum_{\substack{q=dc \\ d \mid (mh-nk)}} \varphi(d)\mu(c) \right)$$

- We then split $S(h,k)$ into three pieces:

$$S(h,k) = \mathcal{L}(h,k) + \mathcal{D}(h,k) + \mathcal{U}(h,k)$$

where

parameter (small power of Q)

$\mathcal{L}(h,k)$ = sum of the terms with $c > C$

The role of C: make bound from large sieve $\ll Q^{2-\varepsilon}$

$\mathcal{D}(h,k)$ = sum of the "diagonal" terms with $c \le C$ and $mh = nk$.

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Using standard methods, recover the 0-swap terms from $D(h,k)$.

Where are the 1-swap terms?

The "long" sum, $\mathcal{L}(h,k)$, $c > C$

$$\mathcal{L}(h,k) := \frac{1}{2} \sum_{\substack{1 \le q < \infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\substack{1 \le m,n < \infty \\ (mn,q)=1}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right)$$

$$\times \left(\sum_{\substack{c > C, d \geq 1 \\ cd = q \\ d \mid mh+nk}} \varphi(d)\mu(c) + \sum_{\substack{c > C, d \geq 1 \\ cd = q \\ d \mid mh-nk}} \varphi(d)\mu(c) \right).$$

- detect $d \mid mh \pm nk$ using orthogonality of characters

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- split $\mathcal{L}(h,k) = \mathcal{L}_0(h,k) + \mathcal{L}_r(h,k)$
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  - $\mathcal{L}_0(h,k)$: contribution of the principal character mod d
  - $\mathcal{L}_r(h,k)$ the rest.
- bound $\mathcal{L}_r(h,k)$ using the large sieve inequality and GLH
- $\mathcal{L}_0(h,k)$ ends up cancelling with a contribution from $\mathcal{U}(h,k)$

Finding the predicted 1-swap terms

$$S(h,k) = \underbrace{D(h,k)}_{\text{short, diagonal}} + \underbrace{\mathcal{L}(h,k)}_{\text{long}} + \underbrace{\mathcal{U}(h,k)}_{\text{short, off diagonal}}$$

0-swap terms

$$\underbrace{\mathcal{L}_0(h,k)}_{\text{(eventually cancels)}} + \underbrace{\mathcal{L}_r(h,k)}_{\text{bounded under GLH}}$$

Finding the predicted 1-swap terms

$$S(h,k) = \underbrace{D(h,k)}_{\text{short, diagonal}} + \underbrace{L(h,k)}_{\text{long}} + \underbrace{\mathcal{U}(h,k)}_{\text{short, off diagonal}}$$

0-swap terms

$$L(h,k) = \underbrace{L_0(h,k)}_{\text{(eventually cancels)}} + \underbrace{L_r(h,k)}_{\text{bounded under GLH}}$$

the 1-swap terms are hidden here

$$\mathcal{U}(h,k) := \frac{1}{2} \sum_{\substack{1 \le q < \infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\substack{1 \le m,n < \infty \\ mh \ne nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right)$$

$$\times \left(\sum_{\substack{1 \le c \le C,\, d \geq 1 \\ cd = q \\ d \mid mh+nk}} \varphi(d)\mu(c) + \sum_{\substack{1 \le c \le C,\, d \geq 1 \\ cd = q \\ d \mid mh-nk}} \varphi(d)\mu(c) \right).$$

Finding the predicted 1-swap terms

$$S(h,k) = D(h,k) + \mathcal{L}(h,k) + \mathcal{U}(h,k)$$

$D(h,k)$: short, diagonal

0-swap terms

$\mathcal{L}(h,k)$: long

$$\mathcal{L}_0(h,k) + \mathcal{L}_r(h,k)$$

$\mathcal{L}_0(h,k)$: (eventually cancels)

$\mathcal{L}_r(h,k)$: bounded under GLH

$\mathcal{U}(h,k)$: short, off diagonal

the 1-swap terms are hidden here

$$\mathcal{U}(h,k) := \frac{1}{2} \sum_{\substack{1 \le q < \infty \\ (q,hk)=1}} W\left(\frac{q}{Q}\right) \sum_{\substack{1 \le m,n < \infty \\ mh \ne nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right)$$

$$\times \left(\sum_{\substack{1 \le c \le C,\, d \ge 1 \\ cd = q \\ d \mid mh + nk}} \varphi(d)\mu(c) + \sum_{\substack{1 \le c \le C,\, d \ge 1 \\ cd = q \\ d \mid mh - nk}} \varphi(d)\mu(c) \right).$$

The large sieve is not strong enough to use as before because d can be as large as Q.

Crux of argument:

We switch to the "complementary modulus"

$$d \mid |mh \pm nk| \text{ , so instead consider } \ell := \frac{|mh \pm nk|}{d}$$

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Split $\mathcal{U}(h,k) = \mathcal{U}_0(h,k) + \mathcal{U}_r(h,k)$

- $\mathcal{U}_0(h,k)$: contribution from principal character mod ℓ

Bound $\mathcal{U}_r(h,k)$ - more delicate than $\mathcal{L}_r(h,k)$

- interdependence of variables and the complexity of the multivariable Mellin transform.
- Closely follow work of Conrey-Iwaniec-Soundararajan (2019).
- This is where we must assume GLH, because we are working with an arbitrarily large number of L-functions

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0-swap terms

$$\mathcal{L}(h,k) = \mathcal{L}_0(h,k) + \mathcal{L}_r(h,k)$$

(eventually cancels)

bounded under GLH

$$\mathcal{U}(h,k) = \mathcal{U}_0(h,k) + \mathcal{U}_r(h,k)$$

the 1-swaps should be here

bounded under GLH

Finding the predicted 1-swap terms

$$S(h,k) = D(h,k) + L(h,k) + \mathcal{U}(h,k)$$

$D(h,k)$: short, diagonal — 0-swap terms

$L(h,k)$: long

$\mathcal{U}(h,k)$: short, off diagonal

$$L(h,k) = L_0(h,k) + L_r(h,k)$$

$L_0(h,k)$: (eventually cancels)

$L_r(h,k)$: bounded under GLH

$$\mathcal{U}(h,k) = \mathcal{U}_0(h,k) + \mathcal{U}_r(h,k)$$

$\mathcal{U}_0(h,k)$: the 1-swaps should be here

$\mathcal{U}_r(h,k)$: bounded under GLH

Focusing on $\mathcal{U}_0(h,k)$

We apply Mellin inversion to W and move a line of integration to write

$$\mathcal{U}_0(h,k) = \mathcal{U}_1(h,k) + \mathcal{U}_2(h,k)$$

where $\mathcal{U}_1(h,k)$ is the residue (from $\zeta(1+w)$) and $\mathcal{U}_2(h,k)$ is the rest.

Finding the predicted 1-swap terms

- After careful manipulation, we realize $\mathcal{U}_1(h,k)$ cancels with $\mathcal{L}_0(h,k)$!

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0-swap terms

$$\mathcal{L}(h,k) = \mathcal{L}_0(h,k) + \underbrace{\mathcal{L}_r(h,k)}_{\text{bounded under GLH}}$$

$$\mathcal{U}(h,k) = U_0(h,k) + \underbrace{U_r(h,k)}_{\text{bounded under GLH}}$$

$$U_0(h,k) = \mathcal{U}_1(h,k) + \mathcal{U}_2(h,k)$$

cancels!

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bounded under GLH

$$\mathcal{U}(h,k) = \mathcal{U}_0(h,k) + \mathcal{U}_r(h,k)$$

bounded under GLH

$$\mathcal{U}_0(h,k) = \mathcal{U}_1(h,k) + \mathcal{U}_2(h,k)$$

cancels!

Ah-ha!

The 1-swaps must be here!

Finding the predicted 1-swap terms

We are left with

$$\mathcal{U}_2(h,k) \approx \frac{Q}{2} \sum_{\substack{1\le c<C \\ (c,hk)=1}} \frac{\mu(c)}{c} \sum_{\substack{1\le m,n<\infty \\ (mn,c)=1 \\ mh\neq nk}} \frac{\tau_A(m)\tau_B(n)}{\sqrt{mn}} V\left(\frac{m}{X}\right) V\left(\frac{n}{X}\right) \sum_{\substack{1\le e<\infty \\ (e,g)=1}} \frac{\mu(e)}{e}$$

$$\times \sum_{a\mid g} \frac{\mu(a)}{\varphi(ea)} \cdot \frac{1}{2\pi i} \int_{(-\varepsilon)} \left(\frac{c\,|mh\pm nk|}{gQ}\right)^{w} \widetilde{W}(1-w)\,\zeta(1+w)\,dw.$$

- Write $\mathcal{U}_2(h,k)$ as an Euler product after separating the variables in $|mh\pm nk|^{w}$ (use a lemma from CIS'19)
- Use Mellin inversion & express $\mathcal{U}_2(h,k)$ as a quadruple integral.

Finding the predicted 1-swap terms

The recipe tells us what the 1-swaps look like, but no information on how to extract them.

Difficulty:

How to bridge the gap?

The asymptotic large sieve tells us where the 1-swap terms are hiding.

Finding the predicted 1-swap terms

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Difficulty:

How to bridge the gap?

The asymptotic large sieve tells us where the 1-swap terms are hiding.

We find the 1-swaps via strategic contour integration and proving identities involving several Euler products so that we can match to what the recipe predicts.

The map of the argument

$$S(h,k) = \underbrace{D(h,k)}_{\text{short, diagonal}} + \underbrace{\mathcal{L}(h,k)}_{\text{long}} + \underbrace{\mathcal{U}(h,k)}_{\text{short, off diagonal}}$$

0-swap terms

$$\mathcal{L}^0(h,k) + \mathcal{L}_r(h,k)$$

bounded under GLH

$$\mathcal{U}^0(h,k) + \mathcal{U}_r(h,k)$$

bounded under GLH

$$\mathcal{U}_1(h,k) + \mathcal{U}_2(h,k)$$

cancels!

The 1-swaps are here!
(match with recipe prediction)

Generalized Lindelöff Hypothesis

The Lindelöff Hypothesis is true, and for all $\varepsilon > 0$ and all nonprincipal characters mod q,

$$L\left(\tfrac{1}{2} + it, \chi\right) \ll \left(q(1 + |t|)\right)^{\varepsilon}.$$

- We assume GLH in a handful of places in the proof to control the large number of zeta- and L-function factors.

Generalized Lindelöff Hypothesis

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Making this precise is work in-progress with student Bowen Li.

Finding "1-swaps" in other families

NSF FRG : Averages of L-functions & Arithmetic Stratification

Conrey-Rodgers ('22+)

- family of quadratic L-functions
- symplectic
- Poisson summation
- assumes GLH

Conrey-Fazzari ('23)

- L-functions assoc. with primitive cusp forms of level 1 in weight aspect
- orthogonal
- Peterrson trace formula
- assumes GLH

Thank you for your attention!