Restricted Arithmetic Quantum Unique Ergodicity

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The Eigenvalue Problem for the Laplacian

(M, g) compact n-dimensional Riemannian manifold; e.g. n-sphere

$$S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = n\right\}.$$

We study Laplacian eigenfunctions:
L^2-normalised $\phi \in L^2(M)$ satisfying

$$\Delta\phi = \lambda\phi,$$
$$\Delta := -\frac{1}{\sqrt{|\det g|}} \sum_{k,\ell=1}^{n} \frac{\partial}{\partial x_k} g^{k\ell} \sqrt{|\det g|} \frac{\partial}{\partial x_\ell}.$$

The Laplacian eigenvalue of ϕ is $\lambda \in [0, \infty)$.
These functions form an orthonormal basis $\{\phi_j\}$ of $L^2(M)$ with $\lambda_j \to \infty$ as $j \to \infty$.

On $\mathbb{R}^n$,

$$\Delta = -\sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

Quantum Unique Ergodicity

Conjecture (Quantum Unique Ergodicity)

For all $\psi \in C_b(M)$,

$$\lim_{j\to\infty} \int_M |\phi_j(x)|^2 \psi(x)\, d\mathrm{vol}(x) = \frac{1}{\mathrm{vol}(M)} \int_M \psi(x)\, d\mathrm{vol}(x).$$

Equivalently,

$$\lim_{j\to\infty} \int_B |\phi_j(x)|^2\, d\mathrm{vol}(x) = \frac{\mathrm{vol}(B)}{\mathrm{vol}(M)}$$

for every continuity set $B \subseteq M$.

This is QUE in *physical space*.
Baby version of a conjecture of Rudnick–Sarnak for *phase space* QUE involving microlocal lifts to S^*M.

Heuristic

L^2-masses of Laplacian eigenfunctions spread out randomly.

Known results on QUE:

- False without negative curvature of M, even if geodesic flow on M is ergodic (Hassell);
- True for *almost all* eigenfunctions (Shnirelman, Colin de Verdière, Zelditch);
- Any weak-* limit has positive entropy; cannot completely concentrate on a geodesic (Anantharaman);
- Any weak-* limit gives positive measure to nonempty open sets (Dyatlov–Jin).

Example: Modular Surface

Interesting setting for number theorists:
Riemannian locally symmetric spaces $M = \Gamma \backslash G/K$;

- G a Lie group,
- K a maximal compact subgroup of G,
- Γ a lattice in G.

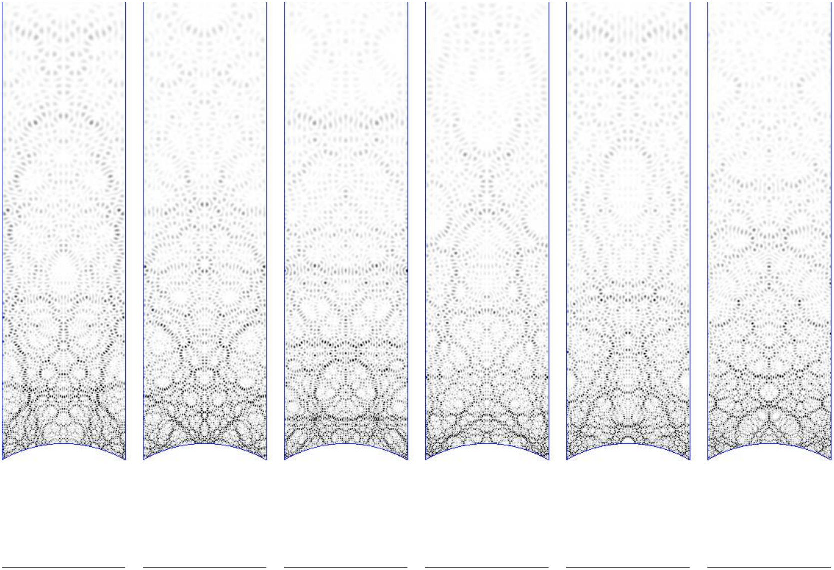
Simplest interesting case: $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

- $G/K \cong \mathbb{H}$, the upper half-plane
$$\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\},$$
- $\Gamma \backslash G/K \cong \Gamma \backslash \mathbb{H}$, the modular surface
$$\Gamma \backslash \mathbb{H} = \left\{z = x + iy \in \mathbb{H} : -\frac{1}{2} < x < \frac{1}{2},\ x^2 + y^2 > 1\right\},$$
- Laplacian eigenfunctions are automorphic forms.

Example: Modular Surface

- $\mathbb{H}$ is a negatively curved hyperbolic surface.
- $\Gamma\backslash\mathbb{H}$ inherits a hyperbolic metric from $\mathbb{H}$.
- The Laplacian is $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$.
- The volume measure on $\Gamma\backslash\mathbb{H}$ is $d\mu(z) = \frac{dx\,dy}{y^2}$; $\mathrm{vol}(\Gamma\backslash\mathbb{H}) = \frac{\pi}{3}$.
- Nonconstant eigenfunctions of Δ on $\Gamma\backslash\mathbb{H}$ are *Maaß forms* ϕ_j with Laplacian eigenvalue $\lambda_j = 1/4 + t_j^2$.
- The space of Maaß forms has an orthonormal basis $\mathcal{B}_0$ consisting of Hecke–Maaß cusp forms.

Example: Modular Surface



Quantum Unique Ergodicity for $\Gamma\backslash\mathbb{H}$

Theorem (Lindenstrauss (2006), Soundararajan (2010))

We have that

$$\lim_{j\to\infty}\int_{\Gamma\backslash\mathbb{H}}|\phi_j(z)|^2\psi(z)\,d\mu(z)=\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_{\Gamma\backslash\mathbb{H}}\psi(z)\,d\mu(z)$$

for all $\psi\in C_b(\Gamma\backslash\mathbb{H})$.

Quantum Unique Ergodicity for $\Gamma\backslash\mathbb{H}$

Theorem (Lindenstrauss (2006), Soundararajan (2010))

We have that

$$\lim_{j\to\infty}\int_B |\phi_j(z)|^2\,d\mu(z) = \frac{\mathrm{vol}(B)}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}$$

for every continuity set $B \subseteq \Gamma\backslash\mathbb{H}$.

Effective QUE

Conjecture (Luo–Sarnak (1995))

As $j \to \infty$, we have that

$$\sup_{B_R(w) \subset \Gamma\backslash\mathbb{H}} \left| \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) - \frac{\mathrm{vol}(B_R(w))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \right| \ll_\varepsilon t_j^{-\frac{1}{2}+\varepsilon}.$$

Conjecture is an effective form of QUE:
(essentially) optimal bounds for the discrepancy.

Theorem (Watson (2002), Young (2016))

Conjecture is true assuming GLH.

Heuristic

GLH $\iff$ central L-values $L(\frac{1}{2}, \Pi)$ are essentially bounded.

Remark

GRH $\implies$ GLH.

Spectral Decomposition of $\Gamma\backslash\mathbb{H}$

L^2-spectral decomposition of $\psi \in L^2(\Gamma\backslash\mathbb{H})$ is

$$\psi(z) = \frac{\langle \psi, 1\rangle}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} + \sum_{f \in \mathcal{B}_0} \langle \psi, f\rangle f(z).$$

- $\langle g_1, g_2\rangle := \int_{\Gamma\backslash\mathbb{H}} g_1(z)\overline{g_2(z)}\, d\mu(z)$;
- $\mathcal{B}_0$ orthonormal basis of Hecke–Maaß cusp forms f.

Remark

There is also a continuous spectrum involving Eisenstein series, which we ignore for ease of exposition.

Idea of Proof of Effective QUE

Take $\psi = 1_{B_R(w)}$ in spectral expansion, multiply by $|\phi_j|^2$, and integrate over $\Gamma\backslash\mathbb{H}$:

$$\sup_{B_R(w)\subset\Gamma\backslash\mathbb{H}} \left| \int_{B_R(w)} |\phi_j(z)|^2 \, d\mu(z) - \frac{\mathrm{vol}(B_R(w))}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \right| = \sup_{B_R(w)\subset\Gamma\backslash\mathbb{H}} \left| \sum_{f\in\mathcal{B}_0} \langle |\phi_j|^2, f\rangle \langle f, 1_{B_R(w)}\rangle \right|.$$

Need to bound RHS.

Strategy: take absolute values and bound each term.

- Good (averaged) bounds for $\langle f, 1_{B_R(w)}\rangle$ known via local Weyl law and properties of Selberg–Harish-Chandra transform;
- Can relate triple product $\langle |\phi_j|^2, f\rangle$ to L-functions.

Upshot

Problem reduces to knowing good bounds for L-functions.

Triple Product Formula

Proposition (Watson (2002), Ichino (2008))

We have that

$$\left|\left\langle |\phi_j|^2, f\right\rangle\right|^2 \approx \frac{L\left(\frac{1}{2}, \operatorname{ad}\phi_j \otimes f\right) L\left(\frac{1}{2}, f\right)}{L(1, \operatorname{ad}\phi_j)^2 L(1, \operatorname{ad} f)} \times \begin{cases} \text{mild polynomial decay} & \text{if } t_f \leq 2t_j, \\ \text{exponential decay} & \text{if } t_f > 2t_j. \end{cases}$$

Remark

Gan–Gross–Prasad for $(\mathrm{SO}(4), \mathrm{SO}(3))$: $\mathfrak{so}_4 \cong \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$, $\mathfrak{so}_3 \cong \mathfrak{sl}_2$.

Assuming GLH, ratio of L-functions is essentially bounded.
Exponential decay means $\sum_{f \in \mathcal{B}_0}$ can be truncated to $t_f \leq 2t_j$.

Upshot

Yields Luo–Sarnak conjecture assuming GLH:
effective QUE with (essentially) optimal error term.

Refinements of QUE

Refinements of QUE beyond effective bounds for the discrepancy:

- Small-scale QUE: how fast can R shrink as t_j grows for QUE to hold on $B_R(w)$?
  - Young (2016): $R \gg t_j^{-1/3}$ under GLH.
  - H. (2018): $R \gg t_j^{-1}$ for a.e. $w \in \Gamma\backslash\mathbb{H}$ under GLH.
- Restricted QUE: does QUE hold when restricted to a submanifold?
  - Toth–Zelditch (2013), Dyatlov–Zworski (2013): RQE holds for negatively curved M and generic hypersurface Σ for almost all eigenfunctions.
  - Young (2016, 2018): RQUE holds for Eisenstein series restricted to vertical geodesics.
  - Hu (2020): a version of RQUE holds in the level (depth) aspect for Hecke–Maaß cusp forms.

Theorem (H. (2024+))

Fix a closed geodesic $\mathcal{C} \subset \Gamma\backslash\mathbb{H}$. Assume GLH. For $\phi_j \in \mathcal{B}_0$,

$$\lim_{j\to\infty} \int_{\mathcal{C}} |\phi_j(z)|^2 \psi(z)\, ds = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\mathcal{C}} \psi(z)\, ds$$

for all $\psi \in C(\mathcal{C})$.

Proof is effective and gives bounds for the discrepancy:

$$\sup_{I\subseteq\mathcal{C}} \left| \int_I |\phi_j(z)|^2\, ds - \frac{\ell(I)}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \right| \ll t_j^{-\delta}.$$

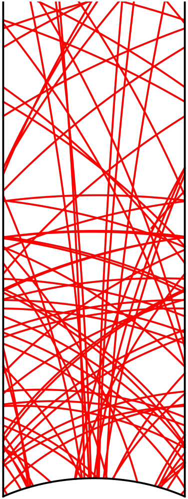
Maybe $\delta = 1/4$ is plausible.

Optimal bound is $\delta = 1/2$; currently seems out of reach.

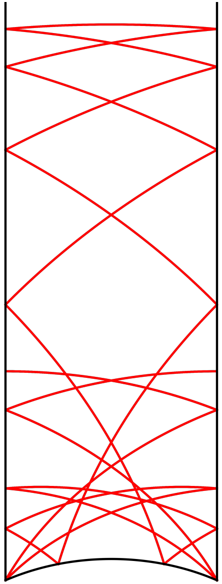
Remarks on the Method

- Proof does not use ergodic theory;
  instead uses period integrals of automorphic forms.
- Need to assume Laplacian eigenfunctions are Hecke eigenfunctions.
- Requirement of GLH could be weakened;
  - need strong bounds for certain fractional moments of L-functions that imply hybrid subconvexity.
- Method works for Hecke–Maaß cusp forms on other congruence subgroups, including compact quotients.
- Method works for vertical geodesics from a *rational* point $x \in \mathbb{Q}$ to $i\infty$
  - conditional partial resolution of a conjecture of Young (2018).
- Method might (?) work for other arithmetic submanifolds:
  - horocycles;
  - geodesic circles centred at Heegner points.

Geodesics on $\Gamma\backslash\mathbb{H}$



Closed Geodesics on $\Gamma\backslash\mathbb{H}$



Closed Geodesics on $\Gamma\backslash\mathbb{H}$

Key properties of closed geodesics:

- Bijective correspondence with narrow ideal classes of real quadratic number fields $\mathbb{Q}(\sqrt{D})$ (*arithmetic* submanifold)
- Length is $2\log\epsilon$, where ϵ is the fundamental unit of $\mathbb{Q}(\sqrt{D})$.
- Infinitely many closed geodesics.
- Union of all closed geodesics is dense in $\Gamma\backslash\mathbb{H}$.
- Topologically equivalent to a circle.
- Period integrals of automorphic forms on closed geodesics are related to L-functions (Waldspurger's formula).

For the proof, we assume for ease of exposition that $h_D^+ = 1$.

Reduction of RQUE

Closed geodesic $\mathcal{C}$ is topologically a circle.

Weyl equidistribution criterion: suffices to show for each $m \in \mathbb{Z}$,

$$\lim_{j\to\infty} \int_{\mathcal{C}} |\phi_j(z)|^2 \psi_m(z)\, ds = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\mathcal{C}} \psi_m(z)\, ds$$
$$= \begin{cases} \dfrac{\ell(\mathcal{C})}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} & \text{if } m = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here $\psi_m(\theta) = e^{2\pi i m\theta}$.

Remark

For discrepancy bounds, additionally need explicit rate of decay for $\int_{\mathcal{C}} |\phi_j|^2 \psi_m\, ds$ in both t_j and m.

First Approach to RQUE

Idea 1 of Proof.

Insert spectral expansion of $|\phi_j|^2 \in L^2(\Gamma\backslash\mathbb{H})$:

$$\int_{\mathcal{C}} |\phi_j(z)|^2 \psi_m(z)\, ds = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\mathcal{C}} \psi_m(z)\, ds + \sum_{f \in \mathcal{B}_0} \langle |\phi_j|^2, f\rangle \int_{\mathcal{C}} f(z)\psi_m(z)\, ds.$$

First term is desired main term.
Need to show second term is small.

Watson–Ichino and GLH imply $\langle |\phi_j|^2, f\rangle$ has mild polynomial decay if $t_f \leq 2t_j$ and exponential decay if $t_f > 2t_j$.

For $\int_{\mathcal{C}} f\psi_m\, ds$, apply Waldspurger's formula to relate to L-functions.

Waldspurger's Formula

Proposition (Waldspurger (1985))

We have that

$$\left|\int_{\mathcal{C}} f(z)\psi_m(z)\,ds\right|^2 \approx \frac{L\left(\frac{1}{2}, f \otimes \Theta_{\psi_m}\right)}{L(1, \operatorname{ad} f)} \times \begin{cases} \text{mild polynomial decay} & \text{if } |m| \leq t_f, \\ \text{exponential decay} & \text{if } |m| > t_f. \end{cases}$$

Remark

Gan–Gross–Prasad for $(\mathrm{SO}(3), \mathrm{SO}(2))$: $\mathfrak{so}_3 \cong \mathfrak{sl}_2$.

Θ_{ψ_m} is a dihedral Maaß form of spectral parameter $\frac{2\pi|m|}{\ell(\mathcal{C})}$: automorphic induction of the Hecke character ψ_m.

Assuming GLH, ratio of L-functions on RHS is essentially bounded.

First Approach to RQUE

Idea 1 of Proof (cont'd).

Want to show that as $j \to \infty$,

$$\sum_{f \in \mathcal{B}_0} \langle |\phi_j|^2, f \rangle \int_{\mathcal{C}} f(z) \psi_m(z)\, ds = o(1).$$

- Take absolute values;
- Apply Watson–Ichino and Waldspurger;
- Truncate sum to $t_f \leq 2t_j$, bound each term assuming GLH, and sum via Weyl law.

Eventually get the upper bound $O(t_j^{1/2})$. □

Much too big!

- Lossy since taking absolute values wastes oscillations of sign of $\langle |\phi_j|^2, f \rangle$ and $\int_{\mathcal{C}} f \psi_m \, ds$.
- After taking absolute values, spectral sum is too long; need to be able to truncate to $t_f = o(t_j^{1/2})$.

Second Approach to RQUE

Idea 2 of Proof.

- Use Parseval for $L^2(\mathcal{C})$:

$$\int_{\mathcal{C}} |\phi_j(z)|^2 \psi_m(z)\, ds = \sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} \phi_j(z)\psi_{m+n}(z)\, ds \overline{\int_{\mathcal{C}} \phi_j(z)\psi_n(z)\, ds}.$$

  - Uses the fact that $\psi_m(z)\psi_n(z) = \psi_{m+n}(z)$ since $e^{2\pi i m\theta} e^{2\pi i n\theta} = e^{2\pi i(m+n)\theta}$.
- Take absolute values and apply Waldspurger.
- Truncate sum to $|n| \leq t_j$, bound each term assuming GLH, and sum.

Eventually get the upper bound $O(1)$. □

Better, but still not quite good enough.

Method cannot even extract a main term when $m = 0$!

Second Approach to RQUE

Second approach can be used to prove good bounds for L^2-restriction problem.

Theorem (Ali (2022))

Unconditionally,

$$\int_{\mathcal{C}} |\phi_j(z)|^2 \, ds \ll_\varepsilon t_j^{2\vartheta+\varepsilon},$$

where $\vartheta = \frac{7}{64}$ is the best known exponent towards the Ramanujan conjecture.

Remark

For arbitrary (nonarithmetic) compact manifolds, instead get $\ll t_j^{1/2}$ (Burq–Gérard–Tzvetkov).

Method below gives correct asymptotic $\int_{\mathcal{C}} |\phi_j|^2 \, ds \sim \frac{\ell(\mathcal{C})}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}$ under the assumption of GLH.

Proof of RQUE

Second approach barely fails:
$O(1)$ instead of $\frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})}\int_{\mathcal{C}}\psi_m\,ds+o(1)$.

First key idea: determine how to extract a main term from

$$\int_{\mathcal{C}}|\phi_j(z)|^2\psi_m(z)\,ds=\sum_{n=-\infty}^{\infty}\int_{\mathcal{C}}\phi_j(z)\psi_{m+n}(z)\,ds\overline{\int_{\mathcal{C}}\phi_j(z)\psi_n(z)\,ds}.$$

Step 1 of proof.

Break up above sum into ranges.

By Waldspurger and GLH, **bulk range** is when $|n|$ is close to t_j (i.e. $t_j^{1-\delta}\leq|n|\leq t_j-t_j^{1-2\delta}$, $\delta>0$ small).

Remaining terms contribute $o(1)$.

Proof of RQUE

Second key idea: modify the Hecke–Maaß cusp form ϕ_j in this period integral of automorphic forms.

Step 2 of proof.

Construct an automorphic form $\widetilde{\phi}_j : \Gamma\backslash \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ that closely approximates ϕ_j along $\mathcal{C}$ (but not necessarily elsewhere in $\Gamma\backslash\mathbb{H}$);

- $\widetilde{\phi}_j$ constructed such that for n in the bulk range,

$$\int_{\mathcal{C}} \phi_j(z)\psi_n(z)\, ds \sim \int_{\mathcal{C}} \widetilde{\phi}_j(z)\psi_n(z)\, ds.$$

- For n *outside* the bulk range, $\int_{\mathcal{C}} \widetilde{\phi}_j\psi_n\, ds$ is exponentially small (whereas $\int_{\mathcal{C}} \phi_j\psi_n\, ds$ is only polynomially small).

Proof of RQUE

Step 3 of proof.

Use Parseval for $L^2(\mathcal{C})$ to write

$$\int_{\mathcal{C}} |\phi_j(z)|^2 \psi_m(z)\, ds = \sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} \phi_j(z)\psi_{m+n}(z)\, ds \overline{\int_{\mathcal{C}} \phi_j(z)\psi_n(z)\, ds},$$

$$\int_{\mathcal{C}} \phi_j(z)\overline{\widetilde{\phi}_j(z)}\psi_m(z)\, ds = \sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} \phi_j(z)\psi_{m+n}(z)\, ds \overline{\int_{\mathcal{C}} \widetilde{\phi}_j(z)\psi_n(z)\, ds}.$$

Expansions essentially equal for n in the bulk range.
Both expansions negligibly small for n outside the bulk range.

Upshot

$$\int_{\mathcal{C}} |\phi_j(z)|^2 \psi_m(z)\, ds = \int_{\mathcal{C}} \phi_j(z)\overline{\widetilde{\phi}_j(z)}\psi_m(z)\, ds + o(1).$$

Proof of RQUE

Step 4 of proof.

Return to first approach using new choice of automorphic form.

Insert spectral expansion of $\phi_j \overline{\widetilde{\phi}_j} \in L^2(\Gamma \backslash \mathrm{SL}_2(\mathbb{R}))$:

$$\begin{aligned}\int_{\mathcal{C}} \phi_j(z) \overline{\widetilde{\phi}_j(z)} \psi_m(z) \, ds &= \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} \int_{\Gamma \backslash \mathrm{SL}_2(\mathbb{R})} \phi_j(z) \overline{\widetilde{\phi}_j(z)} \, d\mu(z) \int_{\mathcal{C}} \psi_m(z) \, ds \\ &\quad + \sum_{f \in \mathcal{B}} \langle \phi_j \overline{\widetilde{\phi}_j}, f \rangle \int_{\mathcal{C}} f(z) \psi_m(z) \, ds.\end{aligned}$$

Via construction of $\widetilde{\phi}_j$, $\int \phi_j \overline{\widetilde{\phi}_j} \, d\mu \sim 1$; gives expected main term.

Remains to show that

$$\sum_{f \in \mathcal{B}} \langle \phi_j \widetilde{\phi}_j, f \rangle \int_{\mathcal{C}} f(z) \psi_m(z) \, ds = o(1).$$

Proof of RQUE

Step 5 of proof.

Again take absolute values and apply period formulæ to relate to L-functions:

- Watson–Ichino for $\langle \phi_j \widetilde{\phi}_j, f \rangle$;
- Waldspurger for $\int_{\mathcal{C}} f \psi_m \, ds$.

Still lossy; wastes oscillations of sign of $\langle |\phi_j|^2, f \rangle$ and $\int_{\mathcal{C}} f \psi_m \, ds$.

Key trick: replacing ϕ_j with $\widetilde{\phi}_j$ gives *same* L-functions but *different* archimedean weight. Delicate analysis shows that archimedean weight has much smaller support:

- essentially the same as previously for $t_f = o(t_j^{1/2})$;
- exponentially small for $t_f \gg t_j^{1/2}$.

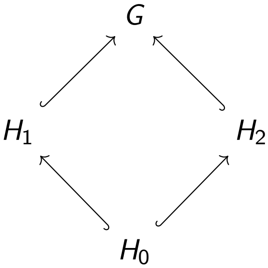
Eventually get the upper bound $o(1)$. □

Upshot

We win since spectral sum is sufficiently short.

Period Integral Framework

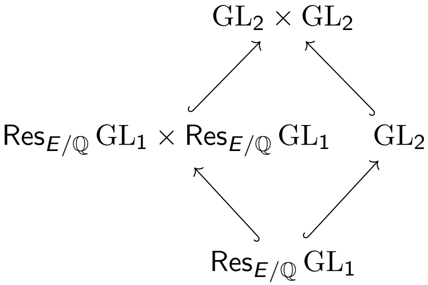
Underlying strong Gelfand formation:



$(G, H_1), (G, H_2), (H_1, H_0), (H_2, H_0)$ are each strong Gelfand pairs.

1. Take automorphic form Φ on G, restrict to H_0, and integrate against an automorphic form ϕ_0 on H_0;
2. On the one hand, expand on H_1 via Parseval, yielding $\sum_{\phi_1} \langle \Phi, \phi_1 \rangle \langle \phi_1, \phi_0 \rangle$;
3. On the other hand, expand on H_2 via Parseval, yielding $\sum_{\phi_2} \langle \Phi, \phi_2 \rangle \langle \phi_2, \phi_0 \rangle$.

Underlying strong Gelfand formation:



1. Take automorphic forms φ_1, φ_2 in an automorphic representation Π on $\mathrm{Z}(\mathbb{A}_\mathbb{Q})\mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$, restrict to $\mathbb{A}_\mathbb{Q}^\times E^\times\backslash\mathbb{A}_E^\times$, and integrate against a Hecke character Ω on $\mathbb{A}_\mathbb{Q}^\times E^\times\backslash\mathbb{A}_E^\times$:
$$\int_{\mathcal{C}} \phi_j(z)\overline{\widetilde{\phi}_j(z)}\psi_m(z)\, ds = \int_{\mathbb{A}_\mathbb{Q}^\times E^\times\backslash\mathbb{A}_E^\times} \varphi_1(x)\overline{\varphi_2(x)}\Omega(x)\, d^\times x.$$

Period Integral Framework

2. On the one hand, expand on $\mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_1 \times \mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_1$ via Parseval, yielding

$$\sum_{n=-\infty}^{\infty} \int_{\mathcal{C}} \phi_j(z)\psi_{m+n}(z)\, ds \overline{\int_{\mathcal{C}} \widetilde{\phi}_j(z)\psi_n(z)\, ds}$$

$$= \sum_{\Omega' \in \widehat{\mathbb{A}_{\mathbb{Q}}^{\times} E^{\times} \backslash \mathbb{A}_E^{\times}}} \int_{\mathbb{A}_{\mathbb{Q}}^{\times} E^{\times} \backslash \mathbb{A}_E^{\times}} \varphi_1(x)\Omega\Omega'(x)\, d^{\times}x \overline{\int_{\mathbb{A}_{\mathbb{Q}}^{\times} E^{\times} \backslash \mathbb{A}_E^{\times}} \varphi_2(x)\Omega'(x)\, d^{\times}x}$$

$$= \sum_{\Omega' \in \widehat{\mathbb{A}_{\mathbb{Q}}^{\times} E^{\times} \backslash \mathbb{A}_E^{\times}}} \frac{L\left(\frac{1}{2}, \Pi \otimes \Omega\Omega'\right)^{1/2} L\left(\frac{1}{2}, \widetilde{\Pi} \otimes \Omega'^{-1}\right)^{1/2}}{L(1, \mathrm{ad}\,\Pi)} \alpha(\Omega').$$

Weight function $\alpha(\Omega')$ explicitly determined in terms of choice of data of φ_1, φ_2 (i.e. local Whittaker functions).

- Proof uses uniqueness of linear functionals:

$$\dim \mathrm{Hom}_{\mathbb{A}_E^{\times}}(\Pi, \Omega^{-1}\Omega'^{-1}) = 1, \qquad \dim \mathrm{Hom}_{\mathbb{A}_E^{\times}}(\widetilde{\Pi}, \Omega') = 1.$$

Period Integral Framework

3. On the other hand, expand on GL_2 via Parseval, yielding

$$\sum_{f \in \mathcal{B}} \langle \phi_j \overline{\widetilde{\phi}_j}, f \rangle \int_{\mathcal{C}} f(z) \psi_m(z) \, ds$$

$$= \sum_{\pi} \sum_{\phi \in \mathcal{B}(\pi)} \int_{\mathrm{Z}(\mathbb{A}_{\mathbb{Q}})\mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})} \varphi_1(h) \overline{\varphi_2(h) \phi(h)} \, dh$$

$$\times \int_{\mathbb{A}_{\mathbb{Q}}^{\times} E^{\times} \backslash \mathbb{A}_E^{\times}} \phi(x) \Omega(x) \, d^{\times} x$$

$$= \sum_{\pi} \frac{L\left(\frac{1}{2}, \Pi \otimes \widetilde{\Pi} \otimes \widetilde{\pi}\right)^{1/2} L\left(\frac{1}{2}, \pi \otimes \Omega\right)^{1/2}}{L(1, \mathrm{ad}\, \Pi)^2 L(1, \mathrm{ad}\, \pi)} \beta(\pi).$$

Weight function $\beta(\pi)$ explicitly determined in terms of of choice of data of φ_1, φ_2 (i.e. local Whittaker functions).

- Proof uses uniqueness of trilinear and linear functionals:

$$\dim \mathrm{Hom}_{\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})}(\Pi \otimes \widetilde{\Pi}, \pi) = 1, \quad \dim \mathrm{Hom}_{\mathbb{A}_E^{\times}}(\pi, \Omega^{-1}) = 1.$$

Period Integral Framework

4. To prove RQUE:
   (i) Take φ_1, φ_2 to be the adèlic lifts of $\phi_j, \widetilde{\phi_j}$, Ω to be the adèlic lift of ψ_m;
   (ii) Using expansion on $\mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_1 \times \mathrm{Res}_{E/\mathbb{Q}} \mathrm{GL}_1$, show that choice of φ_2 ensures that $\alpha(\Omega')$ localises to bulk range, so that this expansion closely approximates that of $\int_{\mathcal{C}} |\phi_j|^2 \psi_m \, ds$ (easy);
   (iii) Using expansion on GL_2, show that $\beta(\pi)$ is small once archimedean data of π is $\gg t_j^{1/2}$, so that this expansion gives desired main term plus $o(1)$ error term (hard).

Thank you!