Non-escape of mass for automorphic forms in hyperbolic 4-manifolds

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(joint work with Zvi Shem-Tov)

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Shanks Conference Series

Quantum unique ergodicity

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QUE conjecture (Rudnick and Sarnak, 1994)

The probability measures $\mu_i = |\phi_i|^2 d\,\mathrm{vol}_X$ converge in the weak-$*$ topology to $d\,\mathrm{vol}_X$.

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Theorem (Lindenstrauss, 2006)

Every weak-* limiting measure of the sequence $\mu_i = |\phi_i|^2 d\,\mathrm{vol}_X$ is of the form $c \cdot d\,\mathrm{vol}_X$ for some $c \in [0, 1]$.

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Theorem (Soundararajan, 2010)

We have $c = 1$, so AQUE holds for $\Gamma \backslash \mathbb{H}_2$.

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- Let $\mathbf{H}$ denote the Hamilton quaternions. Then

$$SV_2(\mathbb{Z}) \simeq \left\{g \in M_2(\mathbf{H}(\mathbb{Z})) : gJg^{*t} = J\right\}, \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Here $(a_0 + a_1 i + a_2 j + a_3 k)^* = a_0 + a_1 i + a_2 j - a_3 k$.

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- On $X_4 = SV_2(\mathbb{Z})\backslash\mathbb{H}_4$: no Watson-Ichino, violations to Ramanujan.

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Theorem (F. and Shem-Tov, 2024)

Let $X_4 = SV_2(\mathbb{Z})\backslash\mathbb{H}_4$ and $\phi_i \in L^2(X)$ be a sequence of Hecke-Maass forms with unit norm. Suppose the probability measures $\mu_i = |\phi_i|^2 d\,\mathrm{vol}_{X_4}$ converge in the weak-$*$ topology. Then the limit is a probability measure.

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- AQUE for X_4 essentially reduces to ruling out measure concentration on orbits of $\mathrm{SL}_2(\mathbb{C})$ inside $SV_2(\mathbb{R})$.

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$$\phi(x+iy) = \sqrt{y} \sum_{0 \neq n \in \mathbb{Z}} a(n) K_{ir}(2\pi|n|y) e(nx).$$

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- Let $\lambda(m)$ denote the eigenvalue of ϕ for T_m. For each prime p,

$$\lambda(m)a(n) = \sum_{d|(m,n)} a\left(\frac{mn}{d^2}\right),$$
$$\lambda(p)a(n) = a(np) + a(n/p),$$
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Theorem (Soundararajan, 2010)

For any $1 \leq y \leq x$,

$$\sum_{n\leq \frac{x}{y}} |a(n)|^2 \leq 10^8 \left(\frac{1+\log y}{\sqrt{y}}\right) \sum_{n\leq x} |a(n)|^2.$$

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- Normalize so that $\|\phi\|_2 = 1$. Fourier-expanding

$$I_T(\phi) := \int_T^\infty \int_0^1 |\phi(x+iy)|^2 \frac{dx\,dy}{y^2}$$

with $T \geq 1$,

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Therefore

$$I_T(\phi) \leq 10^8 \left(\frac{1+\log T}{\sqrt{T}}\right) I_1(\phi) \leq 10^8 \left(\frac{1+\log T}{\sqrt{T}}\right).$$

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Theorem (F. and Shem-Tov, 2024)

There exist absolute constants C and R such that for any $1 \leq y \leq x$,

$$\sum_{|\beta|^2 \leq \frac{x}{y}} |A(\beta)|^2 \leq C \frac{(1 + \log y)^R}{y^{1/8}} \sum_{|\beta|^2 \leq x} |A(\beta)|^2.$$

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- Break the sum $s(x/y)$ into two parts $s^{<K}(x/y)$ and $s^{\geq K}(x/y)$ depending on whether $n \in \mathcal{M}(K)$ or not.

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- This succeeds if $K \ll Ly^{1/4}$.

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- For the elements $a(n)$ in $s^{\geq K}(x/y)$, n is a multiple of some product $d = p_1 \cdots p_K$ of K primes $p_i \asymp \sqrt{y}$. There are $\approx \binom{\sqrt{y}}{K}$ options for $d \approx (\sqrt{y})^K$, and each one contributes

$$\sum_{m \leq \frac{x}{yd}} |a(dm)|^2 \approx |\lambda(d)|^2 \cdot s\left(\frac{x}{yd}\right) \approx L^K \cdot s\left(\frac{x}{y(\sqrt{y})^K}\right).$$

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- This succeeds if $K \geq 20Ly^{1/4}$, which is consistent with the previous restriction $K \ll Ly^{1/4}$.

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$$\sum_{m \leq \frac{x}{yd}} |a(dm)|^2 \approx |\lambda(d)|^2 \cdot s\left(\frac{x}{yd}\right) \approx L^K \cdot s\left(\frac{x}{y(\sqrt{y})^K}\right).$$

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$$s^{\geq K}\left(\frac{x}{y}\right) \ll \binom{\sqrt{y}}{K} L^K \cdot s\left(\frac{x}{y(\sqrt{y})^K}\right) \ll \left(\frac{10\sqrt{y}L}{Ky^{1/4}}\right)^K \frac{s(x)}{\sqrt{y}}.$$

- This succeeds if $K \geq 20Ly^{1/4}$, which is consistent with the previous restriction $K \ll Ly^{1/4}$.
- The case when one must use the $\lambda(p^2)$ is similar.

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- Break the sum

$$S(x/y) := \sum_{|\beta|^2 \leq \frac{x}{y}} |A(\beta)|^2$$

into two parts $S^{<K}(x/y)$ and $S^{\geq K}(x/y)$ depending on whether $n \in \mathcal{M}(K)$ or not.

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$$\frac{1}{p} \sum_{|\beta|^2 \leq z} \left|\sum_{|\alpha|^2=p} A\left(\frac{\alpha\beta\overline{\alpha}}{p}\right)\right|^2 \ll \sum_{|\delta|^2 \leq z} \left(\frac{m_1(\delta)}{p} + m_2(\delta)\right) |A(\delta)|^2 \ll S(z).$$

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Cauchy-Schwarz is a bad move here due to the $A(\alpha\beta\overline{\alpha})$. Instead, observe

$$\lambda_2(p)A(\beta) \approx \lambda_1(p)A(p\beta) - A(p^2\beta) - A(\beta),$$

$$\lambda_2(p)A(\beta) \approx \lambda_2(p)A(p^2\beta) + \lambda_1(p)A(p^3\beta) - A(p^4\beta) - A(p^2\beta).$$

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- One can check that this implies

$$g(y) \leq C(1 + \log y)^R$$

  for some absolute constants C, R (in general they would depend only on the functions b_n).

Thank you!