Moment of cubic L-functions over $\mathbb{F}_q(t)$ at $s = \frac{1}{3}$
Joint work with P. Meisner (Gothenburg)

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Shanks conference on L-functions and automorphic forms
Vanderbilt University, May 2024

Zeta-fonctions and L-functions over $\mathbb{F}_q$

Let $C/\mathbb{F}_q$ be a (smooth, projective) curve, i.e. $f(y,t) = 0$ where $f \in \mathbb{F}_q[y,t]$. Let $\mathcal{Z}(u, C)$ be the zeta-function

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Weil (1949) showed that for any $C/\mathbb{F}_q$ of genus g

$$\mathcal{Z}(u,C) = \frac{P_{2g}(u,C)}{(1-qu)(1-u)} = P_{2g}(u,C)\mathcal{Z}(u,\mathbb{P}^1)$$

where $P_{2g}(u,C)$ is a polynomial of degree $2g$.

Zeta-fonctions and L-functions over $\mathbb{F}_q$

Furthermore,

$$P_{2g}(u, C) = \prod_{j=1}^{2g} \left(1 - q^{\frac{1}{2}} e^{i\theta_{j,C}} u\right)$$

and the roots satisfy $|u| = q^{-\frac{1}{2}}$.

Using $u = q^{-s}$, the roots satisfy $s = \frac{1}{2} + i\theta_{j,C}$ (Riemann Hypothesis).

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Using $u=q^{-s}$, the roots satisfy $s=\frac{1}{2}+i\theta_{j,C}$ (Riemann Hypothesis).

We also write

$$P_{2g}(u,C)=\prod_{j=1}^{2g}\left(1-q^{\frac{1}{2}}e^{i\theta_{j,C}}u\right)=\det\left(1-q^{\frac{1}{2}}\,\Theta_C\,u\right)$$

where Θ_C is the $2g\times 2g$ complex unitary diagonal matrix with diagonal entries $e^{i\theta_{j,C}}$, $j=1,\ldots,2g$.

Families of curves over $\mathbb{F}_q$: Hyperelliptic curves

$$C_f : y^2 = f(t), \quad f(t) \in \mathcal{M}_d \subset \mathbb{F}_q[t], \text{ SF}, \quad 2g = \begin{cases} \deg f - 1 & d \text{ odd} \\ \deg f - 2 & d \text{ even} \end{cases}$$

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where χ_f is the quadratic Dirichlet character over $\mathbb{F}_q[t]$ given by

$$\chi_P : (\mathbb{F}_q[t]/(P))^* \to \{\pm 1\}$$

$$a \mapsto \left(\frac{a}{P}\right) = \begin{cases} 1 & a \equiv \square \bmod P \\ -1 & a \not\equiv \square \bmod P \end{cases}$$

and $\quad \delta_f = \begin{cases} 1 & \chi_f \text{ odd} \\ 0 & \chi_f \text{ even} \end{cases} = \begin{cases} 1 & \deg f \text{ odd} \\ 0 & \deg f \text{ even} \end{cases}$

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$$\mathcal{Z}(u) = \sum_{n \in \mathcal{M}} u^{\deg n} = \sum_{d=0}^{\infty} u^d \sum_{n \in \mathcal{M}_d} 1 = \sum_{d=0}^{\infty} (qu)^d = \frac{1}{1 - qu}$$

$$\mathcal{Z}(u, \mathbb{P}^1) = \frac{1}{(1 - qu)(1 - u)}$$

Families of curves over $\mathbb{F}_q$: ℓ-cyclic cover

For $q \equiv 1 \bmod 2\ell$, ℓ odd prime,

$$C_f : y^\ell = f(t), \quad f(t) \in \mathcal{M}_d \text{ SF}, \quad \frac{2g}{\ell - 1} = \begin{cases} \deg f - 1 & \ell \nmid d \\ \deg f - 2 & \ell \mid d \end{cases}$$

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where χ_P is the ℓ-th power residue symbol defined by

$$\chi_P = \left(\frac{\cdot}{P}\right)_\ell : (\mathbb{F}_q[t]/(P))^* \to \mu_\ell \subset \mathbb{C}^*$$

$$a \mapsto a^{\frac{|P|-1}{\ell}} \bmod P$$

Deligne Equidistribution Theorem and RMT

Theorem (Deligne's ET, 1974)

Let $\mathcal{M}_g(\mathbb{F}_q)$ be the set of isomorphism classes of curves of genus g over $\mathbb{F}_q$.
Let f be a continuous class function on $USP(2g)$.

$$\lim_{q\to\infty}\left(\frac{1}{\#\mathcal{M}_g(\mathbb{F}_q)}\sum_{C\in\mathcal{M}_g(\mathbb{F}_q)} f(\Theta_C)\right) = \int_{USp(2g)} f(A)\,dA$$

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Theorem (Katz-Sarnak, 1999)

Let $f(A) = D(\mu_k(A), \mu_k(GUE))$, where μ_k is the kth consecutive zeroes spacing measure on the $\theta_{j,C}$, $j = 1, \ldots, 2g$.

$$\lim_{g\to\infty}\left(\lim_{q\to\infty}\frac{1}{\#\mathcal{M}_g(\mathbb{F}_q)}\sum_{C\in\mathcal{M}_g(\mathbb{F}_q)} D(\mu_k(\theta_C), \mu_k(GUE))\right) = 0.$$

The mixed trace functions

Let

$$P_\lambda(U) = \prod_{j=1}^{\infty} \mathrm{Tr}(U^j)^{\lambda_j}, \quad \lambda = 1^{\lambda_1} 2^{\lambda_2} \ldots .$$

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Theorem (Meisner, 2021)

For λ such that $|\lambda| = \sum j\lambda_j < g$,

$$\lim_{q\to\infty} \frac{1}{|\mathcal{F}_3(g)|} \sum_{\chi\in\mathcal{F}_3(g)} P_\lambda(\Theta_\chi) = \int_{U(g)} P_\lambda(U)\, dU$$

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$$\lim_{q\to\infty} \frac{1}{|\mathcal{F}_3(g)|} \sum_{\chi\in\mathcal{F}_3(g)} q^{\frac{|\lambda|}{6}} P_\lambda(\Theta_\chi) = \int_{U(g)} P_\lambda(U)\, \overline{\det(1-\wedge^3 U)}\, dU$$

$$\det(1-\wedge^3 U) = \prod_{1\le i_1<i_2<i_3\le g} (1 - x_{i_1}x_{i_2}x_{i_3}),\ x_i \textit{ are the } g \textit{ EVs of } U.$$

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We remark that $q^{\frac{|\lambda|}{6}} P_\lambda(\Theta_C) = P_\lambda(q^{\frac{1}{6}}\Theta_C)$.

The mixed trace functions

Conjecture (Meisner, 2021)

Let f be a continuous class function on $U(g)$. Then

$$\lim_{q\to\infty} \frac{1}{|\mathcal{F}_3(g)|} \sum_{\chi\in\mathcal{F}_3(g)} f(q^{\frac{1}{6}}\Theta_\chi) \sim \int_{U(g)} f(U)\, \overline{\det(1-\wedge^3 U)}\, dU.$$

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Taking $f(U) = \det(1-U)$, we have

$$f(q^{\frac{1}{6}}\Theta_\chi) = \det(1-q^{\frac{1}{6}}\Theta_\chi) = \prod_{j=1}^{g}\left(1-q^{\frac{1}{6}}e^{i\theta_{j,\chi}}\right)$$

$$= \prod_{j=1}^{g}\left(1-q^{\frac{1}{2}}e^{i\theta_{j,\chi}}q^{-\frac{1}{3}}\right) = \mathcal{L}(q^{-\frac{1}{3}},\chi) = L(\tfrac{1}{3},\chi).$$

Theorem (D-Meisner, 2023)

Let q be an odd prime power, $q \equiv 1 \bmod 3$, and

$$\mathcal{H}(g) = \left\{ \chi_D = \left(\frac{\cdot}{D}\right)_3 : D \text{ SF and the genus of } \chi_D \text{ is } g. \right\}$$

Then, as $g \to \infty$,

$$\frac{1}{|\mathcal{H}(3g)|} \sum_{\chi \in \mathcal{H}(3g)} L(\tfrac{1}{3}, \chi) = C_q g + O(1),$$

and taking the q-limit, we get

$$g + O(1) = \int_{U(3g)} \deg(1-U)\overline{\det(1 - \wedge^3 U)}\, dU.$$

We compute the cubic moment at $s = \frac{1}{3}$. At $s = \frac{1}{2}$, Friedberg–Hoffstein–Lieman (2003), Luo (2004), Baier-Young (2010), Rosen (1995) and D-Florea-Lalin (2021, 2022), Guloglu (2023).

Cubic moment for $\frac{1}{3} \leq s < 1$

For any $\frac{1}{3} < s < 1$, we have

$$\frac{1}{|\mathcal{H}(3g)|} \sum_{\chi \in \mathcal{H}(3g)} L(s, \chi) = M(s) + O\left(q^{(\frac{1}{3} - s)(3g+1)} + \dots \right)$$

where

$$M(s) = \zeta_q(3s) \prod_P \left(1 - \frac{1}{|P|^{3s}(|P| + 1)} \right)$$

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$$\sim C_q\, g, \quad q \text{ fixed}, \ g \to \infty$$

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Functional equation for $\mathcal{L}(u, \chi)$

If χ is odd, then

$$\mathcal{L}(u, \chi) = \omega(\chi)(\sqrt{q}u)^g \mathcal{L}\left(\frac{1}{qu}, \overline{\chi}\right)$$

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Proof: $P_{2g}(u, C) = \mathcal{L}(u,\chi)\mathcal{L}(u,\overline{\chi})$, and

$$\mathcal{L}(u,\chi) = \prod_{j=1}^{g}\left(1 - \sqrt{q}e^{i\theta_j}u\right) = (\sqrt{q}u)^g \prod_{j=1}^{g}\left((\sqrt{q}u)^{-1} - e^{i\theta_j}\right)$$

$$= (\sqrt{q}u)^g(-1)^g \prod_{j=1}^{g} e^{i\theta_j} \prod_{j=1}^{g}\left(1 - \frac{e^{-i\theta_j}}{u\sqrt{q}}\right)$$

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Approximate functional equation

Let χ be a primitive character of genus g, and $A = cg, 0 < c < 1$.
If χ is odd, then

$$\mathcal{L}(q^{-s}, \chi) = \sum_{f \in \mathcal{M}_{\leq A}} \frac{\chi(f)}{|f|^s} + \omega(\chi)(q^{\frac{1}{2}-s})^g \sum_{f \in \mathcal{M}_{\leq g-A-1}} \frac{\overline{\chi}(f)}{|f|^{1-s}}$$

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If χ is even, then

$$\mathcal{L}(q^{-s},\chi) = \frac{1}{1-q^{1-s}} \left(\sum_{f \in \mathcal{M}_{\leq A+1}} \frac{\chi(f)}{|f|^s} - q^{1-s} \sum_{f \in \mathcal{M}_{\leq A}} \frac{\chi(f)}{|f|^s} \right)$$
$$+ \omega(\chi) \frac{(q^{\frac{1}{2}-s})^g}{1-q^s} \frac{\zeta_q(2-s)}{\zeta_q(s+1)} \left(\sum_{f \in \mathcal{M}_{g-A}} \frac{\overline{\chi}(f)}{|f|^{1-s}} + \sum_{f \in \mathcal{M}_{g-A-1}} \frac{\overline{\chi}(f)}{|f|^{1-s}} \right)$$

Principal Sum and Dual Sum

$$\begin{aligned}\mathcal{H}(3g) &= \left\{\chi_D = \left(\frac{\cdot}{D}\right)_3 : \text{genus}\,(\chi_D) = 3g, D \text{ SF}\right\}\\ &= \left\{\chi_D = \left(\frac{\cdot}{D}\right)_3 : \deg D = 3g+1, D \text{ SF}\right\}\\ &= \left\{\chi_D = \left(\frac{\cdot}{D}\right)_3 : D \in \mathcal{SF}_{3g+1}\right\}\end{aligned}$$

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$$\mathcal{P}_s(3g, 3A) = \sum_{D \in \mathcal{SF}_{3g+1}} \sum_{f \in \mathcal{M}_{\leq 3A}} \frac{\chi_D(f)}{|f|^s}$$

$$\mathcal{D}_s(3g, 3A) = q^{(\frac{1}{2}-s)3g} \sum_{D \in \mathcal{SF}_{3g+1}} \omega(\chi_F) \sum_{f \in \mathcal{M}_{\leq 3g-3A-1}} \frac{\overline{\chi_D}(f)}{|f|^{1-s}}$$

Principal Sum and Dual Sum

For $\frac{1}{3} < s < 1$,

$$\frac{\mathcal{P}_s(3g,3A)}{|\mathcal{H}(3g)|} \sim M(s) + C_q \frac{q^{(1-3s)A}}{1-q^{3s-1}}$$

$$\frac{P_{\frac{1}{3}}(3g,3A)}{|\mathcal{H}(3g)|} \sim C_q \left(A + \sum_P \frac{\deg(P)}{|P|^2 + |P| - 1} \right)$$

$$\frac{\mathcal{D}_s(3g,3A)}{|\mathcal{H}(3g)|} \sim -C_q \frac{q^{(1-3s)A}}{1-q^{3s-1}}$$

$$\frac{\mathcal{D}_{\frac{1}{3}}(3g,3A)}{|\mathcal{H}(3g)|} \sim C_q \left(g - A - \frac{1}{3}\frac{q+2}{q-1} + \sum_P \frac{\deg(P)}{|P|^3 + 2|P|^2 - 1} \right)$$

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$$= \sum_{f \in \mathcal{M}_{\leq 3A}} \frac{1}{|f|^s} \sum_{D \in \mathcal{SF}_{3g+1}} \chi_f(D)$$

since (recall that $q \equiv 1 \bmod 3$)

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We compute

$$\sum_{D\in\mathcal{M}_d} \chi_f(D)$$

Principal Sum

$$\mathcal{P}_s(3g,3A) = \sum_{D\in\mathcal{SF}_{3g+1}} \sum_{f\in\mathcal{M}_{\leq 3A}} \frac{\chi_D(f)}{|f|^s} = \sum_{f\in\mathcal{M}_{\leq 3A}} \frac{1}{|f|^s} \sum_{D\in\mathcal{SF}_{3g+1}} \chi_D(f)$$

$$= \sum_{f\in\mathcal{M}_{\leq 3A}} \frac{1}{|f|^s} \sum_{D\in\mathcal{SF}_{3g+1}} \chi_f(D)$$

since (recall that $q \equiv 1 \bmod 3$)

$$\chi_D(f) = \left(\frac{f}{D}\right)_3 = \left(\frac{D}{f}\right)_3 = \chi_f(D).$$

We compute

$$\sum_{D\in\mathcal{M}_d} \chi_f(D) = \frac{1}{2\pi i} \int_{|u|=q^{-2}} \frac{\mathcal{L}(u,\chi_f)}{u^{d+1}} du$$

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Theorem (Lindelöf Hypothesis)

For any Dirichlet character $\chi/\mathbb{F}_q(t)$ of modulus h, and any $\varepsilon > 0$,

$$\mathcal{L}(q^{-\frac{1}{2}}e^{i\theta},\chi) \ll_\varepsilon |h|^\varepsilon$$

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"Replacing in the principal sum", we get

$$\text{“}\mathcal{P}_s(d,3A)\text{”} = q^{d+1} \sum_{m\in\mathcal{M}_{\leq A}} \frac{1}{|m|^{3s}} \operatorname*{Res}_{u=q^{-1}} \mathcal{L}(u,\chi_{m^3})$$

$$+ O\left(\sum_{f\in\mathcal{M}_{\leq 3A}} \frac{1}{|f|^s} |f|^\varepsilon q^{\frac{d}{2}}\right)$$

Principal Sum for $s \neq \frac{1}{3}$

Applying Perron for the sum over $m \in \mathcal{M}_{\leq A}$, we get

$$\frac{1}{2\pi i} \int_{|u|=q^{-2}} \frac{\mathcal{G}_s(v)}{v^{A+1}(1-v)} \, dv,$$

where

$$\mathcal{G}_s(v) = \mathcal{Z}\left(\frac{v}{q^{3s}}\right) \prod_P \left(1 - \frac{v^{\deg P}}{|P|^{3s}(|P|+1)}\right)$$

is meromorphic for $|v| \leq q^{3s-\varepsilon}$ with a simple pole at $v = q^{3s-1}$.

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This is the secondary term for $s \neq \frac{1}{3}$.
The main term comes from the pole at $v = 1$.

Principal Sum for $s = \frac{1}{3}$

We compute the residue of the double pole at $v = 1$

$$\operatorname*{Res}_{v=1}\left(\frac{\mathcal{G}_{\frac{1}{3}}(v)}{v^{A+1}(1-v)}\right) = \lim_{v\to 1} \frac{d}{dv}\frac{\mathcal{K}(v)}{v^{A+1}} = -\mathcal{K}(1)(A+1) + \mathcal{K}'(1).$$

where

$$\mathcal{K}(v) = \prod_P \left(1 - \frac{v^{\deg P}}{|P|(|P|+1)}\right) \Longrightarrow \mathcal{K}(1) = C_q$$

$$\frac{\mathcal{K}'(v)}{\mathcal{K}(v)} = \frac{d}{dv}\log \mathcal{K}(v) = -\sum_P \frac{\deg P \; v^{\deg P - 1}}{|P|(|P|+1)}.$$

This gives

$$\frac{\mathcal{P}_{\frac{1}{3}}(3g, 3A)}{|\mathcal{H}(3g)|} \sim C_q\left(A + \sum_P \frac{\deg P}{|P|^2 + |P| - 1}\right)$$

Dual Sum

$$\mathcal{D}_s(3g, 3A) = q^{(\frac{1}{2}-s)3g} \sum_{D \in \mathcal{SF}_{3g+1}} \omega(\chi_D) \sum_{f \in \mathcal{M}_{\leq 3g-3A-1}} \frac{\overline{\chi_D}(f)}{|f|^{1-s}}$$

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where for $D \in \mathcal{SF}_{3g+1}$

$$\omega(\chi_D) = \frac{\overline{\tau(\chi_D|_{\mathbb{F}_q})}}{\sqrt{q}} \frac{g(D)}{q^{\frac{\deg D}{2}}} = \overline{\widetilde{\tau}(\chi_3)}\, \widetilde{g}(D),$$

and for any $F \in \mathcal{M}$, we define

$$g(F) = g(1, F) = \sum_{a \bmod F} \left(\frac{a}{F}\right)_3 e_q\left(\frac{a}{F}\right) \in \mathbb{C}^*$$

$$g(V, F) = \sum_{a \bmod F} \left(\frac{a}{F}\right)_3 e_q\left(\frac{aV}{F}\right) \in \mathbb{C}^*$$

Cubic Gauss Sums

- $g(F) \neq 0 \iff F$ SF, and then $|g(F)| = q^{\frac{\deg F}{2}} = |F|^{1/2}$.
- If $(F_1, F_2) = 1$, then $g(V, F_1F_2) = \left(\frac{F_1}{F_2}\right)_3^2 g(V, F_1)g(V, F_2)$.
- If $(a, F) = 1$, then $g(aV, F) = \overline{\left(\frac{a}{F}\right)_3} g(V, F)$.

$$\mathcal{D}_s(3g, 3A) = \frac{\overline{\widetilde{\tau}(\chi_3)}}{q^{3gs+\frac{1}{2}}} \sum_{f \in \mathcal{M}_{3g-3A-1}} \frac{1}{|f|^{1-s}} \sum_{D \in \mathcal{SF}_{3g+1}} \overline{\left(\frac{f}{D}\right)_3} g(1, D)$$

$$= \frac{\overline{\widetilde{\tau}(\chi_3)}}{q^{3gs+\frac{1}{2}}} \sum_{f \in \mathcal{M}_{3g-3A-1}} \frac{1}{|f|^{1-s}} \sum_{\substack{D \in \mathcal{SF}_{3g+1} \\ (D,f)=1}} g(f, D)$$

Dirichlet series of Cubic Gauss Sums

$$\mathcal{G}(f,u) = \sum_{F \in \mathcal{M}} g(f,F) u^{\deg(F)}$$

$$\psi(f,u) = (1 - u^3 q^3)^{-1} \sum_{F \in \mathcal{M}} g(f,F) u^{\deg(F)}$$

For $i \in \{0, 1, 2\}$,

$$\mathcal{G}_i(f,u) = \sum_{\substack{F \in \mathcal{M} \\ \deg(F) \equiv i \bmod 3}} g(f,F) u^{\deg(F)}$$

$$\psi_i(f,u) = (1 - u^3 q^3)^{-1} \sum_{\substack{F \in \mathcal{M} \\ \deg(F) \equiv i \bmod 3}} g(f,F) u^{\deg(F)}$$

Theorem (Hoffstein, 1992; Patterson, 2007)

For $i \in \{0,1,2\}$, and any $f \in \mathcal{M}$, we have

$$(1-q^4u^3)\,\psi_i(f,u) = |f|\,u^{\deg(f)}\left[a_1(u)\,\psi_{[i]_3}\left(f,\frac{1}{q^2u}\right) + a_2(u)\,\psi_{[\deg f+1-i]_3}\left(f,\frac{1}{q^2u}\right)\right],$$

where

$$\begin{aligned} a_1(u) &= -(q^2u)(qu)^{-[\deg(f)+1-2i]_3}(1-q^{-1}),\\ a_2(u) &= -W_{f,i}(qu)^{-2}(1-q^3u^3)\\ W_{f,i} &= \tau(\chi_3^{2i-1}\overline{\chi}_f). \end{aligned}$$

Cubic Gauss Sums

Theorem (Hoffstein, 1992; Patterson, 2007)

Furthermore,

$$\psi_i(f,u) = \frac{u^i P_i(f,u^3)(1-u^3q^3)}{1-q^4u^3} \iff \mathcal{G}_i(f,u) = \frac{u^i P_i(f,u^3)}{1-q^4u^3}$$

where $P_i(f,x)$ is a polynomial in x of degree at most $[(1+\deg f - i)/3]$ in x.

This shows that $\mathcal{G}_i(f,u)$ is analytic for any $s \in \mathbb{C}$, except for possible poles at $u^3 = q^{-4}$ and $u^3 = q^{-2}$.

Residue at $u^3 = q^{-4}$

Let

$$\rho_i(f) = \lim_{s\to\frac{4}{3}} (1 - q^{4-3s}) q^{is} \psi_i(f, q^{-s}) = P_i(f, q^{-4})$$

Lemme (D-Florea-Lalin, 2022)

Let $f = f_1 f_2^2 f_3^3$ *with* f_1, f_2 *square-free and coprime. Then,* $\rho_i(f) = 0$ *if* $f_2 \neq 1$*, and when* $f_2 = 1$

$$\rho_i(f) = \frac{\overline{g(1, f_1)}}{|f_1|^{\frac{2}{3}}} q^{\frac{4i}{3} - \frac{4}{3}[i - 2\deg(f)]_3} \rho(1, [i - 2\deg(f)]_3)$$

where $\rho(1,0) = 1, \ \ \rho(1,1) = \tau(\chi_3) q, \ \ \rho(1,2) = 0.$

Residue at $u^3 = q^{-4}$

We want

$$\mathcal{G}^{(f)}(f,u) = \sum_{\substack{F \in \mathcal{M} \\ (F,f)=1}} g(f,F) u^{\deg(F)}$$

Lemme (D-Florea-Lalin, 2022)

Let f_3^* be the product of the primes dividing f_3 but not $f_1 f_2$. Then,

$$\mathcal{G}^{(f)}(f,u) = \prod_{P | f_1 f_2} \left(1 - (u^3 q^2)^{\deg(P)}\right)^{-1} \sum_{a | f_3^*} \mu(a) g(f_1 f_2^2, a) u^{\deg(a)}$$

$$\times \prod_{P|a} (1 - (u^3 q^2)^{\deg(P)})^{-1}$$

$$\times \sum_{\ell | a f_1} \mu(\ell)(u^2 q)^{\deg(\ell)} \overline{g(1,\ell)}\, \chi_\ell \left(\frac{a f_1 f_2^2}{\ell}\right) \mathcal{G}\left(\frac{a f_1 f_2^2}{\ell}, u\right).$$

Summing Gauss sums

Proposition (D-Florea-Lalin, 2022)

Let $f = f_1 f_2^2 f_3^3$ *with* f_1 *and* f_2 *square-free and coprime. We have, for any* $\epsilon > 0$

$$\sum_{\substack{F \in \mathcal{M}_d \\ (F,f)=1}} G(f,F) = \delta_{f_2=1}\, \rho(d;f)\, \frac{\overline{G(1,f_1)}}{|f_1|^{2/3}} \frac{q^{4d/3}}{\zeta_q(2)} \prod_{P|f} \left(1 + \frac{1}{|P|}\right)^{-1}$$

$$+ O_\epsilon\left(\delta_{f_2=1} \frac{q^{(\frac{1}{3}+\epsilon)d}}{|f_1|^{\frac{1}{6}}} + q^d |f|^{\frac{1}{4}+\epsilon}\right)$$

where

$$\rho(d;f) = \begin{cases} 1 & d + \deg(f) \equiv 0 \bmod 3 \\ \frac{\tau(\chi_3)}{q^{1/3}} & d + \deg(f) \equiv 1 \bmod 3 \\ 0 & d + \deg(f) \equiv 2 \bmod 3 \end{cases}.$$

Now we sum over f. Summing Gauss sums again!

Now we sum over f. Summing Gauss sums again!

We have to evaluate

$$\int_{|u|=q^{1-s-\varepsilon}} \frac{\mathcal{D}(v)}{v^{3(g-A-1)+1}(1-v^3)} dv$$

- For $\frac{1}{3} < s < 1$, we have simple poles at

$$v = q^{\frac{1}{3}-s}, \quad v = 1, \omega, \omega^2, \quad v = \omega^j q^{s-\frac{4}{3}}, \; j = 0, 1, 2.$$

- For $s = \frac{1}{3}$, we have simple poles at

$$v = 1, \quad v = 1, \omega, \omega^2, \quad v = \omega^j q^{s-\frac{4}{3}}, \; j = 0, 1, 2.$$

Magical cancellation

For $\frac{1}{3} < s < 1 - \varepsilon$,

$$\frac{\mathcal{D}_s(3g, 3A)}{|\mathcal{H}(3g)|} \sim -C_q \frac{q^{(1-3s)A}}{1 - q^{3s-1}}$$

If $s = \frac{1}{3}$, then

$$\frac{\mathcal{D}_{\frac{1}{3}}(3g, 3A)}{|\mathcal{H}(3g)|} \sim C_q g - C_q A + C_q \left(-\frac{1}{3} \frac{q+2}{q-1} + \sum_P \frac{\deg(P)}{|P|^3 + 2|P|^2 - 1} \right)$$