The eighth moment of $\Gamma_1(q)$ L-functions

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Moments of the Riemann zeta function $\zeta(s)$

Let $I_{2k}(T) = \frac{1}{T}\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^{2k} \, dt$.

- $k = 1$: $I_2(T) \sim \log T$. (Hardy and Littlewood, 1918)
- $k = 2$: $I_4(T) \sim 2a_2\frac{(\log T)^4}{4!}$. (Ingham, 1926)
- $k \geq 3$: Asymptotic formulae are not proven. However, we have a good conjecture.

$$I_{2k}(T) \sim g_k a_k \frac{(\log T)^{k^2}}{k^2!}.$$

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- $a_k \frac{(\log T)^{k^2}}{k^2!}$ is easy to understand from $\sum_{n\leq T}\frac{d_k^2(n)}{n}$, but g_k is some constant that remains unsolved.
- When $k \geq 3$, the moments are harder since "off-diagonal terms" also contribute.

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- When $k \geq 3$, the moments are harder since "off-diagonal terms" also contribute.
- Conjecture: (e.g. Keating and Snaith (2000),)

$$g_3 = 42, \quad g_4 = 24024, \quad g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}$$

Dirichlet L-functions

For $\mathrm{Re}(s) > 1$,

$$L(s,\chi) = \sum_{n\geq 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where χ is a primitive Dirichlet character modulo q.
The moments of this family behave similarly to the moments of the $\zeta(1/2 + it)$.

Notation

- $\sum^*$ is the sum over all primitive characters mod q.
- $\phi^*(q)$ is number of primitive characters mod q.
- Moments of $L(s,\chi)$ is defined to be

$$M_{2k}(q) = \frac{1}{\phi^*(q)} \sum_{\chi \pmod q}^{*} |L(1/2,\chi)|^{2k}.$$

Moments of $L(s, \chi)$

$k = 1$**:** $\quad M_2(q) \sim \log q$.

$k = 2$**:** Heath-Brown (1981), Soundararjan (2007), Young (2010) showed that

$$M_4(q) \sim 2b_2 \frac{(\log q)^4}{4!}.$$

$k \geq 3$: Unknown. It is conjectured that

$$M_{2k}(q) \sim g_k b_k \frac{(\log q)^{k^2}}{k^2!},$$

where $g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}$ (same as the constant in the asymptotic formula of Riemann zeta function case).

Upper bounds for moments of Dirichlet L-functions

- $\sum^*_{\chi \pmod q} |L(1/2, \chi)|^{2k} \ll q(\log q)^{k^2}$ under the Generalized Riemann Hypothesis (GRH) (Soundararajan, 2009 and Harper, 2013).
- For $k = 1, 2$, we had asymptotic formula without GRH.

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- For $k = 1, 2$, we had asymptotic formula without GRH.
- By using large sieve inequality, Huxley (1970) showed that for $k = 3, 4$,

$$\sum_{q\sim Q} \sum^*_{\chi \pmod q} |L(1/2,\chi)|^{2k} \ll Q^2(\log Q)^{k^2}.$$

	$\sum^*_{\chi \pmod q}$	$\sum_{q\sim Q} \sum^*_{\chi \pmod q}$
conductor	q	q
family	$\sim q$	$\sim Q^2$

The sixth moment of Dirichlet L-functions

It is conjectured that

$$\sideset{}{^*}\sum_{\chi \pmod q} |L(1/2,\chi)|^6 \sim 42 a_3 \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)} \phi^*(q) \frac{(\log q)^9}{9!}.$$

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Question: Is there an asymptotic formula of the sixth moment for the larger family of Dirichlet L-functions, i.e.

$$\sum_{q \sim Q} \sideset{}{^*}\sum_{\chi \pmod q} |L(1/2,\chi)|^6 \quad ?$$

Conrey, Iwaniec and Soundararajan's work

$$\sum_{q\sim Q}\ \sideset{}{^*}\sum_{\chi \pmod q} \int_{-\infty}^{\infty} \left|L(\frac{1}{2}+it,\chi)\right|^6 \left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^6 dt$$

$$\sim 42 a_3 \sum_{q\sim Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^5}{\left(1+\frac{4}{p}+\frac{1}{p^2}\right)} \phi^*(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^6 dt$$

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- They also state a more precise technical result which gives the asymptotic for the sixth moment including shifts with a power saving error term of size $Q^{2-1/10+\epsilon}$.
- The average over t is introduced to get rid of "unbalanced" sums.

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- They also state a more precise technical result which gives the asymptotic for the sixth moment including shifts with a power saving error term of size $Q^{2-1/10+\epsilon}$.
- The average over t is introduced to get rid of "unbalanced" sums.
- C., Li, Matomäki, Radziwiłł (2023 +): Obtain an asymptotic formula without the average over t.

The eighth moment of Dirichlet L-functions

It is conjectured that

$$\sum_{\chi \pmod q}^{*} |L(1/2,\chi)|^8 \sim 24024 a_4 \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \phi^*(q) \frac{(\log q)^{16}}{16!}.$$

Note that the constant 24024 appears in the leading term of the eighth moment of $\zeta(s)$.

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On GRH, C. and Li (2014) derived asymptotic formula for the eighth moment of a large family of Dirichlet L-functions with extra average over t, i.e.

$$\mathcal{M}_8(Q) = \sum_{q\sim Q} \sum_{\chi \pmod q}^{*} \int_{-\infty}^{\infty} \left| L(\frac{1}{2}+it,\chi)\right|^8 \left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^8 dt.$$

Theorem (C., Li, Matomäki, and Radziwiłł (2023))

We have $\mathcal{M}_8(Q)$ *is*

$$\sim 24024\ a_4 \sum_{q\sim Q} \prod_{p|q} \frac{\left(1-\frac{1}{p}\right)^7}{\left(1+\frac{9}{p}+\frac{9}{p^2}+\frac{1}{p^3}\right)} \phi^*(q) \frac{(\log q)^{16}}{16!} \times \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^8 \, dt$$

$$\sim 24024\ \widetilde{a}_4 Q^2 \frac{(\log Q)^{16}}{16!} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{1/2+it}{2}\right)\right|^8 \, dt.$$

Note: We cannot get power saving error terms. Our error term is of size $Q^2(\log Q)^{15+\epsilon}$.

Holomorphic L-functions

Let q be a prime number. Let $S_k(\Gamma_0(q), \chi)$ be the space of cuspidal holomorphic forms of weight k with respect to the congruence subgroup $\Gamma_0(q)$ and the character $\chi \mod q$.

$$\Gamma_0(q) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod q \right\},$$

and we define $S_k(\Gamma_1(q))$ be the space of cuspidal holomorphic forms of weight k with respect to the congruence subgroup $\Gamma_1(q)$, where

$$\Gamma_1(q) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod q,\ a \equiv d \equiv 1 \mod q \right\}$$

Let $\mathcal{H}_k(q,\chi) \subset S_k(\Gamma_0(q),\chi)$ be the set of orthogonal basis of $S_k(\Gamma_0(q),\chi)$. Let f be a normalized cusp form in $\mathcal{H}_k(q,\chi)$ has a Fourier expansion of the form

$$f(z) = \sum_{n\geq 1} \lambda_f(n) n^{(k-1)/2} e(nz),$$

where $\lambda_f(1) = 1$.

An L-function $L(f,s)$ associated to the normalized cusp form f is defined for $\mathrm{Re}(s) > 1$ as

$$L(f,s) = \sum_{n\geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1},$$

where $\lambda_f(n)$ is the coefficient from the Fourier expansion of f.

Note: We consider q to be prime to eliminate old forms.

The completed L-functions is

$$\Lambda\left(f, \tfrac{1}{2}+s\right) = \left(\tfrac{q}{4\pi}\right)^{\frac{s}{2}} \Gamma\left(s+\tfrac{k}{2}\right) L\left(f, \tfrac{1}{2}+s\right).$$

It satisfies the following functional equations

$$\Lambda\left(f, \tfrac{1}{2}+s\right) = i^k \overline{\eta}_f \Lambda\left(\bar{f}, \tfrac{1}{2}-s\right),$$

where $|\eta_f| = 1$.

Harmonic average:

$$\sum_{f\in\mathcal{H}_k(q,\chi)}^{h} \alpha_f := \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f\in\mathcal{H}_k(q,\chi)} \frac{\alpha_f}{\|f\|^2},$$

where $< f, g >$ is the Petersson inner product on $\Gamma_0(q)\backslash\mathbb{H}$.

Some results on moments of automorphic L-functions

The second moment: $\sum^h_{f\in\mathcal{H}_2(q,\chi_0)} L(f,1/2)^2 \sim \log q$ is obtained by Iwaniec and Sarnak

The fourth moment: Kowalski, Michel and Vanderkam (2000) obtained the result for $\sum^h_{f\in\mathcal{H}_2(q,\chi_0)} L(f,1/2)^4 \sim \frac{1}{60\pi^2}(\log q)^6$

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Higher moment: Asymptotic formulae are unknown.

This family has orthogonal symmetry. Therefore the leading order term is $(\log q)^6$ instead of $(\log q)^4$.

- To obtain good upper bounds/asymptotic formulae for the sixth and the eighth moment of Dirichlet L-functions, we need to enlarge the size of the family we average on.

- In this case, we will also increase the size of family of L-functions to get an asymptotic formula and good upper bounds (without GRH) for the sixth and the eighth moment.

$$S_k(\Gamma_0(q), \chi) \quad \Longrightarrow \quad S_k(\Gamma_1(q)).$$

$$\sideset{}{^h}\sum_{f \in \mathcal{H}_k(q,\chi)} \quad \Longrightarrow \quad \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_k(q,\chi)}$$

Note that the analytic conductor of L-functions in these families $\sim k^2 q$.

The spaces $S_k(\Gamma_0(q))$ vs $S_k(\Gamma_1(q))$

Dimension of the spaces

$$\dim S_k(\Gamma_0(q)) \sim \frac{k-1}{12} q \prod_{p|q}(1+p^{-1}),$$

and

$$\dim S_k(\Gamma_1(q)) \sim \frac{k-1}{24} q^2 \prod_{p|q}(1-p^{-2}).$$

They are connected by

$$S_k(\Gamma_1(q)) = \bigoplus_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} S_k(\Gamma_0(q), \chi).$$

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Dirichlet	**Holomorphic**
Dirichlet characters mod q (size q)	$\Gamma_0(q)$ modular forms
All Dirichlet characters mod $q \sim Q$ (size Q^2)	$\Gamma_1(q)$ modular forms

Upper bounds for the the moment

Recall that for Dirichlet L-functions case, unconditionally, we have correct size of upper bounds for the sixth ($\ell = 3$) and the eighth moments ($\ell = 4$).

$$\sum_{q \sim Q} \sum_{\chi \bmod q}^{*} \left| L\left(\frac{1}{2}, \chi\right) \right|^{2\ell} \ll Q^2 (\log Q)^{\ell^2}.$$

It is natural to ask if there is analogous upper bounds for the sixth and the eighth moment of $\Gamma_1(q)$ L-functions.

$$M_{2\ell}(q) = \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1) = (-1)^k}} \sum_{f \in \mathcal{H}_k(q,\chi)}^{h} |L(f, 1/2)|^{2\ell}$$

for $\ell = 3$ and 4.

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for $\ell = 3$ and 4. This family also admits the unitary symmetry, so it has similar conjectures to the moment of $\zeta(s)$, i.e. we expect that

$$M_{2\ell}(q) \ll (\log q)^{\ell^2}.$$

Upper bounds for the sixth and the eighth moment

- Djankovic (2011) showed that for $k \geq 3$,

$$M_6(q) \ll q^{\epsilon}.$$

  This bound is consistent with the Lindelöf hypothesis on average.

- Stucky (2021) proved the correct size of upper bound

$$M_6(q) \ll (\log q)^9.$$

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- C. and Li (2018) showed that for $k \geq 5$,

$$M_8(q) \ll q^{\epsilon}.$$

The correct size for the upper bound is $(\log q)^{16}$. This problem remains open.

Asymptotic large sieve

A main tool to prove upper bounds is an asymptotic large sieve for the family of $\Gamma_1(q)$ developed by Iwaniec and Xiaoqing Li (2007):

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_k(q,\chi)} \left| \sum_{n \leq N} a_n \lambda_f(n) \right|^2 .$$

Part of the difficulty in this family is from the fact that this asymptotic large sieve is **not perfectly orthogonal**.

Interesting feature of family

"Perfectly orthogonal" large sieve: Let $\mathcal{X}$ be a finite set of "nice" sequences.

$$\sum_{x \in \mathcal{X}} \left| \sum_{n \leq N} a_n x(n) \right|^2 \ll (|\mathcal{X}| + N) \sum_{n \leq N} |a_n|^2.$$

For example, **the large sieve for primitive Dirichlet characters:**

$$\sum_{q \sim Q} \sideset{}{^*}\sum_{\chi \pmod q} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \ll (Q^2 + N) \sum_{n \leq N} |a_n|^2.$$

Iwaniec and Xiaoqing Li proved an asymptotic large sieve which is of a different nature.

More precisely they showed for any $\epsilon > 0$,

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_\chi} \left| \sum_{n \geq 1} a_n \lambda_f(n) \right|^2$$

$$= \text{Main term} + O\left(N^\epsilon \left(\frac{N}{q^2} + \sqrt{\frac{N}{qH}} \right) \right) \sum_n |a_n|^2$$

where

- $\alpha = (a_n)$, where $N < n \leq 2N$, and $1 \leq H \leq \frac{N}{q}$
- If the coefficients a_n are chosen to look like certain Bessel functions twisted by Kloosterman sums, the main term can be large of size $\gg \sqrt{\frac{N}{qh_0}} \sum_n |a_n|^2$
- The component $\sqrt{\frac{N}{qH}}$ cannot be removed.

Asymptotic formula

Similar to Dirichlet L-functions, we compute an asymptotic formula with the average over the critical line. Let

$$\mathcal{I}_{2k}(q) := \frac{2}{\phi(q)} \sum_{\substack{\chi \bmod q \\ \chi(-1)=(-1)^k}} \sum_{f \in \mathcal{H}_k(q,\chi)}^{h} \int_{-\infty}^{\infty} \left| L\left(f, \frac{1}{2}+it\right)\right|^{2k} \left|\Gamma\left(\frac{k}{2}+it\right)\right|^{2k} dt.$$

Theorem (C. and Li, 2016)

For odd integer $k \geq 5$, we have

$$\mathcal{I}_6(q) \sim 42 b_3 \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left|\Gamma\left(\frac{k}{2}+it\right)\right|^6 dt.$$

- We prove a more precise asymptotic formula including shifts with a power saving error term of size $q^{-1/4+\epsilon}$.

The eighth moment of $\Gamma_1(q)$ L-functions

Theorem (C., Dunn, Li and Stucky, 2024+)

For odd integer $k \geq 5$, we have

$$\mathcal{I}_8(q) \sim 24024 b_4 \frac{(\log q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma\left(\frac{k}{2} + it\right) \right|^8 dt.$$

We can obtain an asymptotic formula with the leading term and having the error term of size $O((\log q)^{15+\epsilon})$.

Similar phenomenon

The eighth moments for the enlarged family of Dirichlet *L*-functions and $\Gamma_1(q)$ L-functions require different techniques as their structures are different. However, there are some similar phenomenon appearing in the proof.

- Switching to smaller conductor.
- Truncation of the sum.
  (This step is harder for $\Gamma_1(q)$ *L*-functions.)
- Understanding the sums in narrow regions.
  (The appearance for narrow region in $\Gamma_1(q)$ L-functions is not obvious!)

The eighth moments of Dirichlet L-functions

After approximate functional equation, we consider

$$\sum_q \Psi\left(\frac{q}{Q}\right) \sum_{\chi \pmod q}^{*} \sum_{m,n \le Q^2} \frac{d_4(n) d_4(m)}{\sqrt{mn}} \chi(m)\overline{\chi}(n)$$

where Ψ is a smooth function compactly supported in [1, 2].

Without the integration over t, the main contribution will comes from $mn \ll Q^4$. We need to consider unbalanced sums when one variable is large and another one is small, e.g. $m = Q^3$ and $n = Q$.

After the orthogonality relation of Dirichlet characters, we roughly need to understand the sum of the form

$$\sum_{q} \Psi\left(\frac{q}{Q}\right)\phi(q) \sum_{\substack{m,n\leq Q^2 \\ m\equiv n \mod q}}\sum \frac{d_4(n)d_4(m)}{\sqrt{mn}}$$

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- The diagonal term $m = n$ is easy to understand.
- For the off-diagonal term, use complementary divisor trick to switch to a smaller conductor.

  Write $m - n = hq$, where $h \neq 0$. So $h \asymp \frac{|m-n|}{Q}$, and $m \equiv n \mod h$. [Goal: we want h to be smaller than q.] But

  $$h \ll \frac{Q^2}{Q} = Q$$

  This is not smaller! We need to truncate sums over m, n.

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  $$h \ll \frac{Q^2}{Q} = Q$$

  This is not smaller! We need to truncate sums over m, n.
- Note: if we do not have integration over t, the size of h can be much larger than Q.
  (e.g. h can be $\frac{Q^4-1}{Q} \asymp Q^3$)

We truncate the sums over m, n by the large sieve inequality

$$\sum_{q\sim Q}\sum_{\chi \pmod q}^{*}\left|\sum_{n\le N} a_n\chi(n)\right|^2 \ll (Q^2+N)\sum_{n\le N}|a_n|^2.$$

Now $m, n \ll Q^{2-\epsilon}$

After the truncation, $h \ll \frac{Q^{2-\epsilon}}{Q} = Q^{1-\epsilon}$. [smaller conductor!]

We truncate the sums over m, n by the large sieve inequality

$$\sum_{q\sim Q} \sum_{\chi \pmod q}^{*} \left| \sum_{n\le N} a_n \chi(n) \right|^2 \ll (Q^2+N) \sum_{n\le N} |a_n|^2.$$

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After the truncation, $h \ll \frac{Q^{2-\epsilon}}{Q} = Q^{1-\epsilon}$. [smaller conductor!]

The sum over a narrow region.

It comes from the condition $|m-n| \asymp hQ$.

When h is small, $|m-n| \asymp hQ$ is also small. Hence, for fixed n, the sums over m is restricted to an interval much shorter than Q^2.

The eighth moments of $\Gamma_1(q)$ L-functions

Recall that

$$\mathcal{I}_8(q)$$

$$:= \frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sideset{}{^h}\sum_{f \in \mathcal{H}_k(q,\chi)} \int_{-\infty}^{\infty} \left| L\left(f, \frac{1}{2}+it\right)\right|^8 \left|\Gamma\left(\frac{k}{2}+it\right)\right|^8 dt.$$

Roughly speaking, after the approximate functional equation, we consider

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} \sideset{}{^h}\sum_{f \in \mathcal{H}_k(q,\chi)} \sum\sum_{m,n \ll q^2} \frac{\lambda_f(n) d_4(n)}{n^{1/2}} \frac{\overline{\lambda_f}(m) d_4(m)}{m^{1/2}}.$$

No unbalanced sums.

We then apply Petersson's formula

$$\sum_{f \in \mathcal{H}_k(q,\chi)}^{h} \lambda_f(m)\overline{\lambda_f(n)} = \delta(m,n) + \sigma_\chi(m,n),$$

where $\delta(m,n) = 1$ if $m = n$ and 0 otherwise, and

$$\sigma_\chi(m,n) = 2\pi i^{-k} \sum_{c=1}^{\infty} \frac{1}{cq} S_\chi(m,n;cq) J_{k-1}\left(\frac{4\pi}{cq}\sqrt{mn}\right)$$

where

$$S_\chi(m,n;cq) = \sum_{a \mod cq}^{*} \overline{\chi(a)} e\left(\frac{am + \bar{a}n}{cq}\right).$$

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$$S_\chi(m,n;cq) = \sum_{a \mod cq}^{*} \overline{\chi(a)}e\left(\frac{am+\bar{a}n}{cq}\right).$$

- The **diagonal terms** from $\delta(m,n)$ $(m=n)$ are easy (This contributes to the main term.)
- The **off-diagonal terms** from $\sigma_\chi(m,n)$ will contain another main term, and it is a lot harder.

Next we apply orthogonality relation for Dirichlet characters

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \chi(m)\overline{\chi}(n) = \begin{cases} 1 & \text{if } m \equiv n \mod q \\ (-1)^k & \text{if } m \equiv -n \mod q \\ 0 & \text{otherwise.} \end{cases}$$

Essentially, we need to understand the sum of the form

$$\sum_c \frac{1}{cq} \sideset{}{^*}\sum_{\substack{a \mod cq \\ a\equiv 1 \mod q}} \sum\sum_{n,m \asymp q^2} \frac{d_4(m)}{m^{1/2}} e\left(\frac{am}{cq}\right) \frac{d_4(n)}{n^{1/2}} e\left(\frac{\overline{a}n}{cq}\right) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right)$$

The Bessel function satisfies

$$J_{k-1}(x) \ll \min\{x^{-1/2}, x^{k-1}\}.$$

So we need to consider two cases: $x \gg 1$ and $x \ll 1$.
- When x is small, the Bessel function can be treated as the smooth function.
- When x is big, there is oscillation from the Bessel function.
- When $x \asymp 1$, this is called **the transition region.**

The transition region for $J_{k-1}\left(\frac{4\pi}{cq}\sqrt{mn}\right)$ is when

$$c \asymp \frac{\sqrt{mn}}{q} \asymp q.$$

.

(Recall that here we consider only the case when $m, n \asymp q^2$.)

$$\sum_{n \asymp q^2} \frac{d_4(n)}{n^{1/2}} e\left(\frac{am}{cq}\right).$$

Our next step is to apply Voronoi summation to the sum over m and n. Now the conductor is of size $cq \asymp q^2$. If we applied Voronoi summation, then

Orginal sum		**the dual sum**
q^2	$\rightarrow$	$\frac{(cq)^4}{q^2} \asymp q^6$

The dual sum is longer than the original sum, and more difficult to handle! We try to reduce the conductor in the exponential sum.

Switching to a smaller conductor

Recall that in the case of Dirichlet L-functions, the complementary divisor trick is used to reduce the conductor.

By Chinese Remainder Theorem and reciprocity, we may factor our exponential sum as

$$\sideset{}{^*}\sum_{\substack{a \bmod cq \\ a\equiv 1 \bmod q}} e\left(\frac{am}{cq}\right) = \sideset{}{^*}\sum_{z \bmod c} \sideset{}{^*}\sum_{\substack{y \bmod q \\ y\equiv 1 \bmod q}} e\left(\frac{my\bar{c}}{q} + \frac{mz\bar{q}}{c}\right)$$
$$= \sideset{}{^*}\sum_{z \bmod c} e\left(\frac{m\bar{c}}{q} + \frac{mz\bar{q}}{c}\right)$$
$$= e\left(\frac{m}{cq}\right) \sideset{}{^*}\sum_{z \bmod c} e\left(\frac{m\bar{q}(z-1)}{c}\right)$$

The conductor is of the size $c \asymp q$

Smaller conductor

- Side note: for the sixth moment, the conductor c is $q^{1/2}$. After Voronoi, the dual sum is very short. So we can bound it trivially.

Orginal sum		**the dual sum**
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Orginal sum		**the dual sum**
$q^{3/2}$	$\rightarrow$	$\frac{c^3}{q^{3/2}} \asymp 1$

- For the eighth moment, the conductor c is q. The conductor is NOT reduced! If we apply Voronoi summation, the length of the **dual sum** is around the size

$$\frac{c^4}{q^2} \asymp q^2.$$

The dual is of the same length as the original sum.

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[So, the truncation here is more difficult than the truncation in Dirichlet L-functions.]

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[So, the truncation here is more difficult than the truncation in Dirichlet L-functions.]

- After the truncation, The most important region is when $m, n \asymp q^{2-\epsilon}$. In a simple model, we will do analysis there.
- The transition region of the conductor c is around

$$c \asymp \frac{\sqrt{mn}}{q} \asymp q^{1-\epsilon}.$$

- The conductor is now reduced!

Truncation for the sums over m and n

We want to show that

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} \sum_{f \in \mathcal{H}_\chi}^{h} \left| \sum_{q^{2-\epsilon} < n \le q^2} \frac{d_4(n)\lambda_f(n)}{\sqrt{n}} \right|^2 \ll \sum_{q^{2-\epsilon} < n \le q^2} \frac{d_4^2(n)}{n}$$

$$\ll \epsilon (\log q)^{16},$$

where $\epsilon \ll \frac{1}{(\log q)^{1-\epsilon_1}}$. Dividing the sum over n into dyadic intervals, it is enough to show that for $q^{2-\epsilon} \le N \le q^2$,

$$\frac{2}{\phi(q)} \sum_{\substack{\chi \mod q \\ \chi(-1)=-1}} \sum_{f \in \mathcal{H}_\chi}^{h} \left| \sum_{n \asymp N} \frac{d_4(n)\lambda_f(n)}{\sqrt{n}} \right|^2 \ll \sum_{n \asymp N} \frac{d_4^2(n)}{n}.$$

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Squaring it out and applying Petersson's formula gives

$$\sum_{n \asymp N} \frac{d_4^2(n)}{n} + \text{Off diagonal.}$$

The off-diagonal terms in the truncation and after the truncation are of the same form. For $N \ll q^2$, essentially we would like to understand.

$$\sum_{c} \frac{1}{cq} \sideset{}{^*}\sum_{z \bmod c} \sum\sum_{m,n \asymp N} \frac{d_4(m) d_4(n)}{\sqrt{mn}} e\left(\frac{\bar{q}(z-1)m + \bar{q}(\bar{z}-1)n}{c} \right) \times e\left(\frac{m+n}{cq} \right) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq} \right).$$

We will first consider a simple model to illustrate ideas.

$$\sum_{c \geq 1} \frac{1}{cq} \sideset{}{^*}\sum_{a \bmod c} \sum\sum_{m,n \asymp N} \frac{d_4(m) d_4(n)}{\sqrt{mn}} e\left(\frac{am + \bar{a}n}{c} \right) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq} \right).$$

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The transition region is when $\frac{\sqrt{mn}}{cq} \sim 1 \to c \sim \frac{N}{q}$, so we consider two main cases:
c is large $\frac{N}{q} \ll c$, and c is small $\frac{N}{q} \gg c$

Large c: $\frac{N}{q} \ll c$, so $\frac{N}{cq}$ is small.

Here, since $\frac{N}{cq} \ll 1$, there is no oscillation from the Bessel function $J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right)$. For convenience, we consider $c \asymp \frac{N}{q}$ (transition region). Roughly, we are left to understand

$$\sum_{c \asymp \frac{N}{q}} \frac{1}{cq} \sideset{}{^*}\sum_{a \bmod c} \sum_{m,n \asymp N} \frac{d_4(m) d_4(n)}{\sqrt{mn}} e\left(\frac{am + \bar{a}n}{c}\right)$$

$$\leq \frac{1}{N} \sum_{c \asymp \frac{N}{q}} \sideset{}{^*}\sum_{a \bmod c} \left| \sum_{m \asymp N} \frac{d_4(m)}{\sqrt{m}} e\left(\frac{am}{c}\right) \right|^2$$

$$\ll \frac{1}{N} \left(\frac{N^2}{q^2} + N\right) \sum_{n \asymp N} \frac{d_4^2(n)}{n} \qquad \text{(additive large sieve ineq)}$$

$$\ll \left(\frac{N}{q^2} + 1\right) \sum_{n \asymp N} \frac{d_4^2(n)}{n} \ll \sum_{n \asymp N} \frac{d_4^2(n)}{n}$$

since $N \leq q^2$.

Small c: $\frac{N}{q} > c$, so $\frac{N}{cq}$ is large.

$$J_{k-1}(2\pi x) = \frac{1}{\sqrt{\pi x}} \mathrm{Re}\left[W(2\pi x) e\left(x - \tfrac{k}{4} + \tfrac{1}{8}\right)\right],$$

where $W_k^{(j)}(x) \ll_{j,k} x^{-j}$.

- When x is large (c is small for our case), there is oscillation from the exponent term.
- We then separate variables inside the exponent term $e\left(\frac{\sqrt{mn}}{cq}\right)$ via Mellin inversion. We need to bound

$$\frac{1}{\sqrt{\frac{N}{Cq}}} \int_{t \asymp T} \frac{1}{\sqrt{t}} \sum_{c \asymp C} \frac{1}{cq} \sideset{}{^*}\sum_{a \bmod c} \left| \sum_{m \asymp N} \frac{d_4(m)}{m^{1/2+it}} e\left(\frac{am}{c}\right)\right|^2 dt$$

for $C \ll \frac{N}{q}$ and $T \ll \frac{N}{Cq}$.

Applying the hybrid large sieve gives

$$\frac{1}{\sqrt{NTCq}}\left(TC^2+N\right)\sum_{n\asymp N}\frac{d_4^2(n)}{n}$$
$$\ll\left(1+\sqrt{\frac{N}{Cq}\frac{1}{T}}\right)\sum_{n\asymp N}\frac{d_4^2(n)}{n},$$

by using that $C \ll \frac{N}{q}$, $T \ll \frac{N}{Cq}$ and $N \leq q^2$.

The term $\sqrt{\frac{N}{qCT}}$ can be too large since C and T can be small.

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The term $\sqrt{\frac{N}{qCT}}$ can be too large since C and T can be small.

Recall the asymptotic large sieve:

$$\text{main term } + O\left(N^{\epsilon}\left(\frac{N}{q^2}+\sqrt{\frac{N}{qH}}\right)\right)\|\alpha\|^2.$$

Question: What should we do?

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Answer: Voronoi summation

This uses information about the coefficients $d_4(n)$. In particular, $d_4(n)$ is not correlated to Bessel function twisted with Kloosterman sums in certain ranges. Hence we break the non-orthogonal nature of the family we saw in the large sieve of Iwaniec and Li.

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- The hybrid conductor of Voronoi summation is CT.
- The dual sum is of length $\frac{(CT)^4}{N}$.

After Voronoi summation, we need to bound

$$\frac{1}{\sqrt{NCqT}} \int_{t \asymp T} \sum_{c \asymp C} \sum_{a \bmod c}^{*} \left| \sum_{m \ll \frac{(CT)^4}{N}} \frac{d_4(m)}{m^{1/2-it}} e\left(\frac{-am}{c}\right) \right|^2 dt$$

for $C \ll \frac{N}{q}$ and $T \ll \frac{N}{Cq}$.

Now we apply the hybrid large sieve and obtain that it is bounded by

$$\frac{1}{\sqrt{NCqT}}\left(TC^2+\frac{(CT)^4}{N}\right)\sum_{n\asymp\frac{(CT)^4}{N}}\frac{d_4^2(n)}{n}$$

$$\ll\sum_{n\asymp N}\frac{d_4^2(n)}{n}.$$

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$$\ll\sum_{n\asymp N}\frac{d_4^2(n)}{n}.$$

Recall three common phenomenon with Dirichlet L-functions.

- Switching to smaller conductor. ✓
- Truncation of the sum. ✓
- Understanding the sums in narrow regions.

Recall **the off-diagonal terms** are

$$\sum_{c}\frac{1}{cq}\sum_{z \bmod c}^{*}\sum_{m,n\asymp q^{2-\epsilon}}\sum\frac{d_4(m)d_4(n)}{\sqrt{mn}}e\left(\frac{\bar{q}(z-1)m+\bar{q}(\bar{z}-1)n}{c}\right)$$
$$\times e\left(\frac{m+n}{cq}\right)J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right).$$

- The harder case is when c is small, i.e., $c \ll \frac{N}{q}$. So we focus only for this case.
- The phase functions in the exponential function and the Bessel function are big.
- We will need Voronoi summation for the sums over m, n

Add coprimality relation for $m, n, z-1$ and c. Note that

$$(z-1, c) = (\bar{z}-1, c) = \delta.$$

Essentially, we would like to understand

$$\sum_{c} \frac{1}{cq} \sum_{\beta \bmod c} \sideset{}{^*}\sum_{\substack{j \bmod c \\ (j+\beta, c)=1}} \sum_{m,n \asymp q^{2-\epsilon}} \sum \frac{d_4(m) d_4(n)}{\sqrt{mn}} e\left(\frac{\bar{q}jm - \bar{q}\overline{(j+\beta)}n}{c}\right)$$

$$\times \sum_{\delta \equiv \beta \bmod c} \frac{1}{\delta} e\left(\frac{m+n}{cq\delta}\right) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq\delta}\right).$$

- Use the method by Iwaniec and Li to deal with the sum over δ. [Start from Poisson summation in δ.] This is to separate variables in m and n.

Let $H = \frac{N}{q} = \frac{\text{size of } \max\{m,n\}}{q}$. This comes from the phase integral (with respect to y) after Poisson summation in δ.

$$\sum_h F_1(h) \int_0^\infty g(y) e(f(y,h))\, dy,$$

where

$$f(y,h) = \frac{hy}{c} + \frac{m+n}{cqy} \pm \frac{2\sqrt{mn}}{cqy}.$$

If $|h| \gg H$, then we can integral by part many times and get small contribution. So we consider only when $|h| \ll H$.

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If $|h| \gg H$, then we can integral by part many times and get small contribution. So we consider only when $|h| \ll H$.

If there is no integration over critial line, we have unbalanced sums, and H is large.

Let Φ be a compactly supported smooth function. For simplicity, we need to understand

$$\sum_{1<c\ll q^{1-\epsilon}} \frac{H}{c^2 q} \sum_{\ell \neq 0} \sideset{}{^*}\sum_{\substack{j \bmod c \\ (j+\ell,c)=1}} \int_0^\infty \Phi(y) e\left(\frac{\ell H y}{c}\right)$$

$$\times \sum_m \frac{d_4(m)}{\sqrt{m}} \Phi\left(\frac{m}{N}\right) e\left(\frac{\bar{q} j m}{c}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{Hmy}{q}}\right)$$

$$\times \sum_n \frac{d_4(n)}{\sqrt{n}} \Phi\left(\frac{n}{N}\right) e\left(\frac{-\bar{q}\overline{(j+\ell)} n}{c}\right) J_{k-1}\left(\frac{4\pi}{c}\sqrt{\frac{Hny}{q}}\right) dy.$$

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When c is small, it is advantageous to apply the voronoi summation to the sums over m, n.

Voronoi summation

For $(c,d)=1$

$$\sum_{n=1}^{\infty} d_k(n) e\left(\frac{dn}{c}\right)\psi\left(\frac{n}{N}\right) = \mathrm{res}_{s=1} .. + \underbrace{\sum_{n \ll \frac{c^k}{N}}}_{\text{dual sum}} .$$

- The residues at $s=1$ from both sums over m, n contribute to the main terms.
- The rest gives error terms.

After this process, the dual sums over m, n will be

$$\sum_{m,n} a_n a_m \int_0^\infty \int_0^\infty \int_0^\infty \Phi(y)\Phi(x_1)\Phi(x_2)\, dy\, dx_1\, dx_2$$

$$\times\, e\left(\frac{\ell H y}{c} + \underbrace{\frac{2}{c}\sqrt{\frac{HNy}{q}}(\sqrt{x_1} - \sqrt{x_2})}_{\text{J-Bessel functions}} - \underbrace{\frac{\alpha N^{\frac{1}{4}}}{c}(m^{\frac{1}{4}} x_1^{\frac{1}{4}} - n^{\frac{1}{4}} x_2^{\frac{1}{4}})}_{\text{from voronoi}} \right)$$

Recall that $H = \frac{N}{q}$.

- When considering the integration over y, we have saddle point when x_1 is close to x_2
- From the integrals over x_1 and x_2, we have that m and n must be close to each other. (the sum over narrow regions.)

Higher moments

Question: How about a higher moment, e.g. the 10th moment?

Higher moments

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It's still unknown. One possibility is to enlarge the size of a family of L-functions, so the size is Q^3 while the conductor remains $\asymp Q$. In particular, we consider

$$\sum_{q\sim Q} \sum_{\substack{\chi \mod q \\ \chi(-1)=(-1)^k}} \sum_{f\in\mathcal{H}_k(q,\chi)}^{h} \int_{-\infty}^{\infty} \left| L\left(f, \frac{1}{2}+it\right)\right|^{2k} \left|\Gamma\left(\frac{k}{2}+it\right)\right|^{2k} dt$$

for $k=5$ (the 10th moment) and $k=6$ (the 12th moment).

Thank you very much!