Moments of Rankin-Selberg L-functions in the prime-power level aspect

Alia Hamieh
(Joint work with Fatma Çiçek)

University of Northern British Columbia

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Moments of $\zeta(\frac{1}{2}+it)$

We set

$$I_k(T) = \int_0^T |\zeta(\tfrac{1}{2}+it)|^{2k}\, dt.$$

Lindelöf Hypothesis

$$\Updownarrow$$

for any $\epsilon > 0,\ I_k(T) = O(T^{1+\epsilon})\ \ \forall\, k \in \mathbb{N}.$

A Folklore Conjecture

It is believed that

Conjecture

$$I_k(T) = \int_0^T \left|\zeta(\tfrac{1}{2} + it)\right|^{2k} dt \sim c_k T \left(\log T\right)^{k^2},$$

for some unspecified constant c_k.

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- *Hardy-Littlewood (1918):* $I_1(T) \sim T \log T$.
- *Ingham (1926):* $I_1(T) = T P_1(\log T) + O(T^{\frac{1}{2}+\epsilon})$.

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- *Ingham (1926):* $I_2(T) \sim \frac{1}{2\pi^2} T \left(\log T\right)^4$.
- *Heath-Brown (1979):* $I_2(T) = T P_4(\log T) + O(T^{\frac{7}{8}+\epsilon})$.

Asymptotic Bounds

We have the lower bound

- *Radziwiłł-Soundararajan (2013):*
  For all $k > 0$, we have $I_k(T) \gg T\,(\log T)^{k^2}$.

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and the upper bounds

- *Soundararajan (2008):*
  Under RH, for any $\epsilon > 0$ *we have* $I_k(T) \ll T(\log T)^{k^2+\epsilon}$.
- *Harper (2013):*
  Under RH, we have $I_k(T) \ll T(\log T)^{k^2}$.

Conjectural Asymptotic Formulae

Conjecture (Conrey-Ghosh, 1998)

$$I_3(T) \sim \frac{g_3}{9!} a_3 \cdot T(\log T)^9,$$

where $g_3 = 42$ *and* $a_3 = \prod_p \left(1 - p^{-1}\right)^4 \left(1 + 4p^{-1} + p^{-2}\right)$.

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Conjecture (Conrey-Gonek, 2001)

$$I_4(T) \sim \frac{g_4}{16!} a_4 \cdot T(\log T)^{16},$$

where $g_4 = 24024$ *and*
$a_4 = \prod_p \left(1 - p^{-1}\right)^9 \left(1 + 9p^{-1} + 9p^{-2} + p^{-3}\right)$.

Asymptotic Formulae for Higher Moments?

Conjecture (Keating and Snaith, 2000)

$$I_k(T) = \int_0^T \left|\zeta(\tfrac{1}{2}+it)\right|^{2k} dt \sim \frac{g_k}{(k^2)!} \cdot a_k \cdot T(\log T)^{k^2},$$

where

$$g_k = (k^2)! \prod_{j=0}^{k-1} \frac{j!}{(j+k)!}$$

and

$$a_k = \prod_p \left(1 - \frac{1}{p}\right)^{(k-1)^2} \sum_{j=0}^{k-1} \binom{k-1}{j}^2 p^{-j}.$$

Conditional Results

- *Ng (2021): Under a ternary additive divisor conjecture*

$$I_3(T) \sim \frac{g_3}{9!} a_3 \cdot T(\log T)^9.$$

- *Ng-Shen-Wong (2022+): Under the Riemann Hypothesis and a quaternary additive divisor conjecture*

$$I_4(T) \sim \frac{g_4}{16!} a_4 \cdot T(\log T)^{16}.$$

Moments of $L(\frac{1}{2}, \chi)$

Questions regarding the asymptotic behavior of $\zeta(\frac{1}{2} + it)$ have q-aspect analogues concerning $L(\frac{1}{2}, \chi)$ as χ varies over primitive characters modulo q and $q \to \infty$.

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Questions regarding the asymptotic behavior of $\zeta(\frac{1}{2}+it)$ have q-aspect analogues concerning $L(\frac{1}{2}, \chi)$ as χ varies over primitive characters modulo q and $q \to \infty$.
Let

$$M_k(q) = \frac{1}{\phi^*(q)} \sum_{\chi \pmod q}^{*} \left| L(\tfrac{1}{2}, \chi) \right|^{2k}.$$

Conjecture (Conrey-Farmer-Keating-Rubinstein-Snaith, 2005)

$$M_k(q) \sim \frac{g_k}{(k^2)!} b_k (\log q)^{k^2}, \quad \text{for any } k \in \mathbb{N}^*.$$

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$$M_2(q) = \frac{2}{4!} b_2(q) \left(\log q\right)^4 + O\left(2^{\omega(q)} \frac{q}{\phi^*(q)} \left(\log q\right)^3\right),$$

where $b_2(q) = \frac{4!}{2} \frac{1}{2\pi^2} \prod_{p|q} \frac{(1-\frac{1}{p})^3}{(1+\frac{1}{p})}$.

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- *Young (2011):* $M_2(q) = P_4(\log q) + O(q^{-\frac{1}{80}+\frac{\theta}{40}+\epsilon})$, *as* $q \to \infty$ *in the primes.*

Chandee-Li-Matomaki-Radziwill (2023+): Asymptotic formula for

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \pmod q} \int_{\mathbb{R}} \left|\Gamma\left(\tfrac{1}{4}+\tfrac{it}{2}\right) L\left(\tfrac{1}{2}+it,\chi\right)\right|^{8}\, dt,$$

and

$$\sum_{q\leq Q}\ \sideset{}{^*}\sum_{\chi \pmod q} \left|L\left(\tfrac{1}{2},\chi\right)\right|^{6}.$$

Khan-Milićević-Ngo (2016): Suppose $q = p^\nu$ for some fixed odd prime p. Let $d|p-1$, and let $\mathcal{O}$ be the set of primitive characters $\mod q$ of order $p^{\nu-1}d$. We have

$$\frac{1}{\mathcal{O}}\sum_{\chi\in\mathcal{O}}\left|L(\tfrac{1}{2},\chi)\right|^2 = \frac{p-1}{p}\left(\log\left(\tfrac{q}{\pi}\right) + \frac{\Gamma'}{\Gamma}\left(\tfrac{1+2\iota}{4}\right) + 2\gamma + 2\tfrac{\log p}{p-1}\right) + O\left(q^{-\frac{1}{4}+\epsilon}\right).$$

Space of Cusp Forms

- The space of cusp forms of weight k for $\Gamma_0(q)$ and $\chi \pmod q$ is denoted by $S_k(\Gamma_0(q), \chi)$.
  If χ_0 is the principal character, we set $S_k(q) = S_k(\Gamma_0(q), \chi_0)$, and we have
  $$\dim_{\mathbb{C}} S_k(q) \sim \frac{k-1}{12} q \prod_{p|q} \left(1 + \frac{1}{p}\right).$$

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- A cusp form f has a Fourier series expansion
  $$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} q^n, \qquad \text{where } q = e^{2\pi i z}.$$
  It is called normalized if $\lambda_f(1) = 1$.

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  It is called normalized if $\lambda_f(1) = 1$.
- $S_k(\Gamma_0(q), \chi)$ is equipped with Petersson's inner product

$$\langle f, g \rangle_k = \int\int_{\Gamma_0(q)\backslash\mathbb{H}} f(z)\overline{g(z)} y^{k-2} dx dy.$$

- The Hecke operators $\{T_n\}_{(n,q)=1}$ are normal with respect to the inner product; more precisely, $T^* = \chi(n)T_n$ for $(n,q) = 1$
- One can find an orthogonal basis $H_k(\Gamma_0(q), \chi)$ of $S_k(\Gamma_0(q), \chi)$, formed of eigenvectors of all $\{T_n\}_{(n,q)=1}$.

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- We have

$$S_k(\Gamma_0(q), \chi) = S_k^{\text{new}}(\Gamma_0(q), \chi) \oplus S_k^{\text{old}}(\Gamma_0(q), \chi),$$

  where $S_k^{\text{old}}(\Gamma_0(q), \chi)$ is generated by the forms $f(dz)$ with $f \in S_k(q', \chi')$, $dq'|q$, $q' \neq q$, and χ' inducing χ.
- A cusp form in $S_k(\Gamma_0(q), \chi)$ is said to be primitive if it is a normalized Hecke eigenform in the new space.
- Let $H_k^*(\Gamma_0(q), \chi)$ be the set of primitive forms in $S_k(\Gamma_0(q), \chi)$.

Modular L-functions

For $f \in H_k^*(\Gamma_0(q), \chi)$, the L-function attached to f is defined as:

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}.$$

- $L(s,f)$ converges absolutely for $\Re(s) > 1$,
- admits an analytic continuation to $\mathbb{C}$ and
- satisfies a functional equation: $\Lambda(s,f) = \epsilon_f \Lambda(1-s,\overline{f})$ with $|\epsilon_f| = 1$.

Moments of $L(\frac{1}{2}, f)$

We consider harmonic averages of the form

$$\sum_{f}^{h} \alpha_f = \frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{f} \frac{\alpha_f}{\langle f, f \rangle}.$$

We define the 2ℓ-th moment

$$M_\ell^h(q, \chi) = \sum_{f \in H_k^*(\Gamma_0(q), \chi)}^{h} |L(\tfrac{1}{2}, f)|^{2\ell}.$$

Second Moments in the Level Aspect

Theorem (Duke, 1995)

For $k = 2$ and q prime, we have

$$M_1^h(q) = \sum_{f \in H_2^*(q)}^{h} L(\tfrac{1}{2}, f)^2 = \log q + O(q^{-\frac{1}{2}} \log q).$$

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For any fixed even k and q squarefree

- Iwaniec-Sarnak (2000): $M_1^h(q) \sim \log q$.

4th Moments in the Level Aspect

Theorem (Kowalski-Michel-Vanderkam, 2000)

For $k = 2$, as $q \to \infty$ through prime numbers

$$M_2^h(q) = P(\log q) + O_\epsilon(q^{-\frac{1}{12}+\epsilon}),$$

where P is a degree 6 polynomial with leading coefficient $\frac{1}{60\pi^2}$.

4th Moments in the Level Aspect

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- Balkanova-Frolenkov (2017): improved error term for $M_2^h(q)$ of size $O_\epsilon(q^{-\frac{25}{228}+\epsilon})$.
- Balkanova (2016): k is a fixed even integer and $q = p^v$ for a fixed prime p as $v \geq 3$. We have

$$M_2^h(q) = R(\log q) + O_{\epsilon,k,p}(q^{-\frac{1}{4}+\epsilon} + q^{-\frac{k-1-2\theta}{8-8\theta}+\epsilon}),$$

where R is a degree 6 polynomial .

Higher Moments

$$\mathcal{M}_\ell^h(q) = \frac{2}{\phi(q)} \sum_{\chi \pmod q} \sum_{f \in H_k^*(\Gamma_0(q),\chi)}^{h} |L(\tfrac{1}{2}, f)|^{2\ell}.$$

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For fixed odd integer $k \geq 5$, as $q \to \infty$ through the primes

- Djanković (2011): $\mathcal{M}_3^h(q) \ll q^\epsilon$.
- Stucky (2021): $\mathcal{M}_3^h(q) \ll (\log q)^9$.
- Chandee-Li (2017):

$$\frac{2}{\phi(q)} \sum_{\chi \pmod q} \sum_{f \in H_k^*(\Gamma_0(q),\chi)}^h \int_{-\infty}^{\infty} \left|\Gamma\left(\tfrac{k}{2} + it\right) L\left(\tfrac{1}{2} + it, f\right)\right|^6 \, dt$$

$$\sim \frac{42}{9!} b_3 (\log q)^9 \int_{-\infty}^{\infty} \left|\Gamma\left(\tfrac{k}{2} + it\right)\right|^6 dt. \quad (1)$$

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- Chandee-Li (2020): $\mathcal{M}_4^h(q) \ll q^\epsilon$.
- Chandee-Dunn-Li-Stucky (2024+): analogue of (1) for the 8-th moment.

Rankin-Selberg Convolutions of Modular Forms

Assume $(q, N) = 1$, and let $f \in H_k^*(q)$ and $g \in H_r^*(N)$ be primitive forms. We set

$$L(s, f \otimes g) = \zeta^{qN}(2s) \sum_{m \geq 1} \frac{a_f(m) a_g(m)}{m^s}, \qquad \text{for } \Re(s) > 1.$$

- analytic continuation to $\mathbb{C}$ unless $f = g$.
- functional equation $\Lambda(s, f \otimes g) = \Lambda(1 - s, f \otimes g)$, where

$$\Lambda(s, f \otimes g) = \left(\frac{qN}{4\pi^2}\right)^s \underbrace{\Gamma\left(s + \tfrac{|k-r|}{2}\right) \Gamma\left(s + \tfrac{k+r}{2} - 1\right)}_{\Gamma_g(s)} L(s, f \otimes g).$$

Moments in the Prime Level Aspect

Let $g \in H_r^*(N)$. For a fixed even $k < 12$, consider

$$M_\ell^h(q;g) = \sum_{f \in H_k^*(q)}^{h} \left(L\left(\tfrac{1}{2}, f \otimes g\right)\right)^\ell .$$

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As $q \to \infty$ through primes

- Luo (1999): $M_1^h(q;g) = \prod_{p|N}(1-p^{-1}) \log q + O_g(1)$.

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- Luo (1999): $M_1^h(q;g) = \prod_{p|N}(1-p^{-1}) \log q + O_g(1)$.
- Kowalski-Michel-Vanderkam (2002):

$$M_2^h(q;g) = P(\log q) + O_{g,\epsilon}\left(q^{-\frac{1}{12}+\epsilon}\right), \quad \deg(P) = 3.$$

Twists of Modular L-functions

- Luo-Ramakrishnan (1997): for $g \in H_r^*(N)$, p odd prime , m such that $p \nmid m$, and $\alpha \geq 0$, we have

$$\frac{1}{p^\nu} \sideset{}{^*}\sum_{\chi \mod p^\nu} \chi(m) L(\tfrac{1}{2}+\alpha, g \otimes \chi) \sim \tfrac{1}{p}(1-\tfrac{1}{p}) \frac{\lambda_g(m)}{m^{\frac{1}{2}+\alpha}}, \quad \text{as } \nu \to \infty.$$

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$$\frac{1}{p^\nu} \sideset{}{^*}\sum_{\chi \mod p^\nu} \chi(m) L(\tfrac{1}{2} + \alpha, g \otimes \chi) \sim \tfrac{1}{p}(1 - \tfrac{1}{p}) \frac{\lambda_g(m)}{m^{\frac{1}{2}+\alpha}}, \quad \text{as } \nu \to \infty.$$

- Bettin (2017): for $q = p^\nu$ with $\nu \geq 3$, $\chi \pmod N$ primitive and $\alpha \in \mathbb{C}$ with $\Re(\alpha) \ll \frac{1}{\log q}$, we have

$$\sum_{f \in H_2^*(q)}^{h} \lambda_f(m) L\left(\tfrac{1}{2} + \alpha, f \otimes \chi\right) = (1 - \tfrac{1}{p}) \frac{\chi(m)}{m^{\frac{1}{2}+\alpha}} + O\left(\frac{(Nm(1+|\Im(\alpha)|))^{\frac{1}{2}}}{q^{1-\epsilon}}\right),$$

provided $p \nmid mM$.

1st Moment of $L(\frac{1}{2}, f \otimes g)$ as f varies over $H_k^*(p^\nu)$

Theorem

Let $k, r \in 2\mathbb{N}$ be such that $k > 2r - 2$. Let $A > 0$ be such that $1 < A \leq \frac{k}{2} - 1 - \epsilon$ with $\epsilon > 0$. Then for $g \in H_r^*(D)$ with D being an odd prime and $(mD, p) = 1$, we have

$$\sum_{f \in H_k^*(p^\nu)}^{h} \lambda_f(m) L\left(\tfrac{1}{2}, f \otimes g\right)$$

$$= 2\left(1 - \frac{1}{p}\right)^2 \left(1 - \tfrac{1}{D}\right) \frac{\lambda_g(m)}{\sqrt{m}}$$

$$\times \left\{\gamma_0 - \log(4\pi) - \tfrac{1}{2}\log m + \nu \log p + \log D + \tfrac{1}{2}\log(k^2 - r^2)\right.$$

$$\left. + O\left(\frac{\log p}{p}\right) + O\left(\frac{\log D}{D}\right) + O\left(\frac{k}{k^2 - r^2}\right)\right\}$$

$$+ O\left(\sqrt[4]{m}(p^\nu D)^{\left(-\frac{5}{32}+\epsilon\right)}\right) + O\left(D^{\frac{\epsilon+1}{2}} m^{\frac{2A+\epsilon+1}{2}} p^{-2A(\nu-1)}\right).$$

Twisted 2nd Moment of $L(\frac{1}{2}+\omega, f\otimes g)$ as f varies over $H_k^*(p^\nu)$

Theorem

Let $k, r \in 2\mathbb{N}$ such that $k+r \geq 8$. For $g \in H_r^*(1)$, p odd prime, and $(m,p)=1$ we have

$$\sum_{f\in H_k^*(p^\nu)}^{h} \lambda_f(m) L\left(\tfrac{1}{2}+\omega_1, f\otimes g\right) L\left(\tfrac{1}{2}+\omega_2, f\otimes g\right)$$

$$\sim \left(\frac{4\pi^2}{p^\nu}\right)^{\omega_1+\omega_2} \frac{\Gamma_g\left(\frac{1}{2}-\omega_1\right)}{\Gamma_g\left(\frac{1}{2}+\omega_2\right)} \left(T_{1,\omega_1,\omega_2}+T_{1,\omega_1,-\omega_2}+T_{1,-\omega_1,\omega_2}+T_{1,-\omega_1,-\omega_2}\right)$$

$$+\left(\frac{4\pi^2}{p^\nu}\right)^{2\omega_1} \frac{\Gamma_g\left(\frac{1}{2}-\omega_1\right)}{\Gamma_g\left(\frac{1}{2}+\omega_2\right)} \left(T_{2,\omega_1,\omega_2}+\left(\frac{4\pi^2}{p^\nu}\right)^{2\omega_2} T_{2,\omega_1,-\omega_2}\right)$$

$$+\frac{\Gamma_g\left(\frac{1}{2}-\omega_1\right)}{\Gamma_g\left(\frac{1}{2}+\omega_2\right)} \left(T_{2,-\omega_1,\omega_2}+\left(\frac{4\pi^2}{p^\nu}\right)^{2\omega_2} T_{2,-\omega_1,-\omega_2}\right)$$

where

$$T_{1,\omega_1,\omega_2} = \tfrac{L(1,\mathrm{sym}^2 g)}{\omega_1-\omega_2}\left(1-\tfrac{1}{p}\right)\frac{\xi(1/2+\omega_1-\omega_2)^5 P_g(\omega_1)}{\zeta(2)\xi(1/2)^5 P_g(\omega_2)} \times \frac{\Gamma_g(1/2-\omega_1)}{\Gamma_g(1/2+\omega_2)}\zeta_{p^\nu}(1-2\omega_1)\zeta_{p^\nu}(1+2\omega_1) \times \sum_{de=m}\frac{(e/d)^{\omega_1}}{\sqrt{m}}\sum_{ab=d}\frac{\mu(a)\lambda_g(b)}{a}\Upsilon(p,ae,1)$$

and

$$T_{2,\omega_1,\omega_2} = \left(1-\tfrac{1}{p}\right)\frac{L(1-\omega_1+\omega_2, g\otimes g)}{\zeta(2-2\omega_1+2\omega_2)} \times \frac{\Gamma_g(1/2+\omega_2)}{\Gamma_g(1/2+\omega_1)}\zeta_{p^\nu}(1-2\omega_1)\zeta_{p^\nu}(1+2\omega_2) \times \sum_{de=m}\frac{e^{\omega_1}d^{-\omega_2}}{\sqrt{m}}\sum_{ab=d}\frac{\mu(a)\lambda_g(b)}{a^{1-\omega_1+\omega_2}}\Upsilon(p,ae,1-\omega_1+\omega_2)$$

2nd Moment of $L(\frac{1}{2}, f \otimes g)$ as f varies over $H_k^*(p^\nu)$

Corollary

Let $k, r \in 2\mathbb{N}$ such that $k + r \geq 8$. For $g \in H_r^*(1)$, we have

$$\sum_{f \in H_k^*(p^\nu)}^{h} L\left(\tfrac{1}{2}, f \otimes g\right)^2 \sim R_g(\nu)$$

for a polynomial $R_g(\nu)$ of degree 3 with leading coefficient $\frac{2}{\pi^2}\frac{(p-1)^2}{p^2}(\log p)^3 L\big(\operatorname{sym}^2 g, 1\big)$. All the other coefficients of R_g have been explicitly computed.

Approximate Functional Equation (AFE)

$$L\left(\tfrac{1}{2}+\omega_j, f\otimes g\right) = p^{-\nu\omega_j}\sum_{n\geq 1}\frac{\lambda_f(n)\lambda_g(n)}{\sqrt{n}}V_{g,\omega_j}\left(\frac{n}{p^\nu}\right)$$

$$+\, p^{-\nu\omega_j}\varepsilon_{\omega_j}(f\otimes g)\sum_{n\geq 1}\frac{\lambda_f(n)\lambda_g(n)}{\sqrt{n}}V_{g,-\omega_j}\left(\frac{n}{p^\nu}\right).$$

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Here

$$V_{g,\omega_j}(y) = \frac{1}{2\pi i}\int_{(3)}(4\pi^2)^{-s+\omega_j}\frac{\Gamma_g\left(\frac{1}{2}+s\right)}{\Gamma_g\left(\frac{1}{2}+\omega_j\right)}\left(\frac{\xi\left(\frac{1}{2}+s-\omega_j\right)}{\xi\left(\frac{1}{2}\right)}\right)^5\frac{P_g(s)}{P_g(\omega_j)}$$
$$\times\,\zeta_{p^\nu}(1+2s)y^{-s}\frac{ds}{s-\omega_j},$$

and

$$\varepsilon_{\omega_j}(f\otimes g) = \frac{(4\pi^2)^{2\omega_j}\Gamma_g\left(\frac{1}{2}-\omega_j\right)}{\Gamma_g\left(\frac{1}{2}+\omega_j\right)}.$$

Applying AFE gives

$$\sum_{f\in H_k^*(p^\nu)}^{h} \lambda_f(m) L\left(\tfrac{1}{2}+\omega_1, f\otimes g\right) L\left(\tfrac{1}{2}+\omega_2, f\otimes g\right)$$

$$= M_1 + \varepsilon_{\omega_1}(f\otimes g)M_2 + \varepsilon_{\omega_2}(f\otimes g)M_3 + \varepsilon_{\omega_1}(f\otimes g)\varepsilon_{\omega_2}(f\otimes g)M_4.$$

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We shall focus on

$$M_1 = p^{-\nu(\omega_1+\omega_2)} \sum_{d|m} \sum_{n_1,n_2\geq 1} \frac{\lambda_g(n_1)\lambda_g(n_2 d)}{\sqrt{n_1 n_2 d}} V_{g,\omega_1}\Big(\frac{n_1}{p^\nu}\Big) V_{g,\omega_2}\Big(\frac{n_2 d}{p^\nu}\Big)$$
$$\times \sum_{f\in H_k^*(p^\nu)}^{h} \lambda_f(n_1)\lambda_f\Big(\frac{n_2 m}{d}\Big).$$

Petersson's Trace Formula and Rouymi's Observation

Petersson's Trace Formula states

$$\begin{aligned}\Delta_{p^\nu}(a,b) &= \sum_{f\in H_k(p^\nu)}^{h} \lambda_f(a)\lambda_f(b)\\ &= \delta_{a,b} + 2\pi i^{-k}\sum_{p^\nu|c}\frac{S(a,b;c)}{c}J_{k-1}\left(\frac{4\pi\sqrt{ab}}{c}\right).\end{aligned}$$

We also use the following observation by Rouymi's (2011). For $\nu \geq 3$, we have

$$\begin{aligned}\Delta^*_{p^\nu}(a,b) &= \sum_{f\in H^*_k(p^\nu)}^{h} \lambda_f(a)\lambda_f(b)\\ &= \begin{cases}\Delta_{p^\nu}(a,b) - \frac{\Delta_{p^{\nu-1}}(a,b)}{p} & \text{if } p\nmid ab,\\ 0 & \text{else.}\end{cases}\end{aligned}$$

Applying Petersson's trace formula and Rouymi's observation yields

$$M_1 = M_1^D + M_{1,1}^{ND} + M_{1,2}^{ND},$$

with

$$M_1^D = \Big(1 - \frac{1}{p}\Big)\frac{p^{-\nu(\omega_1+\omega_2)}}{\sqrt{m}} \sum_{de=m} \sum_{ab=d} \frac{\mu(a)\lambda_g(b)}{a} \times \sum_{\substack{n\geq 1,\\ (n,p)=1}} \frac{\lambda_g(nae)\lambda_g(n)}{n} V_{g,\omega_1}\Big(\frac{nae}{p^\nu}\Big) V_{g,\omega_2}\Big(\frac{nad}{p^\nu}\Big).$$

After introducing dyadic partitions of unity in the n_1 and n_2 variables and manipulating the non-diagonal terms, we get

$$M_{1,1}^{ND} = 2\pi i^{-k} p^{-\nu(\omega_1+\omega_2)} \sum_{de=m} \frac{1}{\sqrt{d}} \sum_{ab=d} \frac{\mu(a)\lambda_g(b)}{\sqrt{a}} \sum_{p^\nu | c} \frac{1}{c^2} T(c)$$

and

$$M_{1,2}^{ND} = -\frac{2\pi i^{-k}}{p} p^{-\nu(\omega_1+\omega_2)} \sum_{de=m} \frac{1}{\sqrt{d}} \sum_{ab=d} \frac{\mu(a)\lambda_g(b)}{\sqrt{a}} \sum_{p^{\nu-1} | c} \frac{1}{c^2} T(c),$$

where

$$\begin{aligned} T(c) &= \sum_{M,N \geq 1} c \sum_{\substack{n_1,n_2 \geq 1 \\ (n_1 n_2, p)=1}} \lambda_g(n_1)\lambda_g(n_2) F_{M,N}(n_1,n_2) J_{k-1}\Big(\frac{4\pi\sqrt{n_1 a e n_2}}{c}\Big) \\ &\qquad \times S\big(n_1, aen_2; c\big) \\ &= \sum_{M,N\geq 1} T_{M,N}(c). \end{aligned}$$

Large Sieve Inequality

Deshouillers-Iwaniec (1983), Matomaki (2009):
Let r, s and d be pairwise coprime positive integers, with r, s squarefree. Let M, N, C be positive real numbers and g be a smooth function supported on $[M, 2M] \times [N, 2N] \times [C, 2C]$ with

$$\left| \frac{\partial^{j+k+\ell}}{\partial m^j \partial n^k \partial c^\ell} g(m,n,c) \right| \leq M^{-j} N^{-k} C^{-\ell} \quad \text{for } 0 \leq j, k, \ell \leq 2.$$

Let $X_d = \frac{\sqrt{dMN}}{sC\sqrt{r}}$. Then

$$\sum_m a_m \sum_n b_n \sum_{(c,r)=1} g(m,n,c) S(dm\bar{r}, \pm n, sc)$$

$$\ll_\epsilon C^\epsilon d^\theta sC\sqrt{r} \frac{(1+X_d^{-1})^{2\theta}}{1+X_d} \left(1 + X_d + \sqrt{\frac{M}{rs}}\right) \left(1 + X_d + \sqrt{\frac{N}{rs}}\right) ||\mathbf{a}||_\mathbf{2} ||\mathbf{b}||_\mathbf{2}.$$

Remove the co-primality condition in $T_{M,N}(c)$ to write it as

$$T_{M,N}(c) = TS^-(c,0) - \lambda_g(p)TS^-(c,1) + TS^-(c,2).$$

For example,

$$TS^-(c,0)$$
$$= \sum_{n_2 \geq 1} \lambda_g(n_2) \sum_{\substack{f \,(\text{mod } c), \\ (f,c)=1}} e\Big(\frac{aen_2 f}{c}\Big) c \sum_{n_1 \geq 1} \lambda_g(n_1) e\Big(\frac{n_1 \overline{f}}{c}\Big) \tilde{F}_{M,N}(n_1,n_2),$$

where we set

$$\tilde{F}_{M,N}(n_1,n_2) = F_{M,N}(n_1,n_2) J_{k-1}\Big(\frac{4\pi\sqrt{aen_1n_2}}{c}\Big).$$

Voronoi Summation

By Voronoï summation formula, we have

$$c\sum_{n_1\geq 1}\lambda_g(n_1)e\Big(\frac{n_1\overline{f}}{c}\Big)\tilde{F}_{M,N}(n_1,n_2)$$

$$=2\pi i^{-r}\sum_{n_1\geq 1}\lambda_g(n_1)e\Big(-\frac{n_1 f}{c}\Big)\int_0^\infty \tilde{F}_{M,N}(x,n_2)J_{r-1}\Big(\frac{4\pi\sqrt{n_1 x}}{c}\Big)\,dx\,.$$

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Using this in $TS^-(c,0)$, we get

$$TS^-(c,0)$$
$$=2\pi i^{-r}\phi(c)\sum_{n\geq 1}\lambda_g(n)\lambda_g(aen)\int_0^\infty \tilde{F}_{M,N}(x,n)J_{r-1}\Big(\frac{4\pi\sqrt{aenx}}{c}\Big)\,dx$$
$$+2\pi i^{-r}\sum_{h\neq 0}\sum_{n\geq 1}\lambda_g(n)\lambda_g(aen-h)$$
$$\times\sum_{\substack{f \,(\mathrm{mod}\; c),\\ (f,c)=1}} e\Big(\frac{fh}{c}\Big)\int_0^\infty \tilde{F}_{M,N}(x,n)J_{r-1}\Big(\frac{4\pi\sqrt{(aen-h)x}}{c}\Big)\,dx\,.$$

For $B = 0, 1, 2$ we get

$$TS^{-}(c,B)$$

$$= \underbrace{2\pi i^{-r}\phi(c)\sum_{n\geq 1}\lambda_g(n)\lambda_g(aenp^B)\int_0^\infty \tilde{F}_{M,N}(x,np^B)J_{r-1}\left(\frac{4\pi\sqrt{aenp^B x}}{c}\right)dx}_{\text{main term contribution}}$$

$$+ \underbrace{2\pi i^{-r}\sum_{h\neq 0} S(0,h;c){T_h}^{-}(c,B)}_{\text{error contribution}}$$

with

$${T_h}^{-}(c,B) = \sum_{n\geq 1}\lambda_g(n)\lambda_g(aenp^B - h) \times \int_0^\infty \tilde{F}_{M,N}(x,np^B)J_{r-1}\left(\frac{4\pi\sqrt{(aenp^B-h)x}}{c}\right)dx$$

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Estimate contribution of ${T_h}^{-}(c,B)$ using estimates on shifted convolution sums.

DFI type Shifted Convolution Sums

Kowalski-Michel-Vanderkam (2002):
Suppose that $f(x,y)$ is a smooth test function on $\mathbb{R}^+ \times \mathbb{R}^+$ that satisfies the condition

$$x^j y^k f^{(jk)}(x,y) \ll_{j,k} \left(1+\frac{x}{X}\right)\left(1+\frac{y}{Y}\right)P^{j+k} \quad \text{for all } \; j,k \geq 0$$

for some $X, Y, P \geq 1$. Let $(\alpha, \beta) = 1$ and $h \neq 0$. We have

$$D_f^{\pm}(\alpha,\beta;h) := \sum_{\alpha m \pm \beta n = h} \lambda_g(m)\lambda_g(n) f(\alpha m, \beta n) \ll_{g,\epsilon} P^{\frac{5}{4}}(X+Y)^{\frac{1}{4}}(XY)^{\frac{1}{4}+\epsilon}.$$

Thank you for listening!