The prime geodesic theorem in arithmetic progressions

(joint work with Ikuya Kaneko and Dimitrios Chatzakos)

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Conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$

Question

How to count the conjugacy classes of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$?

Hint

Γ acts on the Riemann sphere by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}, \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad z \in \mathbb{C} \cup \{\infty\}.$$

For $c \neq 0$, the fixed point equation $cz^2 + (d-a)z - b = 0$ is quadratic with discriminant $(d-a)^2 + 4bc = (a+d)^2 - 4$, hence the type of the transformation is governed by the trace $t = a + d$.

For $|t| < 2$ the transformation is elliptic with one fixed point in $\mathcal{H}$ and another one in $\overline{\mathcal{H}}$. For $|t| = 2$ the transformation is either the identity or it is parabolic with a single fixed point in $\mathbb{Q} \cup \{\infty\}$. For $|t| > 2$ the transformation is hyperbolic with two fixed points in $\mathbb{R}$.

Elliptic and parabolic conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$

Plan

We shall count the conjugacy classes of $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ according to their traces t. Without loss of generality, $t \geqslant 0$.

Consider an elliptic conjugacy class of Γ of trace $t = 0$ or $t = 1$.
The corresponding fixed points in $\mathcal{H}$ form the Γ-orbit of $\frac{t+\sqrt{t^2-4}}{2}$,
and the conjugacy class is represented by

$$\begin{pmatrix} t & -1 \\ 1 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix}.$$

Consider a parabolic conjugacy class of trace $t = 2$.
The corresponding fixed points form the Γ-orbit of ∞,
and the conjugacy class is represented by

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \quad \text{for a unique} \quad n \in \mathbb{Z}.$$

Hyperbolic conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$ (1 of 2)

A hyperbolic conjugacy class of trace $t \geqslant 3$ corresponds bijectively to a positive integer u and a Γ-class of quadratic forms in $\mathbb{Z}[x, y]$ of discriminant $(t^2 - 4)/u^2$. Moreover, it corresponds bijectively to an oriented closed geodesic of length $2 \log \left(\frac{t+\sqrt{t^2-4}}{2} \right)$ in $\Gamma\backslash\mathcal{H}$.

Here are some details. Pick an element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ from the conjugacy class. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ fixes the quadratic form $cx^2 + (d - a)xy - by^2$ of discriminant $t^2 - 4$. Now $u = \gcd(c, d - a, b)$ only depends on the conjugacy class, and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ fixes the primitive quadratic form

$$Ax^2 + Bxy + Cy^2 = \frac{cx^2 + (d - a)xy - by^2}{u}$$

of discriminant $B^2 - 4AC = (t^2 - 4)/u^2$. Hence in fact

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (t - Bu)/2 & -Cu \\ Au & (t + Bu)/2 \end{pmatrix}.$$

Hyperbolic conjugacy classes of $\mathrm{SL}_2(\mathbb{Z})$ (2 of 2)

On the other hand, if we consider the oriented geodesic in $\mathcal{H}$

$$\text{going from} \quad \frac{-B-\sqrt{B^2-4AC}}{2A} \quad \text{to} \quad \frac{-B+\sqrt{B^2-4AC}}{2A},$$

then we find that the representative element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (t-Bu)/2 & -Cu \\ Au & (t+Bu)/2 \end{pmatrix}$$

moves the points forward by hyperbolic distance $2\log\left(\frac{t+\sqrt{t^2-4}}{2}\right)$ on this geodesic.

Summary

For each discriminant $(t^2-4)/u^2$, we exhibited $h((t^2-4)/u^2)$ oriented closed geodesics of length $2\log\left(\frac{t+\sqrt{t^2-4}}{2}\right)$ in $\Gamma\backslash\mathcal{H}$.

An analogue of Chebyshev's counting function

In analogy with the Chebyshev counting function for prime powers, it is natural to count the oriented closed geodesics of $\Gamma\backslash\mathcal{H}$ (or equivalently the hyperbolic conjugacy classes of Γ) by considering them up to $\log x$ in length and weighting each of them by the length of the underlying primitive closed geodesic.

By Dirichlet's class number formula, the resulting sum equals

$$\Psi_\Gamma(x) = 2 \sum_{3 \leqslant t \leqslant x^{1/2}+x^{-1/2}} \sqrt{t^2-4}\, L(1, t^2-4),$$

where $L(s, t^2-4)$ is Zagier's L-series:

$$L(s, t^2-4) = \sum_{(t^2-4)/u^2 \equiv 0,1 \pmod 4} L(s, \chi_{(t^2-4)/u^2}) u^{1-2s}.$$

Initially observed by Kuznetsov (1978) and Bykovskiĭ (1994).

Zagier's L-series

Writing $t^2 - 4 = D\ell^2$, where D is a fundamental discriminant,

$$L(s, t^2-4) = \prod_{\boldsymbol{p}} \left(\sum_{0 \leqslant m < v_{\boldsymbol{p}}(\ell)} \boldsymbol{p}^{m(1-2s)} + \frac{\boldsymbol{p}^{v_{\boldsymbol{p}}(\ell)(1-2s)}}{1 - \chi_D(\boldsymbol{p})\boldsymbol{p}^{-s}} \right)$$

$$= \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{\#\left\{x \pmod{2q} : x^2 \equiv t^2 - 4 \pmod{4q}\right\}}{q^s}.$$

We used $\boldsymbol{p}$ in the Euler product as p will be a fixed prime later.

$L(s, t^2-4)$ satisfies GRH if and only if $L(s, \chi_D)$ does.

The completed series

$$\Lambda(s, t^2-4) = (t^2-4)^{s/2}\pi^{-s/2}\Gamma(s/2)L(s, t^2-4)$$

is entire and invariant under $s \leftrightarrow 1 - s$.

Prime geodesic theorem (1 of 2)

Theorem (Conrey–Iwaniec 2000)

For $\theta = 1/6$ and some $A > 0$ we have that

$$L(s, t^2 - 4) \ll_\varepsilon (t^2 - 4)^{\theta+\varepsilon} |s|^A, \qquad \mathrm{Re}(s) = 1/2.$$

Theorem (Soundararajan–Young 2013)

For $\sqrt{x} \leqslant u \leqslant x$ we have that

$$\Psi_\Gamma(x + u) - \Psi_\Gamma(x) = u + O_\varepsilon(u^{1/2} x^{1/4+\theta/2+\varepsilon}).$$

The proof is nontrivial, e.g. it uses that the coefficient of q^{-s} in $L(s, t^2 - 4)$ equals

$$\sum_{q_1^2 q_2 = q} \frac{1}{q_2} \sum_{k \pmod{q_2}} e\left(\frac{kt}{q_2}\right) S(k^2, 1; q_2).$$

Prime geodesic theorem (2 of 2)

Setting $u = x$ in the above mentioned short-interval estimate of Soundararajan–Young (2013), and applying a dyadic decomposition, we obtain a version of the prime geodesic theorem:

$$\Psi_\Gamma(x) = x + O_\varepsilon(x^{3/4+\theta/2+\varepsilon}).$$

Originally Selberg (1956) treated $\Psi_\Gamma(x)$ with his trace formula. In fact Iwaniec (1984) proved the following spectral counterpart of the Kuznetsov–Bykovskiĭ formula. For $1 \leqslant T \leqslant \sqrt{x}/\log^2 x$ we have

$$\Psi_\Gamma(x) = x + 2\mathrm{Re} \sum_{0 < t_j \leqslant T} \frac{x^{1/2+it_j}}{1/2+it_j} + O\left(\frac{x}{T}\log^2 x\right).$$

This readily yields the error term $O_\varepsilon(x^{3/4+\varepsilon})$ in the PGT, which was subsequently improved by Iwaniec (1984), Luo–Sarnak (1995), Cai (2002), Soundararajan–Young (2013), and Kaneko (2024).

New result

Inspired by the prime number theorem for arithmetic progressions, we restrict the trace t in our count to a residue class (modulo a prime for simplicity):

$$\Psi_\Gamma(x; p, a) = 2 \sum_{\substack{3 \leqslant t \leqslant x^{1/2} + x^{-1/2} \\ t \equiv a \pmod{p}}} \sqrt{t^2 - 4}\, L(1, t^2 - 4).$$

Our main result was conjectured by Golovchanskiĭ–Smotrov (1999):

Theorem (Chatzakos–Harcos–Kaneko 2023)

Let $p \geqslant 3$ be a prime. Then we have that

$$\Psi_\Gamma(x; p, a) = \begin{cases} \frac{1}{p-1} \cdot x + O_\varepsilon(x^{3/4+\theta/2+\varepsilon}) & \text{if } \left(\frac{a^2-4}{p}\right) = 1, \\ \frac{1}{p+1} \cdot x + O_\varepsilon(x^{3/4+\theta/2+\varepsilon}) & \text{if } \left(\frac{a^2-4}{p}\right) = -1, \\ \frac{p}{p^2-1} \cdot x + O_{p,\varepsilon}(x^{3/4+\theta/2+\varepsilon}) & \text{if } \left(\frac{a^2-4}{p}\right) = 0. \end{cases}$$

Sketch of the proof (1 of 5)

Let $L^p(s, t^2-4)$ denote $L(s, t^2-4)$ without the Euler factor at $\boldsymbol{p} = p$. The idea is to consider the sum

$$\Psi_\Gamma^\star(x; p^n, r) = 2 \sum_{\substack{3 \leqslant t \leqslant x^{1/2}+x^{-1/2} \\ t \equiv r \pmod{p^n}}} \sqrt{t^2-4}\; L^p(1, t^2-4).$$

Mimicking Soundararajan–Young (2013), we find that

$$\Psi_\Gamma^\star(x; p^n, r) = \frac{x}{p^n} + O_\varepsilon(x^{3/4+\theta/2+\varepsilon}).$$

Now if $t \equiv a \not\equiv \pm 2 \pmod{p}$, then writing $t^2 - 4 = D\ell^2$ as before (with D a fundamental discriminant), we see that $p \nmid \ell$ and

$$\chi_D(p) = \left(\frac{D}{p}\right) = \left(\frac{D\ell^2}{p}\right) = \left(\frac{t^2-4}{p}\right) = \left(\frac{a^2-4}{p}\right).$$

Hence the result follows for $a \not\equiv \pm 2 \pmod{p}$, because in that case

$$\Psi_\Gamma(x; p, a) = \left(1 - \left(\frac{a^2-4}{p}\right) p^{-1}\right)^{-1} \Psi_\Gamma^\star(x; p, a).$$

Sketch of the proof (2 of 5)

We need to work harder when $a \equiv \pm 2 \pmod{p}$. Without loss of generality, $a = \pm 2$. We decompose

$$\Psi_\Gamma(x; p, a) = \sum_{k=1}^{\infty} \Psi_\Gamma(x; p, a; k),$$

where

$$\Psi_\Gamma(x; p, a; k) = 2 \sum_{\substack{3 \leqslant t \leqslant x^{1/2} + x^{-1/2} \\ v_p(t-a) = k}} \sqrt{t^2 - 4}\, L(1, t^2 - 4).$$

The idea behind this decomposition is that, as we shall see, the Euler factor at p of $L(s, t^2 - 4)$ is constant within $\Psi_\Gamma(x; p, a; k)$.

Note that $p^k > t - a$ implies $\Psi_\Gamma(x; p, a; k) = 0$. Also, the condition $v_p(t - a) = k$ constrains t to $p - 1$ residue classes modulo p^{k+1}, and it yields $v_p(t^2 - 4) = k$.

Sketch of the proof (3 of 5)

If $k = 2n - 1$ is odd, then $p \mid D$ and $v_p(\ell) = n - 1$, hence

$$L(s, t^2 - 4) = \frac{1 - p^{n(1-2s)}}{1 - p^{1-2s}} L^p(s, t^2 - 4),$$

yielding

$$\begin{aligned}\Psi_\Gamma(x; p, a; 2n-1) &= \frac{p-1}{p^{2n}} \cdot \frac{1 - p^{-n}}{1 - p^{-1}} \cdot x + O_{p,\varepsilon}(x^{3/4+\theta/2+\varepsilon}) \\ &= (p^{1-2n} - p^{1-3n})x + O_{p,\varepsilon}(x^{3/4+\theta/2+\varepsilon}).\end{aligned}$$

It is important that the implied constant is independent of n.

Sketch of the proof (4 of 5)

If $k = 2n$ is even, then $p \nmid D$ and $v_p(\ell) = n$, hence

$$L(s, t^2 - 4) = \left(\frac{1 - p^{n(1-2s)}}{1 - p^{1-2s}} + \frac{p^{n(1-2s)}}{1 - \chi_D(p) p^{-s}} \right) L^p(s, t^2 - 4).$$

Writing $t = a + p^{2n} r$, we get $t^2 - 4 = 2ap^{2n} r + p^{4n} r^2$, hence

$$\chi_D(p) = \left(\frac{D}{p} \right) = \left(\frac{D\ell^2 p^{-2n}}{p} \right) = \left(\frac{2ar}{p} \right).$$

So among the $p - 1$ choices for t modulo p^{2n+1}, half the time $\chi_D(p)$ equals $+1$, and half the time it equals -1. Therefore,

$$\begin{aligned} \Psi_\Gamma(x; p, a; 2n) &= \frac{p-1}{p^{2n+1}} \left(\frac{1 - p^{-n}}{1 - p^{-1}} + \frac{(1/2)p^{-n}}{1 - p^{-1}} + \frac{(1/2)p^{-n}}{1 + p^{-1}} \right) x + \ldots \\ &= \left(p^{-2n} - \frac{p^{-3n}}{p+1} \right) x + O_{p,\varepsilon}(x^{3/4+\theta/2+\varepsilon}). \end{aligned}$$

As before, the implied constant is independent of n.

Sketch of the proof (5 of 5)

In the end,

$$\Psi_\Gamma(x; p, \pm 2) = c_p x + O_{p,\varepsilon}(x^{3/4+\theta/2+\varepsilon}),$$

where

$$c_p = \sum_{n=1}^{\infty} \left(p^{1-2n} - p^{1-3n} + p^{-2n} - \frac{p^{-3n}}{p+1} \right) = \frac{p}{p^2-1}.$$

Thanks for your attention!