The Shape of the Universe

Stacy Hoehn

Vanderbilt University
stacy.hoehn@vanderbilt.edu

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Examples of Homeomorphic Objects:
A doughnut and a coffee cup are homeomorphic.
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A torus and a sphere are not homeomorphic.
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The torus and the sphere are both called surfaces (or 2-manifolds) because they share this property.
Going Up a Dimension

Definition

If the area near any point in a space looks like a solid 3-dimensional ball, the space is called a 3-manifold.
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Examples:
- $\mathbb{R}^3$
- The Universe
The Surface of the Earth

The surface of the Earth is a surface (2-manifold).

How can we eliminate the infinite plane and torus as possibilities for the shape of the surface of the Earth? What other surfaces are there?
The Torus

To help us visualize the other surfaces (and eventually 3-manifolds), we will first view the torus a little bit differently. We will construct a torus by gluing together opposite edges of a square.
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This square, with its opposite sides identified, helps us depict the torus in the plane.
Tic-Tac-Toe on the Torus

Does anyone win?
What would you see if you were a two-dimensional being living in a torus?
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You would see copies of yourself in every direction, as far as your eye could see!
A Möbius band is constructed from a square by gluing the left side to the right side of the square after performing a half-twist.
A Möbius band contains an orientation-reversing curve. Clockwise becomes counterclockwise along this curve!
A Klein bottle is constructed from a square by gluing together the left and right edges the same way as for a torus, but now the top edge is flipped before being glued to the bottom edge.
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The Klein bottle is a surface.
Tic-Tac-Toe on the Klein Bottle

Does anyone win?
What would you see if you were a two-dimensional being living in a Klein bottle?

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The Klein Bottle (continued)

What would you see if you were a two-dimensional being living in a Klein bottle?

You would see copies of yourself in every direction, but sometimes you would be flipped!
The Klein bottle contains an orientation-reversing curve since it contains a Möbius band.

Surfaces that contain an orientation-reversing curve are called nonorientable. Surfaces that do not contain an orientation-reversing curve are called orientable.
No matter where we have been in the universe so far, if we choose a spot and travel out from it a short distance in all directions, we enclose a space that resembles a solid 3-dimensional ball. Thus, the universe appears to be some 3-manifold. But which 3-manifold is it?
Scientists have measured the amount of cosmic microwave background radiation in the universe, and they have found that it is distributed surprisingly uniformly.
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This limits the geometries (notions of distance, angles, and curvature) that can be placed on the universe’s 3-manifold to the following:

- **spherical geometry** with positive curvature
- **Euclidean geometry** with zero curvature
- **hyperbolic geometry** with negative curvature.
In Euclidean geometry, the sum of the angles in a triangle is 180 degrees. Meanwhile, in spherical geometry, the sum of the angles is more than 180 degrees, and in hyperbolic geometry, the sum of angles is less than 180 degrees.
Geometry and the Eventual Fate of the Universe

- **Spherical Geometry** ⇒ The universe will eventually recollapse.

  - Euclidean Geometry ⇒ The universe will continue to expand forever, but just barely (i.e. the rate of expansion will approach 0.)

  - Hyperbolic Geometry ⇒ The universe will continue to expand forever, gradually approaching a (positive) constant rate of expansion.
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Data from a NASA probe in 2001 suggests that the curvature of the universe is very close to 0. This either means that we live in a Euclidean universe or we live in a spherical or hyperbolic universe with extremely low curvature.
If we assume that the universe is a Euclidean 3-manifold, does this help us determine which 3-manifold the universe is?

Theorem
There are exactly 18 Euclidean 3-manifolds.
- 6 are compact (finite) and orientable
- 4 are compact (finite) and nonorientable
- 4 are noncompact (infinite) and orientable
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Euclidean 3-Manifolds

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The 8 nonorientable Euclidean 3-manifolds all contain an orientation-reversing loop. If you were to fly from Earth along such a loop, you would eventually return home with your orientation reversed. It would appear that you had returned to a mirror image of Earth.

If the universe was nonorientable, there would be strange physical consequences that have not yet been observed. While they could be happening outside of our field of vision, it is unlikely that our universe is nonorientable. It is more likely that the universe is one of the 10 orientable Euclidean 3-manifolds.
The simplest orientable, compact, Euclidean 3-manifold is the 3-torus. It is a generalization of the torus in a higher dimension.

Instead of gluing together opposite edges of a square, the opposite faces of a cube are joined.
The 3-Torus (continued)

If you were somehow in the 3-torus and looked around, you would see copies of yourself in each direction, and past these copies, other copies would be visible as far as the eye could see.
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If the universe is a 3-torus, you could fly from Earth in a particular direction and, without ever changing course, eventually return home.
In the quarter-twist and half-twist 3-manifolds, four of the faces of the cube are glued together just as for the 3-torus.

The front and back faces, however, are glued together after a twist of 90 degrees (quarter-twist) or 180 degrees (half-twist).
In the quarter-twist and half-twist 3-manifolds, four of the faces of the cube are glued together just as for the 3-torus.

The front and back faces, however, are glued together after a twist of 90 degrees (quarter-twist) or 180 degrees (half-twist).

If you were inside the cube for the quarter-twist manifold and stared out the front or back face, you would see copy after copy of yourself, each one a 90-degree rotation of the preceding copy.
The sixth-twist and third-twist 3-manifolds are both obtained by gluing faces on a hexagonal prism instead of a cube. Each parallelogram face is glued to the face directly opposite it.

The two hexagonal faces are then glued together after a twist of 60 degrees (sixth-twist) or 120 degrees (third-twist).
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If you looked out of one of the hexagonal faces of the prism for the sixth-twist manifold, you would see copy after copy of yourself, each rotated 60 degrees more than the preceding copy.
The last compact, orientable, Euclidean 3-manifold is the Double Cube manifold.

You would see yourself with a very peculiar perspective in this 3-manifold!
It is likely that the universe has the shape of one of the six compact, orientable, Euclidean 3-manifolds that we just described. However, there are also 4 non-compact, orientable, Euclidean 3-manifolds.
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Many cosmologists believe that the universe is not infinite in nature, but we still must consider these 4 non-compact options as possibilities until there is substantial evidence against them.
Can We Narrow Down the Possibilities Even Further?

The simplest procedure is to look for copies of our galaxy, the Milky Way, in the night sky. If we find copies, we can look at their pattern to determine the gluing diagram for the universe.

Possible Problems:

Light travels at a finite speed, so looking out into the universe, we are looking back in time. Even if we someday find a copy of our galaxy, we may not recognize it because it might have looked different in its younger years.

The fundamental domain for the universe is huge (possibly bigger than our sphere of vision) and is continuing to expand.
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