Lab 6 (Capacitors) Practice Problem Sheet

Here are some problems to prepare you for the quiz on Lab 6 (capacitors). If you are comfortable with these problems, you should easily do well on the quiz. Some solutions you can use to check your work are at the end of this document.

Helpful things to keep in mind for RC circuits:

Equation for stuff going down:

$$D(t) = Ae^{-t/\tau}$$

Equation for stuff going up:

$$U(t) = B\left(1 - e^{-t/\tau}\right)$$

τ is the time constant ($\tau = RC$ for circuits with just batteries, resistors, and capacitors), which controls how quickly these equations rise and fall. A bigger τ means it takes longer for these equations to go up or down, while a shorter τ means that they go up or down relatively quickly.

A and B just denote arbitrary constants that we will replace with whatever constant is appropriate for the quantity we are using the equations to describe. For example, if we are using these equations to describe voltage, we will see that we should choose V_0 as our constant in front.

GOLDEN EQUATIONS:

$$(i)\ C = \frac{Q(t)}{V_C(t)} \qquad (iii)\ V_R(t) = I(t)R$$

$$(ii)\ V_0 = V_C(t) + V_R(t) \qquad (iv)\ I(t) = \frac{dQ}{dt}$$

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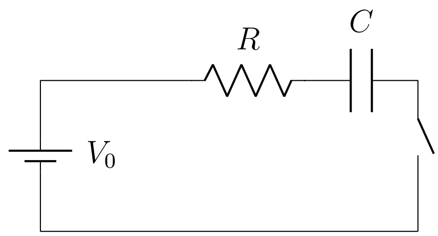
1. **Deriving the equations for a charging capacitor (without calculus).**

You've got a battery, resistor, and capacitor hooked up in series, as shown in the circuit diagram below.



Initially, the switch is open and the capacitor is uncharged. At time $t = 0$, you close the switch, which completes the circuit and allows the capacitor to start charging up.

In this question, we'll first guess what $Q(t)$ is, and then use it to find $V_C(t)$, $V_R(t)$, and $I(t)$.

(a) What is the equation for the (absolute value of) charge $Q(t)$ on each capacitor plate? Your thought process should have three steps.

- First, does it go up or down? (that determines which equation you use)
- Second, what's the time constant τ?
- Third, what's the constant in front? (use the fact that $C = Q_{final}/V_0$, since $V_C \to V_0$)

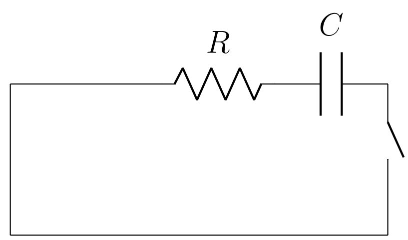
(b) We know that $C = Q(t)/V_C(t)$, where C (the capacitance of the capacitor) is always constant, and $V_C(t)$ is the voltage across the capacitor over time. Using this relationship, and the equation for $Q(t)$ you got in the previous part, what is the voltage across the capacitor $V_C(t)$ over time?

(c) The second of Kirchhoff's laws says that the sum of the voltages around a closed loop is zero. For us, this means that $V_0 = V_C(t) + V_R(t)$, where V_0 (the voltage of the battery) is always constant. Using this relationship, and the equation for $V_C(t)$ you got in the previous part, what is the voltage across the resistor $V_R(t)$ over time?

(d) Ohm's law still works! It says that $V_R(t) = I(t)R$ for all times, with R always constant. Using this relationship, and the equation for $V_R(t)$ you got in the previous part, what is the current through the resistor (and hence, the current through everything, since current is the same in each part of a series circuit) $I(t)$ over time?

2. **Deriving the equations for a discharging capacitor (without calculus).**

You've got a resistor and capacitor hooked up in series, as shown in the circuit diagram below.



Initially, the switch is open and the capacitor is *fully charged*. At time $t = 0$, you close the switch, which completes the circuit and allows the capacitor to start *discharging*.

After doing the previous question, use this question to check your understanding. Keep in mind that **we have removed the battery**, so $V_0 = 0$.

(a) What is the equation for the (absolute value of) charge $Q(t)$ on each capacitor plate?

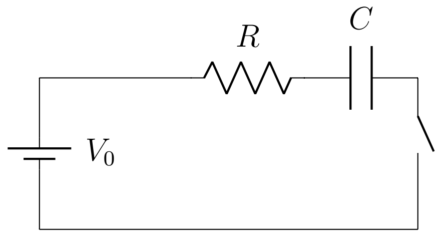
(b) What is the voltage across the capacitor $V_C(t)$ over time?

(c) What is the voltage across the resistor $V_R(t)$ over time?

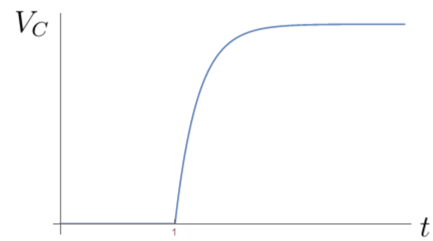
(d) What is the current through the resistor and capacitor $I(t)$ over time?

3. **The time constant for RC circuits.**

You've hooked up a battery, resistor, and capacitor in series, as in the circuit diagram below. You put the probes of a voltmeter on both sides of the capacitor to measure the voltage across it over time; you close the circuit at $t = 1$, and obtain the voltage graph on the right.



Circuit diagram for your set-up.



Measured voltage across the capacitor over time.

You use a fitting program to fit your result to the function

$$A\left(1 - e^{-B(t-t_0)}\right) + C \ ,$$

and find that the best fit parameter values are $A = 3$, $B = 0.891$, $t_0 = 1$, and $C = 0$.

(a) If you originally measured the voltage across the capacitor in volts, what is the final voltage across the capacitor?

(b) What is the time constant τ, assuming you tracked time during your voltage measurement in seconds?

(c) Suppose that you used a $22,000$ μF capacitor in your circuit. Given the time constant you found in the previous part, what is the resistance of the resistor you used?

(d) You take out the resistor you used, and replace it with two resistors in series: one 10 Ω resistor, and one 20 Ω resistor. If you keep the $22,000$ μF capacitor the same, what is the new time constant τ?

(e) Imagine you did similar measurements using a capacitor with an unknown capacitance C, and a resistor with an unknown resistance R. You find that the circuit's time constant is 0.9 s. Then you add a 10 Ω resistor to the circuit (in series with everything else), and find that the circuit's new time constant is 1.1 s. Using this information only, what is R, and what is C?

4. **Bonus: Deriving the equation for charge on a capacitor plate.**

This is just for fun, if you happen to find solving differential equations fun (in which case you're a little weird). The equations that we use to describe stuff going up and stuff going down don't come from nowhere; you can actually derive them if you know how to solve basic differential equations.

The setup is just as before: we've got a battery with potential V_0, a resistor with resistance R, and a capacitor with capacitance C, all in series with each other. The circuit is open and the capacitor is uncharged until $t = 0$, when we close a switch and let the capacitor begin charging.

(a) Using Kirchhoff's second law, the definition of capacitance, Ohm's law, and $I = \frac{dQ}{dt}$, write down a first order differential equation in terms of Q.

(b) Write the differential equation you found in the form

$$\frac{dQ}{dt} + AQ = B \ .$$

What are the constants A and B in terms of V_0, R, and C?

(c) There's two common ways to solve simple first order differential equations like these. One is to multiply both sides by e^{At} to obtain

$$Be^{At} = e^{At}\frac{dQ}{dt} + Ae^{At}Q = \frac{d}{dt}\left(e^{At}Q\right) \ .$$

Then, to solve the differential equation, we can just integrate both sides, because

$$\int \frac{d}{dt}\left(e^{At}Q\right) = e^{At}Q + \text{const.}$$

What do you get when you integrate both sides of

$$\frac{d}{dt}\left(e^{At}Q\right) = Be^{At}$$

and rearrange to solve for $Q(t)$? Remember that we still have an arbitrary constant of integration.

(d) Apply our boundary condition $Q(0) = 0$ (since the capacitor is initially uncharged). What is your final equation for $Q(t)$?

(e) What do you get if you use the boundary condition $Q(0) = CV_0$ instead? What does this physically mean?

Partial solutions.

1. (a) Charge goes up for a charging capacitor, so our equation is

$$Q(t) = Q_f \left(1 - e^{-t/RC}\right) = CV_0 \left(1 - e^{-t/RC}\right) \; .$$

(b) $V_C(t) = V_0 \left(1 - e^{-t/RC}\right)$

3. (a) ~ 3 V

(b) 1.12 s

(c) 51 Ω

(d) 0.66 s

(e) 0.02 F and 45Ω

4. (a) Rewriting $V_0 = V_C(t) + V_R(t)$ gives you

$$V_0 = \frac{dQ}{dt} R + \frac{1}{C} Q \; .$$

(b) Rewriting the above expression gives

$$\frac{dQ}{dt} + \frac{1}{RC} Q = \frac{V_0}{R}$$

with $A = 1/RC$ and $B = V_0/R$.

(c) $Q(t) = CV_0 + (const.)e^{-t/RC}$

(d) $Q(t) = CV_0 \left(1 - e^{-t/RC}\right)$

(e) $Q(t) = CV_0$

Meaning: if the capacitor starts off fully charged, it does not charge more. Nothing happens, in fact—no current runs through the circuit at all!