Lab 9 (AC Circuits) Practice Problem Sheet

Here are some problems to prepare you for the quiz on Lab 9 (AC circuits). If you are comfortable with these problems, you should easily do well on the quiz. Some solutions you can use to check your work are at the end of this document.

These problems are a little bit wordy, but that's mainly because I'm using them to try and help you learn stuff rather than test your understanding.

Helpful things to keep in mind for AC circuits:

Ohm's law $V = IR$ generally doesn't work for AC circuits (because certain circuit components can cause I to become out of phase with V), so we define a new quantity—the impedance Z—so that $V = IZ$ is true. The upside is that something that *looks like* Ohm's law is now true (allowing us to analyze AC circuits similarly to how we've analyzed DC circuits so far), but the downside is that we now have to be much more careful about where 'resistance' comes from in our circuits. In particular, now both inductors and capacitors contribute a sort of resistance.

Z is defined as the complex number

$$Z = R + i(X_L - X_C) \ ,$$

where $X_L = \omega L$ is called the *inductive reactance*, and is the 'resistance' associated with inductors, while $X_C = 1/(\omega C)$ is called the *capacitive reactance*, and is the 'resistance' associated with capacitors.

Z is a complex number because it contains two bits of information: the magnitude $|Z|$, which tells us how much impedance there is (since an imaginary amount of Ohms makes no sense); and the phase shift ϕ, which tells us how much I is phase shifted away from V. The formulas are

$$\begin{aligned} Z &= |Z|e^{i\phi} \\ |Z| &= \sqrt{R^2 + (X_L - X_C)^2} \\ \tan\phi &= \frac{X_L - X_C}{R} \ . \end{aligned}$$

When you plot the 'impedance' as a function of frequency, you're really plotting $|Z|$.

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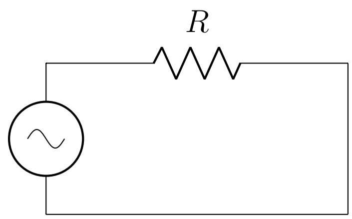
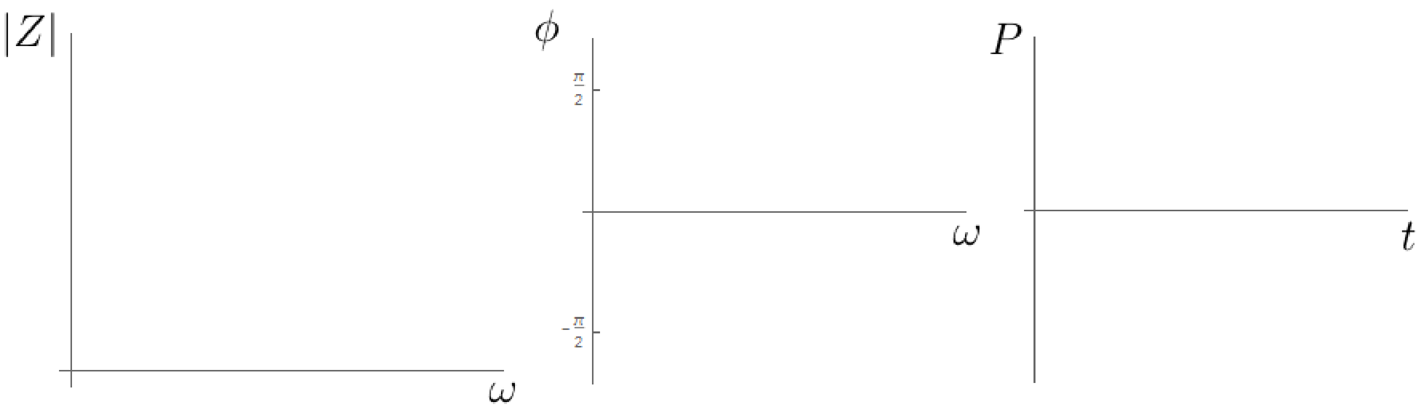
1. **Resistors in AC circuits.**

You've got an AC voltage source with frequency ω and a resistor with resistance R hooked up in series, as shown in the circuit diagram below.





(a) What is the impedance $|Z|$ for this circuit? Plot $|Z|$ as a function of frequency.

(b) What is the phase shift ϕ for this circuit? Plot ϕ as a function of frequency.

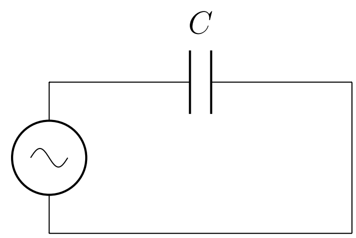
(c) Does $V = IR$ for this circuit? Why or why not?

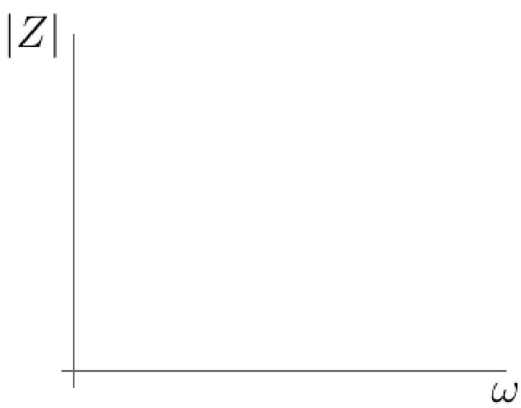
(d) Suppose $V(t) = V_0 \sin(\omega t)$. Based on your previous answer, come up with a simple expression for $I(t)$, and use it to write power P as a function of time. Plot P as a function of time.

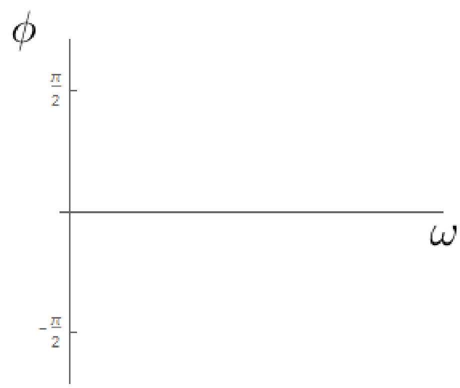
(e) $P > 0$ means power is being consumed, while $P < 0$ means that power is being generated. What does your answer from the previous part physically mean?

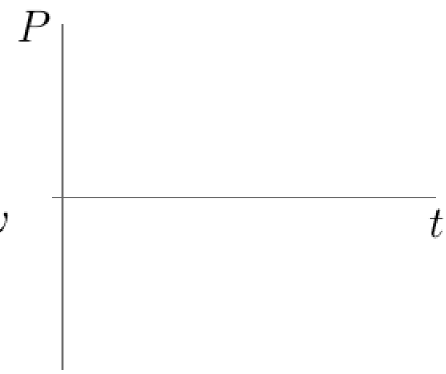
2. **Capacitors in AC circuits.**

You've got an AC voltage source with frequency ω and a capacitor with capacitance C hooked up in series, as shown in the circuit diagram below.









(a) What is the impedance $|Z|$ for this circuit? Plot $|Z|$ as a function of frequency.

(b) What is the phase shift ϕ for this circuit?[1] Plot ϕ as a function of frequency.

(c) Does $V = IR$ for this circuit? Why or why not?

(d) $\phi > 0$ means that I peaks slightly *later* than V, and $\phi < 0$ means that I peaks slightly *earlier* than V. Based on your answer to (b), which way is I shifted in this case? Can you think of an intuitive explanation for this?[2]

(e) In an AC circuit, the capacitor is constantly switching between charging and discharging, since the direction current is being pushed in constantly switches from clockwise to counter-clockwise. What does this suggest about power as a function of time? Plot P as a function of frequency. How is this different from when there is just a resistor?[3]

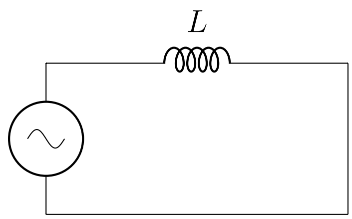
[1]Hint: What is $\tan^{-1}(-\infty)$?

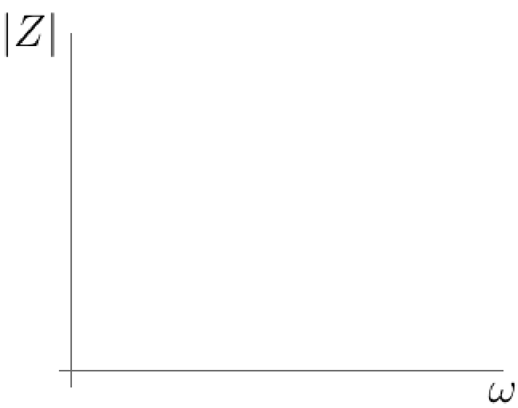
[2]Hint: Remember that high V_C means the capacitor is charged, and that low V_C means the capacitor is uncharged. Is I high just after a capacitor is fully charged, or just before?

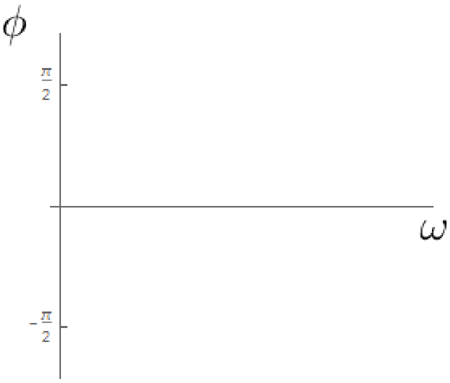
[3]Hint: What is the relationship between charging/discharging and power consumption/generation?

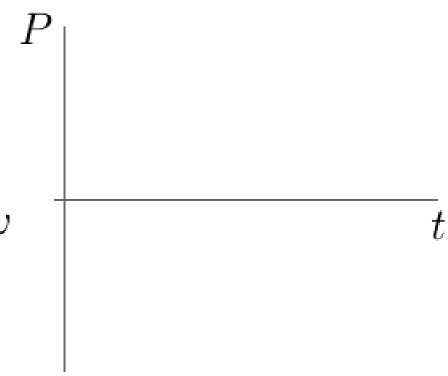
3. **Inductors in AC circuits.**

You've got an AC voltage source with frequency ω and an inductor with inductance L hooked up in series, as shown in the circuit diagram below.









(a) What is the impedance $|Z|$ for this circuit? Plot $|Z|$ as a function of frequency.

(b) What is the phase shift ϕ for this circuit?[4] Plot ϕ as a function of frequency.

(c) Does $V = IR$ for this circuit? Why or why not?

(d) $\phi > 0$ means that I peaks slightly *later* than V, and $\phi < 0$ means that I peaks slightly *earlier* than V. Based on your answer to (b), which way is I shifted in this case? Can you think of an intuitive explanation for this?[5]

(e) In an AC circuit, the inductor is constantly switching between resisting current changes and not doing much, since current constantly switches between changing quickly and not changing very quickly. What does this suggest about power as a function of time? Plot P as a function of frequency. How is this different from when there is just a resistor?[6]

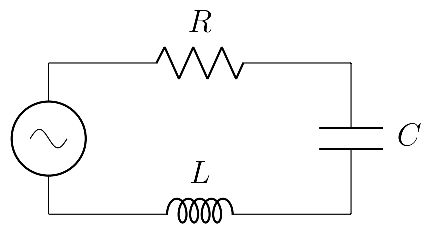
[4]Hint: What is $\tan^{-1}(\infty)$?

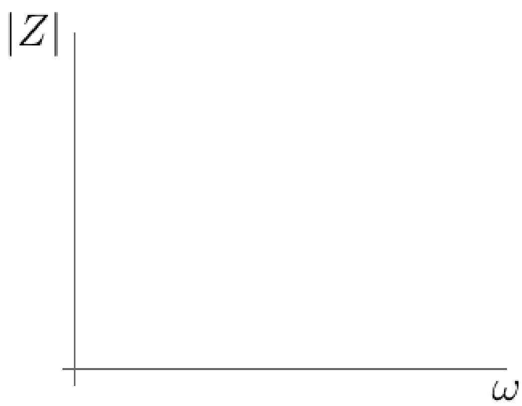
[5]Hint: How do inductors respond to changes in current? How quickly can a change in voltage cause a change in current?

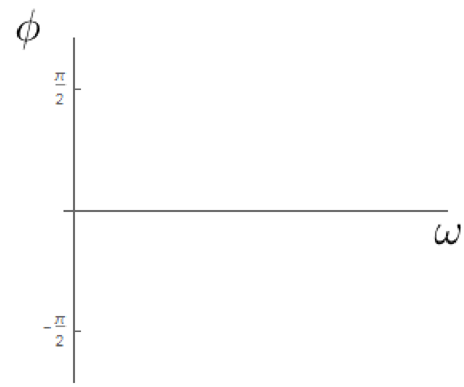
[6]Hint: What is the relationship between resisting current/not resisting current and power consumption/generation?

4. **AC circuits with resistors, capacitors, and inductors.**

You've got an AC voltage source with frequency ω, a resistor with resistance R, a capacitor with capacitance C, and an inductor with inductance L hooked up in series, as shown in the circuit diagram below.







(a) What is the impedance $|Z|$ for this circuit? Plot $|Z|$ as a function of frequency. Which circuit component most affects $|Z|$ when ω is small? Which circuit component most affects $|Z|$ when ω is large?

(b) What is the phase shift ϕ for this circuit? Plot ϕ as a function of frequency. What is ϕ when ω is small? What is ϕ when ω is large? How do your answers compare to the phase shifts from the previous two problems?

(c) At which frequency is $|Z|$ as small as possible? What is the value of $|Z|$ at this frequency, and what is ϕ at this frequency?

5. **The low and high frequency limits.**

It is frustrating to try and memorize how X_L and X_C depend on frequency without having some intuition for them. One way to intuitively understand their behavior is by looking at their low and high frequency limits. In other words, for very small or very large ω, how big are X_L and X_C, and why?

In the low frequency limit, AC circuits behave like DC circuits. Physically, what is happening is that the voltage is changing much more slowly than the time it takes for capacitors to charge or for inductors to stop resisting changes in current.

(a) You can quantify the time it takes for voltage to change using the period $\frac{2\pi}{\omega}$ of the voltage oscillations. In particular, the period is the time it takes for the voltage to go up and down exactly once. What length of time do we use to quantify how quickly a capacitor charges/discharges? What length of time do we use to quantify how quickly an inductor settles down?

(b) In an AC circuit with just a capacitor, what length of time has to be much smaller than which other length of time for us to be in the 'low frequency' limit? In an AC circuit with just an inductor, what length of time has to be much smaller than which other length of time for us to be in the 'low frequency' limit?

In the low frequency limit, the impedance reflects how easily current flows in the corresponding DC circuit after you have waited a while. High impedance means current does not flow much in the DC circuit, while low impedance means it flows easily in the DC circuit.

(c) How do X_C and X_L reflect what happens in RC and RL circuits after a long time, respectively, in the low frequency limit?

In the high frequency limit, voltage switches between pushing current clockwise and counterclockwise very quickly.

(d) How do X_C and X_L reflect what happens in the high frequency limit? Remember that capacitors need current to be flowing in the same direction for a while to charge up, and that inductors strongly resist quick changes in current.

Partial solutions.

1. (a) $|Z| = R$

(b) $\phi = \tan^{-1}(0) = 0$

(c) $\phi = 0$, so V and I are in phase. $V = IZ = I|Z|e^{i\phi} = IRe^0 = IR$

(d) Ohm's law is true, so $I(t) = V/R = \frac{V_0}{R}\sin(\omega t)$. Then $P = IV = \frac{V_0^2}{R}\sin^2(\omega t)$. Notice that $P \geq 0$ since $\sin^2(\omega t) \geq 0$.

2. (a) $|Z| = X_C = \frac{1}{\omega C}$

(b) Taking $R \to 0$ gives us $\tan\phi = -\infty$, and $\phi = \tan^{-1}(-\infty) = -\frac{\pi}{2}$.

(c) Ohm's law does not hold, because there is a nonzero phase shift.

(d) $\phi < 0$, so I peaks a little bit before V peaks. Here's how to think about this. Suppose we start watching the system when current is at a peak. The capacitor must be uncharged (because current is very low when a capacitor is fully charged) at this time. As the capacitor charges up, V goes up and I goes down. When the capacitor is fully charged, we hit a V peak, and $I = 0$.

3. (a) $|Z| = X_L = \omega L$

(b) Taking $R \to 0$ gives us $\tan\phi = \infty$, and $\phi = \tan^{-1}(\infty) = \frac{\pi}{2}$.

(c) Ohm's law does not hold, because there is a nonzero phase shift.

(d) $\phi > 0$, so I peaks a little bit after V peaks. If voltage is at a peak, since inductors cause currents to change more slowly than otherwise, it will take a little bit of time before current is also at a peak.

5. (a) We use the time constants we defined earlier: $\tau_{RC} = RC$ and $\tau_{LR} = L/R$.

(b) We need the period of oscillations (the time it takes for voltage to change a lot) to be much greater than the time it takes for a capacitor to charge, or for the time it takes for an inductor to settle down. In the RC circuit case, we want the period $2\pi/\omega$ to be much greater than RC; in the RL circuit case, we want the period $2\pi/\omega$ to be much greater than L/R.